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BROOKHAVEN NATIONAL LABORATORY
Associated Universities, Inc.
Upton, L.I., N.Y.

THE DEFLECTING MODE IN THE CIRCULAR IRIS-LOADED WAVEGUIDE
OF A RF PARTICLE SEPARATOR

H. Hahn

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I. Introduction

The rf particle separator, proposed in 1959 by W.K.H. Panofsky [1] and now in preparation for the Brookhaven Alternating Gradient Synchrotron, requires a rf structure which gives a transverse impulse to a passing relativistic particle.

In order to produce an accumulative transverse deflection of a traveling charged particle with an electromagnetic field, it is necessary that the field contains a synchronous component and in principle, waveguides and cavities are equivalent with respect to the particle dynamics. It was pointed out by H.G. Hereward [2], that the electric and magnetic deflection of a transverse electric mode (i.e., with no electric field component parallel to the direction of the particle velocity) cancel exactly at all particle velocities. The deflecting force of a transverse magnetic mode on a synchronous particle with the velocity v is proportional to the factor $1 - (v/c)^2$ and vanishes therefore in the case of relativistic particles.

It is impossible to achieve a synchronous deflection in an ordinary cavity. Nevertheless, it is possible to deflect particles by a TM - mode. Again, a TE - mode cannot be used [3]. W.K.H. Panofsky proposed a circular cylindrical cavity which is fed at two points to excite a circularly-polarized TM_{111} mode [4]. P.R. Phillips used a TM_{012} mode in a rectangular cavity and was able to

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obtain a particle deflection [5]. P. Lapostolle showed, that a TM_{110} - mode in a circular cavity can give equally a deflection [6]. In both cavities (TM_{012} , TM_{110}) the fields have no dependence in direction of the traveling particles. The phase velocity is therefore infinite and the limiting factor $1 - (v/c)^2$ does not appear. However, the deflection is now proportional to the transit time factor $\sin(\omega\tau/2)$, where τ is the transit time and ω the rf frequency. For nonsynchronous deflection it is therefore impossible to increase the maximum transverse impulse in a single cavity by a greater length. The use of a multiple-cavity structure is however possible.

Deflecting electromagnetic modes, which are neither TE or TM and structures to support them were indicated by J.P. Blewett [7,8] and H.G. Hereward [2]. A proper choice of the dimensions allows a phase velocity equal or close to light velocity and a synchronous deflection is possible. Some of the proposed rf structures have the disadvantage of vanishing group velocity, which makes the excitation of the wanted mode critical. An extensive study of deflecting modes in rectangular coordinates was made by R. Gabillard [9].

The interpretation of the beam blow-up (pulse shortening) in electron linear accelerators requires the existence of a backward-wave in which TE and TM modes are combined in such a way as to leave a residual transverse deflection on a synchronous relativistic particle [10]. The usefulness of this field configuration as a deflecting mode was recognized at Stanford [16]. For the time being, an iris-loaded waveguide seems to be the best deflecting structure available and it will be used for the projected rf separator in Brookhaven.

The aim of this report is to show an analysis of the deflecting mode in an iris-loaded waveguide which is not based on the concept of TE and TM modes as generating fields. The hybrid solutions employed are derived from the transverse

components of the electric and magnetic Hertzian vectors. Their use avoids the singularities in the field expressions for phase velocities equal to the velocity of light, which must occur with TE and TM modes. The fundamental properties of the hybrid solutions are exposed. The lowest mode of the hybrid group is the wanted deflecting mode. The exact relationship between frequency, phase velocity and geometry of the structure can be derived for the deflecting mode in the form of an infinite determinant. A simple approximate solution, which is sufficient for a mode identification is indicated.

