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MAGNETIC MATERIALS IN TWO-DIMENSIONAL FIELDS

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Introduction

The purpose of this report is to formulate and illustrate a complex "magnetic interface theorem" for treating two-dimensional magnetic fields at a cylindrical interface between media of differing permeability.

The interface theorem, like the analogous current sheet theorem of AADD-102, is useful for treating two-dimensional field problems by exploiting the theory of functions of a complex variable.

Complex Representation

Consider a two-dimensional vector field $\underline{B}$. Its components, $B_X = B_X(X,Y)$ and $B_Y = B_Y(X,Y)$, are real algebraic functions of the real coordinates X and Y , and are assumed to be independent of the third space coordinate perpendicular to the X, Y plane.

We define Z and B as the complex combinations

$$Z = X + i Y \quad (1)$$

$$B = B_Y + i B_X \quad (2)$$

One way to assay B for analyticity as a function of Z is to evaluate the line integral $\oint B dz$ around a closed path in the Z -plane. Let the path be

$$Z = z(s) = x(s) + i y(s) \quad (3)$$

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as a function of the distance s along the path. If θ is the inclination of the tangent to the path at some point,

$$dz = \left(\frac{dx}{ds} + i \frac{dy}{ds} \right) ds = \left(\cos \theta + i \sin \theta \right) ds = e^{i\theta} ds$$

and hence

$$\begin{aligned} B dz &= \left(B_Y + i B_X \right) \left(\cos \theta + i \sin \theta \right) ds \\ &= \left[\left(B_Y \cos \theta - B_X \sin \theta \right) + i \left(B_X \cos \theta + B_Y \sin \theta \right) \right] ds \\ &= \left(B_n + i B_s \right) ds, \end{aligned} \quad (4)$$

where B_n and B_s are the components of $\underline{B}$ normal and tangential to the path as illustrated in Figure 1. Thus we see that $\oint \underline{B} dz$ taken in the right-handed sense around a closed contour gives, as the real part, the total flux of $\underline{B}$ into the area and, as the imaginary part, the circulation of $\underline{B}$ around the contour.

Note that the alternative choice, $B_X + i B_Y$, for (2) does not lead to this useful interpretation.

Fig. 1. Components of $\underline{B}$ at a point on the curve $Z = z(s)$.

By Green's and Stokes' theorems we see that

$$\int B dz = - \iint \left(\frac{\partial B_X}{\partial X} + \frac{\partial B_Y}{\partial Y} \right) dXdY + i \iint \left(\frac{\partial B_Y}{\partial X} - \frac{\partial B_X}{\partial Y} \right) dXdY \quad (5)$$

where the double integrals are taken over the area enclosed by the path (3).

When the vector $\underline{B}$ is the magnetic induction (gauss), and we write dI and dJ abamperes, respectively, for the true and Amperian currents normal to the Z -plane through the area element $dXdY$, we find, since $\text{div } \underline{B} = 0$ and $(\text{curl } \underline{B})_Z dXdY = 4\pi (dI + dJ)$ by Maxwell's equations,

$$\oint B dz = 4\pi i (I + J), \quad (6)$$

where I is the total true current through the area enclosed by the contour and J the total Amperian current through the enclosed area. Note that I and J are real and that $2(I + J)$ is the total residue of B within the contour.

We may perform the same analysis with the vector $\underline{H} = H_Y + i H_X$ representing the magnetic field strength (oersteds).

By Maxwell's equations $(\text{curl } \underline{H})_Z dXdY = 4\pi dI$ and as sources for $\underline{H}$ we visualize an infinite column with cross section $dXdY$ of north pole strength, dP , such that

$$\text{div } \underline{H} dXdY = 4\pi dP.$$

Then

$$\oint H dz = 4\pi i (I + i P). \quad (7)$$

Thus for the H field a "pole filament" plays the role of an imaginary current filament and the total residue of $\underline{H} = H_Y + i H_X$ within a contour is $2(I + i P)$ through the enclosed area.