II. The Hertzian Vectors

The electromagnetic fields in vacuo may be written in terms of the electric and magnetic Hertzian vectors, $\vec{\Pi}_e$ and $\vec{\Pi}_h$, as follows:

$$\begin{aligned}\vec{E} &= \nabla \times \nabla \times \vec{\Pi}_e - jk \nabla \times Z_0 \vec{\Pi}_h \\ Z_0 \vec{H} &= jk \nabla \times \vec{\Pi}_e + \nabla \times \nabla \times Z_0 \vec{\Pi}_h\end{aligned}$$

where the Hertzian potentials are solutions of the vector Helmholtz equation

$$(\nabla^2 + k^2)\vec{\Pi} = 0$$

The time dependence is always assumed to be $e^{j\omega t}$ and the equations are written in the MKSA unit system. The wave number k and the characteristic impedance Z_0 have the usual meaning

$$k = \frac{\omega}{c} \qquad Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

The most general solution of the vector Helmholtz equation requires three independent vector solutions $\vec{P}$, $\vec{Q}$, $\vec{L}$. The solutions can always be brought in a form so that

$$\nabla \times \vec{L} \equiv 0; \nabla \times \vec{P} \neq 0; \nabla \times \vec{Q} \neq 0.$$

The most general solution for the Hertzian vectors is therefore a linear combination of $\vec{P}$ and $\vec{Q}$. It is convenient, if $\vec{P}$ has a longitudinal component only and $\vec{Q}$ transverse components only. Vector solutions in this special form were derived for circular cylinder coordinates (r, θ, z) in reference [11]:

$$\vec{P} = \begin{bmatrix} 0 \\ 0 \\ \frac{J_v(\gamma r)}{\gamma^v} e^{-jv\theta} \end{bmatrix} e^{-j\beta z}$$

$$\vec{Q} = \begin{bmatrix} \frac{J_{v+1}(\gamma r)}{\gamma^{v+1}} e^{-jv\theta} \\ \frac{J_{v+1}(\gamma r)}{\gamma^{v+1}} j e^{-jv\theta} \\ 0 \end{bmatrix} e^{-j\beta z}$$

$$\vec{Q}' = \begin{bmatrix} \frac{J_{v-1}(\gamma r)}{\gamma^{v-1}} e^{-jv\theta} \\ -\frac{J_{v-1}(\gamma r)}{\gamma^{v-1}} j e^{-jv\theta} \\ 0 \end{bmatrix} e^{-j\beta z}$$

with $k^2 = \beta^2 + \gamma^2$. The alternative solution $\vec{Q}$ is irrelevant for our problem, as the fields derived give no transverse deflection.

It is a well known fact, that the most general representation of an electromagnetic field requires only two independent vector solutions [12]. The two vector solutions may be obtained from the two Hertzian vectors with longitudinal components only. It is equally possible to use instead the two Hertzian vectors with transverse components only. In the first case, the fields are developed in terms of TE and TM modes; in the second case, the fields are developed in terms of two hybrid solutions. The choice of the generating solutions will be made in each individual case to yield the most convenient expressions.

III. Transverse and Hybrid Solutions

The expressions for the electric and magnetic field components which can be derived from the two solutions $\vec{P}$ and $\vec{Q}$ for the electric and magnetic Hertzian vectors are tabulated in Table 1. The electric Hertzian vector $\vec{\Pi}_e = \vec{P}$ leads to TM modes, the magnetic Hertzian vector $\vec{\Pi}_h = j\vec{P}$ to TE modes. It is proposed to call the fields derived from $\vec{\Pi}_e = j\vec{Q}$ as hybrid electric (HE), the fields derived from $\vec{\Pi}_h = \vec{Q}$ as hybrid magnetic (HM) solutions. The hybrid solutions obtained and used in this report are different from the symmetric hybrid waves introduced by G. Goubau [13].

The inspection of the field expressions reveals several interesting features. The 0.- order ($v = 0$) hybrid solutions, HM_0 and HE_0 , split both in two independent TM_0 and TE_0 solutions. Modes without θ -dependence are therefore always transverse. This statement holds in circular waveguides with iris loading.

At cutoff, i.e., for $\beta \rightarrow 0$ and $\gamma \rightarrow k$, the HM and HE solutions tend to become TM and TE fields.

The limiting case when the phase velocity approaches light velocity, i.e., $\beta \rightarrow 1$ and $\gamma \rightarrow 0$, is of special interest. TM and TE solutions become both TEM waves with no longitudinal field components. In particular the TM_1 and TE_1 solution approach both a plane wave. On the other hand, the hybrid solutions have six non-vanishing finite field components which are given in Table 2.

The Lorentz force on a charged particle with the velocity $\vec{v}$ equals

$$\vec{F} = q \left(\vec{E} + \frac{\vec{v}}{c} \times \vec{Z}_0 \vec{H} \right).$$

The particle is assumed to travel synchronously ($v = v_{ph}$) in direction of the wave propagation

$$\frac{\vec{v}}{c} = \frac{v}{c} \vec{e}_z = \frac{k}{\beta} \vec{e}_z \quad \text{with} \quad \beta = \frac{\omega}{v_{ph}}$$

The force components are then given by

$$\vec{F} = q \begin{bmatrix} E_r - \frac{k}{\beta} Z_0 H_\theta \\ E_\theta + \frac{k}{\beta} Z_0 H_r \\ E_z \end{bmatrix}$$

Using the field expressions of Table 1, it can be easily verified that the TE modes exert no forces on a particle.

$$\vec{F} \neq 0$$

The maximum transverse force exerted by a TM mode is found to be

$$\vec{F}_T = q \begin{bmatrix} - \left\{ 1 - \left(\frac{v}{c} \right)^2 \right\} \beta \frac{J_v(\gamma r)}{\gamma^{v-1}} e^{-jv\theta} \\ \left\{ 1 - \left(\frac{v}{c} \right)^2 \right\} \beta v \frac{J_v(\gamma r)}{\gamma^v r} j e^{-jv\theta} \end{bmatrix}$$

and vanishes, as reported, for relativistic particles.

The HE field exerts a transverse force

$$\vec{F}_T = q \begin{bmatrix} \left\{ v \frac{J_v(\gamma r)}{\gamma^v r} - \gamma^2 \frac{J_{v+1}(\gamma r)}{\gamma^{v+1}} \right\} e^{-jv\theta} \\ - v \frac{J_v(\gamma r)}{\gamma^v r} j e^{-jv\theta} \end{bmatrix}$$

The HM field exerts a transverse force

$$\vec{F}_T = q \frac{k}{\beta} \begin{bmatrix} \left\{ v \frac{J_v(\gamma r)}{\gamma^v r} - \gamma^2 \frac{J_{v+1}(\gamma r)}{\gamma^{v+1}} \right\} e^{-jv\theta} \\ - v \frac{J_v(\gamma r)}{\gamma^v r} j e^{-jv\theta} \end{bmatrix}$$

The maximum transverse force on a relativistic particle, due to HE and HM fields, does not vanish and its value is

$$\vec{F}_T = (a_0 + b_0) q \frac{r^{v-1}}{2^v (v-1)!} \begin{bmatrix} e^{-jv\theta} \\ -j e^{-jv\theta} \end{bmatrix}$$

where a_0 and b_0 are the relative amplitudes of HE and HM fields.

The 1. order hybrid modes ($v = 1$) are of special interest, as their combination leads to the deflecting mode. Here the transverse force is aberration free, i.e., the force vector is constant for all r and θ :

$$\vec{F}_T = \frac{a_0 + b_0}{2} q \begin{bmatrix} \cos \theta \\ -\sin \theta \end{bmatrix}$$

The deflecting mode has therefore on a synchronous, relativistic particle the same effect as a uniform, static electric field of the equivalent field strength

$$E_o = \frac{a_o + b_o}{2}$$

The complex Poynting vector $\vec{S}$ is defined as

$$Z_o \vec{S} = \frac{1}{2} (\vec{E} \times Z_o \vec{H}^*)$$

and the time average power flux density S_z in propagation direction is given by

$$Z_o S_z = \frac{1}{2} (E_r \cdot Z_o H_\theta^* - E_\theta \cdot Z_o H_r^*)$$

where the asterisk signifies the complex conjugate value.

If the phase velocity equals c , S_z of the hybrid solutions is found to be

$$\begin{aligned} Z_o S_z = & \frac{a_o^2}{2} \left(\frac{kr^v}{2^{v+1} v!} \right)^2 \left\{ \left(\frac{kr}{v+1} \right)^2 + \frac{2v}{v+1} \cos 2v\theta \right\} \\ & + \frac{b_o^2}{2} \left(\frac{kr^v}{2^{v+1} v!} \right)^2 \left\{ \left(\frac{kr}{v+1} \right)^2 - \frac{2v}{v+1} \cos 2v\theta \right\} \\ & + \frac{a_o b_o}{2} \left(\frac{r^{v-1}}{2^v (v-1)!} \right)^2 \left\{ 2 \left(\frac{k^2 r^2}{2v^2 + 2} \right)^2 - 1 \right\} \end{aligned}$$

In contrast to the transverse modes, the power flux density has always negative values in vicinity of the z axis ($r = 0$). The hybrid modes are therefore inherently backward waves, even if the integrated power flow assumes positive values. The hybrid solutions are not orthogonal and the cross product $a_o b_o$ does not vanish.