We may also consider the vector $\underline{M}$ representing magnetic moment per unit volume; in electromagnetic (gaussian) units

$$\underline{B} = \underline{H} + 4\pi \underline{M}.$$

With

$$\underline{M} = \underline{M}_Y + i \underline{M}_X = \left[\left(\underline{B}_Y - \underline{H}_Y \right) + i \left(\underline{B}_X - \underline{H}_X \right) \right] / 4\pi \quad (8)$$

we find from (4) and (6):

$$\oint \underline{M} dz = \frac{1}{4\pi} \iint \left[-\operatorname{div} \underline{M} + i (\operatorname{curl} \underline{M})_Z \right] dXdY \\ = P + i J \quad (9)$$

Constant Permeability

In a simply connected region of constant permeability μ , $\underline{B} = \mu \underline{H}$ and $\operatorname{curl} \underline{B} = \operatorname{curl} \mu \underline{H} = \mu \operatorname{curl} \underline{H}$; therefore $dJ = 0$, i.e., the Amperian currents are zero, and in the absence of true currents, $\oint \underline{B} dz = 0$ for any closed path within the region. Then

$$\underline{B} = \underline{B}(Z) = \mu \underline{H}(Z) = \frac{4\pi\mu}{\mu - 1} \underline{M}(Z)$$

and the indefinite integrals

$$\int \underline{B} dZ = \mu \int \underline{H} dZ = \frac{4\pi\mu}{\mu - 1} \int \underline{M} dZ = W(Z) - W_0 \quad (10)$$

are all analytic functions of Z within any simply connected region of constant permeability.

Interface Between Constant Permeability Regions

Let the cylindrical interface between two constant permeability regions intersect the Z -plane in the curve $Z = z$ shown in Figure 2. Quantities to the right and left with respect to a chosen direction along the interface curve are denoted by the subscripts R and L.

Consider integrals, like (6) and (7), of $\underline{B}$ and $\underline{H}$ around a closed path enclosing

Fig. 2. Contour around a magnetic interface interval.

the interface interval from z_0 to $z = z_0 + \Delta z$, as indicated in Figure 1, and let the enclosed Amperian current and pole strength be $\Delta J = J - J_0$ and $\Delta P = P - P_0$, respectively. For simplicity we assume there is no true current flowing in the interface, i.e., $\Delta I = 0$. Each path integral consists of two parts, and each part runs wholly within a constant permeability region so that, by (10), each part depends only on the end points z_0 and z . Hence

$$\int_{z_0}^z B_R dZ = \mu_R \int_{z_0}^z H_R dZ = W_R(z) - W_R(z_0)$$

$$\int_z^{z_0} B_L dZ = \mu_L \int_z^{z_0} H_L dZ = -W_L(z) + W_L(z_0)$$

The integrals, (4) and (6), are obtained by addition of the appropriate parts,

$$\oint B dZ = W_R(z) - W_L(z) + C_1 = 4\pi i \Delta J = \text{imaginary} \quad (11a)$$

$$\oint H dZ = \frac{W_R(z)}{\mu_R} - \frac{W_L(z)}{\mu_L} + C_2 = -4\pi \Delta P = \text{real}, \quad (11b)$$

where

$$C_1 = - [W_R(z_0) - W_L(z_0)]$$

$$C_2 = - [\frac{W_R(z_0)}{\mu_R} - \frac{W_L(z_0)}{\mu_L}]$$

are constants, independent of z .

Equations (11) constitute a magnetic interface theorem analogous to the current sheet theorem of AADD-102 connecting two-dimensional magnetic fields on the two sides of a cylindrical surface.

Circular Cylinder Multipole

The application of these theorems may be illustrated in the solution of the problem shown in Figure 3. The problem is to find the current required on

Fig. 3. Circular current sheet and magnetic interface.

a circular cylinder current sheet at $r = a$, when surrounded by a magnetic material of constant permeability μ in the region $r \geq b$, to produce an interior $2n$ -pole field

$$W_1(z) = z^n$$

in the region $r < a$.