The transverse and hybrid solutions of Maxwell's equations have been found without considering the boundary conditions. But a solution has physical significance only if a supporting metallic structure can be found. In circular cylindrical waveguides, the HE and HM solutions do not satisfy the boundary conditions. The only possible modes are TE and TM fields.

It will be shown that in an iris-loaded waveguide a combination of HE and HM solutions does satisfy the boundary conditions. One hybrid solution alone cannot exist. The lowest passband hybrid solution with $v = 1$ will be used as deflecting mode (HEM_{11}).

IV. Field Expansions for the Hybrid Modes

It remains to prove that hybrid modes do satisfy the boundary conditions in the given structure and to derive the relation between frequency, phase velocity and guide dimensions. It seems impossible to find a solution for the field distribution in closed form. The alternative is to develop the solution in the form of an infinite series. This method was suggested by W.C. Hahn for the analysis of cavity resonators [14]. It was modified by J.S. Bell and W. Walkinshaw for the calculation of the accelerating mode in iris-loaded waveguides [15].

The periodicity of the iris-loaded waveguide makes it sufficient to consider one cell only (Fig. 1). The cell is cut into two spaces with simple boundary conditions.

Series expansions of the fields are found for the slot region (region II) and in the center region (region I). Maxwell's equations show that it is necessary and sufficient to match E_z , H_z , E_θ , H_θ at the common boundary between region I and II, leading to an infinite set of homogenous linear equations. The secular determinant of this set of equations gives the required frequency condition. Approximate solutions are found by limiting the number of elements in the underlying matrix.

Fig. 1

1. The Fields in Region II

The fields in region II are obtained as an infinite Fourier series of all possible modes, satisfying individually the boundary conditions at $z = \pm d/2$ and $r = b$. The complete field expressions require two sets of arbitrary real coefficients (c_m , c'_m and d_m , d'_m), corresponding to TM and TE modes. All abbreviations used are defined in the appendix and the θ - dependence is not explicitly written. The fields take then the form

$$E_z^{II} = c_0 Z_0(kr) + \sum_1^{\infty} c_m Z_\nu(\Gamma_m r) \cos \eta_m z + \sum_1^{\infty} c'_m Z_\nu(\Gamma'_m r) j^{-1} \sin \eta'_m z$$

$$Z_0 H_z^{II} = \sum_1^{\infty} d_m B_\nu(\Gamma_m r) j^{-1} \sin \eta_m z + \sum_1^{\infty} d'_m B_\nu(\Gamma'_m r) \cos \eta'_m z$$

$$\begin{aligned}
 E_{\theta}^{II} &= +j \sum_1^{\infty} \left\{ c_m \frac{\eta_m}{\Gamma_m} \frac{v Z_v(\Gamma_m r)}{(\Gamma_m r)} + d_m \frac{k}{\Gamma_m} \dot{B}_v(\Gamma_m r) \right\} j^{-1} \sin \eta_m z \\
 &+ j \sum_1^{\infty} \left\{ c'_m \frac{\eta'_m}{\Gamma'_m} \frac{v Z_v(\Gamma'_m r)}{(\Gamma'_m r)} + d'_m \frac{k}{\Gamma'_m} \dot{B}_v(\Gamma'_m r) \right\} \cos \eta'_m z \\
 Z_{\theta}^{II} &= -j c_0 \dot{Z}_v(kr) - j \sum_1^{\infty} \left\{ c_m \frac{k}{\Gamma_m} \dot{Z}_v(\Gamma_m r) + d_m \frac{\eta_m}{\Gamma_m} \frac{v B_v(\Gamma_m r)}{(\Gamma_m r)} \right\} \cos \eta_m z \\
 &- j \sum_1^{\infty} \left\{ c'_m \frac{k}{\Gamma'_m} \dot{Z}_v(\Gamma'_m r) + d'_m \frac{\eta'_m}{\Gamma'_m} \frac{v B_v(\Gamma'_m r)}{\Gamma'_m r} \right\} j^{-1} \sin \eta'_m z
 \end{aligned}$$

2. The Fields in Region I

The fields in region I are obtained as an infinite Fourier series of waves propagating in z - direction with different phase velocities. The periodicity of the structure indicates the use of Floquet's theorem to determine the possible propagation constants

$$\beta_n = \beta_0 + \frac{2\pi}{w} n$$

The individual term need not satisfy by itself the boundary conditions on the irises at $r = a$, and therefore the fields may be developed in terms of HE and HM modes. The most general expression requires again two infinite sets of arbitrary real coefficients (a_n, b_n) . The fields in region I may be written as

$$E_z^I = \sum_{-\infty}^{+\infty} \left(a_n \beta_n \frac{J_v(\gamma_n r)}{\gamma_n} + b_n k \frac{J_v(\gamma_n r)}{\gamma_n} \right) e^{-j\beta_n z}$$

$$Z_o H_z^I = - \sum_{-\infty}^{+\infty} \left(a_n k \frac{J_v(\gamma_n r)}{\gamma_n^v} + b_n \beta_n \frac{J_v(\gamma_n r)}{\gamma_n^v} \right) e^{-j\beta_n z}$$

$$E_\theta^I = j \sum_{-\infty}^{+\infty} \left\{ a_n \left(k^2 \frac{J_{v+1}(\gamma_n r)}{\gamma_n^{v+1}} - v \frac{J_v(\gamma_n r)}{\gamma_n^v r} \right) + b_n k \beta_n \frac{J_{v+1}(\gamma_n r)}{\gamma_n^{v+1}} \right\} e^{-j\beta_n z}$$

$$Z_o H_\theta^I = j \sum_{-\infty}^{+\infty} \left\{ a_n k \beta_n \frac{J_{v+1}(\gamma_n r)}{\gamma_n^{v+1}} + b_n \left(k^2 \frac{J_{v+1}(\gamma_n r)}{\gamma_n^{v+1}} - v \frac{J_v(\gamma_n r)}{\gamma_n^v r} \right) \right\} e^{-j\beta_n z}$$

V. Matching of the Fields at the Common Boundary

The four sets of coefficients require four infinite sets of equations which are obtained by matching of four field components at $r = a$. A possible choice is $E_z, H_z, E_\theta, H_\theta$. Maxwell's equations ensure then the simultaneous matching of H_r and E_r , what can be seen from the following relations

$$-jkZ_o H_r = \frac{1}{a} \frac{\partial E_z}{\partial \theta} - \frac{\partial E_\theta}{\partial z}$$

$$jkE_r = \frac{1}{a} \frac{\partial Z_o H_z}{\partial \theta} - \frac{\partial Z_o H_\theta}{\partial z}$$

1. Matching of E_z at $r = a$.

E_z^I has to satisfy the boundary condition on the irises

$$E_z^I = 0 \quad \text{for} \quad \frac{d}{2} < |z| < \frac{w}{2}$$

and the following relation holds

$$\int_{-w/2}^{+w/2} E_z^I e^{j\beta_n z} dz = \int_{-d/2}^{+d/2} E_z^I e^{j\beta_n z} dz$$

At the common boundary between region I and region II we have $E_z^I = E_z^{II}$ and therefore

$$\int_{-w/2}^{+w/2} E_z^I e^{j\beta_n z} dz = \int_{-d/2}^{+d/2} E_z^{II} e^{j\beta_n z} dz$$

By using the orthogonality of the Fourier terms, the first set of equations is obtained in the general form

$$u_n a_n + v_n b_n = (E_c)_{nm} c_m + (E'_c)_{nm} c'_m$$

2. Matching of E_θ at $r = a$.

An analogous procedure applied to E_θ yields the second set of equations in the general form

$$x_n a_n + y_n b_n = (F_c)_{nm} c_m + (F'_c)_{nm} c'_m + (F_d)_{nm} d_m + (F'_d)_{nm} d'_m$$

3. Matching of H_z at $r = a$.

The two sets of equations obtained by matching E_z and E_θ express each coefficient in region I in terms of the coefficients in region II. The matching of H_z and H_θ , on the other hand, can be done in such a way to express the coefficients in region II as a linear combination of the coefficients in region I.