Since we are dealing with a pure multipole we may expect the exterior field W_3 to fall off with z^{-n} and, therefore, that W_2 would be a mixture of z^n and z^{-n} . Hence we postulate

$$W_2(z) = \alpha z^n + \beta z^{-n}$$

$$W_3(z) = \gamma z^{-n}$$

(12)

and seek to find the coefficients α , β , and γ in terms of a , b , μ and n .

With $Z = r e^{i\theta}$ the current sheet theorem at $r = a$ and the magnetic interface theorem at $r = b$ yield the relations

$$\begin{aligned} (\alpha a^n e^{in\theta} + \beta a^{-n} e^{-in\theta}) - a^n e^{in\theta} &= 4\pi i I(\theta) \\ \gamma b^{-n} e^{-in\theta} - (\alpha b^n e^{in\theta} + \beta b^{-n} e^{-in\theta}) &= 4\pi i J(\theta) \\ \frac{\gamma}{\mu} b^{-n} e^{-in\theta} - (\alpha b^n e^{in\theta} + \beta b^{-n} e^{-in\theta}) &= -4\pi P(\theta) \end{aligned} \quad (13)$$

Hence the real parts of the first two equations and the imaginary part of the last must be zero for all θ :

$$\begin{aligned} \cos n\theta (\alpha a^n + \beta a^{-n} - a^n) &= 0 \\ \cos n\theta (\gamma b^{-n} - \alpha b^n - \beta b^{-n}) &= 0 \\ \sin n\theta (-\frac{\gamma}{\mu} b^{-n} - \alpha b^n + \beta b^{-n}) &= 0 \end{aligned}$$

Hence we find

$$\begin{aligned} \alpha &= \frac{\mu - 1}{2\mu} b^{-2n} \gamma \\ \beta &= \frac{\mu + 1}{2\mu} \gamma \\ \gamma &= \frac{2\mu}{(\mu + 1) a^{-2n} + (\mu - 1) b^{-2n}} \end{aligned} \quad (14)$$

where, for convenience, α and β have been expressed in terms of the solution for γ given in the last line. From (13) we then obtain

$$\begin{aligned} 4\pi I(\theta) &= - \left(\frac{\mu + 1}{\mu} \right) a^{-n} \gamma \sin n\theta \\ 4\pi J(\theta) &= - \left(\frac{\mu - 1}{\mu} \right) b^{-n} \gamma \sin n\theta \\ 4\pi P(\theta) &= \left(\frac{\mu - 1}{\mu} \right) b^{-n} \gamma \cos n\theta \end{aligned} \quad (15)$$

for the required current, $I(\theta)$, at $r = a$ and the resulting Amperian current, $J(\theta)$, and pole strength, $P(\theta)$, at $r = b$.

By (10) the magnetic inductions, $B = B_Y + i B_X = dW/dZ$, and magnetic field strengths, $H = H_Y + i H_X$, in the three regions are

$$\begin{aligned} B_1 = H_1 &= n Z^{n-1} \\ B_2 = H_2 &= n (\alpha Z^{n-1} - \beta Z^{-n-1}) \\ B_3 = \mu H_3 &= -n \gamma Z^{-n-1} \end{aligned} \quad (16)$$

From the first of equations (15) we may compare the current sheet required at $r = a$ to produce the internal field $W_1 = Z^n$ when tightly surrounded by material of permeability μ (for $b = a$, $\gamma = a^{2n}$):

$$4\pi I(\theta) = - \left(\frac{\mu + 1}{\mu} \right) a^n \sin n\theta$$

with that required in the absence of permeable material ($\mu = 1$ or, alternatively, $b \rightarrow \infty$):

$$4\pi I(\theta) = - 2 a^n \sin n\theta$$

In other words, a tight magnetic shell of large extent reduces the required current sheet by the factor $(\mu + 1)/2\mu$ to produce the same internal field; for large μ this factor approaches one half.

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