By integrating the equations

$$\int_{-d/2}^{+d/2} H_z^{II} \begin{Bmatrix} j \sin \eta_m z \\ \cos \eta'_m z \end{Bmatrix} dz = \int_{-d/2}^{+d/2} H_z^I \begin{Bmatrix} j \sin \eta_m z \\ \cos \eta'_m z \end{Bmatrix} dz$$

the third set of equations is obtained in the general form

$$g_{m m}^d = (G_a)_{im} a_i + (G_b)_{im} b_i$$

$$g_{m m}^{d'} = (G'_a)_{im} a_i + (G'_b)_{im} b_i$$

4. Matching of H_θ at $r = a$.

By integrating the equations

$$\int_{-d/2}^{+d/2} H_\theta^{II} \begin{Bmatrix} \cos \eta_m z \\ j \sin \eta'_m z \end{Bmatrix} dz = \int_{-d/2}^{+d/2} H_\theta^I \begin{Bmatrix} \cos \eta_m z \\ j \sin \eta'_m z \end{Bmatrix} dz$$

the fourth set of equations is obtained in the general form

$$(h_c)_m c_m + (h_d)_m d_m = (H_a)_{im} a_i + (H_b)_{im} b_i$$

$$(h'_c)_m c'_m + (h'_d)_m d'_m = (H'_a)_{im} a_i + (H'_b)_{im} b_i$$

VI. An Approximation for the Frequency Condition.

The matching of the 4 fields gave 4 sets of linear homogenous equations for the 4 sets of coefficients. The starting field equations were written in a way to make all coefficients and equations real. It is possible to eliminate the coefficients c_m, c'_m, d_m, d'_m of region II. Then only two sets of equations for the coefficients a_i, b_i of region I will remain. They may be written as

$$U_{ni} a_i + V_{ni} b_i = 0$$

$$-\infty < n, i < +\infty$$

$$X_{ni} a_i + Y_{ni} b_i = 0$$

Each element is the sum of an infinite series

$$U_{ni} = \sum_{m=0}^{\infty} U_{nim} \text{ etc.}$$

The expressions for the elements U_{ni} , V_{ni} , X_{ni} , Y_{ni} are therefore rather long and they are here omitted. A vanishing secular determinant of this system of equations provides the necessary relationship between guide dimensions (a, b, w, d), wave number k and propagation constant β_0 .

Approximate solutions for the frequency condition are found by limiting the number of terms in the Fourier series for the fields. A first order approximation is obtained by taking $n = i = 0$ and $m = 0$ only. This is equivalent to take in region I the fundamental HE and HM component and in region II the fundamental TM-mode. The matrix of the deflecting mode ($v = 1$) is reduced to 4 elements:

$$U_{00} = \beta a \frac{J_1(\gamma a)}{\gamma a} + \frac{d}{w} (ka) (\beta a) \frac{J_2(\gamma a)}{(\gamma a)^2} \frac{Z_1(ka)}{Z_1(ka)} C_{00}^2$$

$$V_{00} = ka \frac{J_1(\gamma a)}{\gamma a} + \frac{d}{w} \left\{ (ka)^2 \frac{J_2(\gamma a)}{(\gamma a)^2} - \frac{J_1(\gamma a)}{\gamma a} \right\} \frac{Z_1(ka)}{Z_1(ka)} C_{00}^2$$

$$X_{00} = (ka)^2 \frac{J_2(\gamma a)}{(\gamma a)^2} - \frac{J_1(\gamma a)}{\gamma a}$$

$$Y_{00} = (ka) (\beta a) \frac{J_2(\gamma a)}{\gamma a}$$

It is possible to demonstrate the physical significance of the equations obtained in the simple case of the "smooth approximation", i.e., if the waveguide has a very large number of infinitely thin irises per wavelength. In this case

$$\frac{d}{w} \rightarrow 1 \quad \text{and} \quad C_{oo} \rightarrow 1.$$

The equation

$$X_{oo} a_o + Y_{oo} b_o = 0$$

may then be obtained directly by exerting the boundary condition $E_\theta = 0$ at $r = a$ on the two fields in region I.

The equation

$$U_{oo} a_o + V_{oo} b_o = 0$$

is obtained by impedance matching at $r = a$ according to

$$\frac{E_z^I}{Z_o H_\theta^I} = \frac{E_z^{II}}{Z_o H_\theta^{II}}$$

The remaining H_z - component in region I has to be matched by currents in the irises.

The approximate frequency condition is given by the relation

$$U_{oo} Y_{oo} - V_{oo} X_{oo} = 0$$

After some manipulations, this expression takes the form

$$ka Z_1(ka) \frac{J_1(\gamma a)}{\gamma a} J_1(\gamma a) + \frac{d}{w} C_{oo}^2 Z_1(ka) \left\{ (\beta a)^2 \frac{J_2(\gamma a)}{(\gamma a)^2} J_o(\gamma a) - J_1^2(\gamma a) \right\} = 0.$$

At cutoff, i.e., for $\beta = 0$, the expression takes the form of a product. The two factors correspond to the TE_1 and TM_1 mode:

$$\dot{J}_1(ka) \left\{ \dot{Z}_1(ka) J_1(ka) - \frac{d}{w} C_{oo}^2 Z_1(ka) \dot{J}_1(ka) \right\} = 0$$

TE_1 TM_1

In the limiting case for $v_{ph} = c$, i.e., for $\gamma = 0$, the frequency condition takes the form:

$$\dot{Z}_1(ka) + \frac{d}{w} C_{oo}^2 Z_1(ka) \left\{ \frac{ka}{2} - \frac{1}{ka} \right\} = 0$$

VII. Normalization

To achieve simple notation, the fields were developed in terms of ~~un~~normalized functions. The question of normalization is without importance for the 1st order approximation obtained. The validity of this approximate solution was checked by comparison of computed and measured values. The infinite determinant, however, has to satisfy several conditions [17] to give a convergent solution, and a proper normalization is necessary.

VIII. The Power Flow

The time-average total power flow P_z through the waveguide is given by the integral

$$Z_o P_z = \int_0^a \int_0^{2\pi} Z_o S_z r dr d\theta$$

S_z is evaluated at $z = w/2$.

Using the approximation of (VI) for the deflecting mode propagating with light velocity, the coefficients a_o and b_o are related by

$$\frac{b_o}{a_o} = \frac{1 - (ka/2)^2}{(ka/2)^2}$$

The equivalent electric field strength E_o is given by

$$E_o = \frac{a_o + b_o}{2} = \frac{1}{2} \left(\frac{2}{ka} \right)^2 a_o$$

After some manipulations, the power flow through the waveguide is found to be approximately

$$Z_o P_z = \frac{\pi a^2}{4} \left\{ \frac{1}{6} (a_o + b_o)^2 \left(\frac{ka}{2} \right)^4 - \frac{1}{2} a_o b_o \right\}$$

or alternatively

$$Z_o P_z = \frac{1}{2} \pi a^2 E_o^2 \left(\frac{ka}{2} \right)^2 \left\{ \frac{4}{3} \left(\frac{ka}{2} \right)^2 - 1 \right\}$$

The power flow assumes negative values if

$$ka \leq \sqrt{3}$$

The maximum deflecting field strength E_o is therefore determined if the klystron power P_z is known. The power transported by the higher harmonics will however reduce the actual deflecting force.

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Distribution: AD B1, B2

	TM	TE	HM	HE	θ	z
$E_r \sim$	$-j\beta \frac{J_v(\gamma r)}{\gamma^{v-1}}$	$-jkv \frac{J_v(\gamma r)}{\gamma^v r}$	$jk\beta \frac{J_{v+1}(\gamma r)}{\gamma^{v+1}}$	$j \left\{ v \frac{J_v(\gamma r)}{\gamma^v r} + \beta^2 \frac{J_{v+1}(\gamma r)}{\gamma^{v+1}} \right\}$	$e^{-jv\theta}$	$e^{-j\beta z}$
$E_\theta \sim$	$j\beta v \frac{J_v(\gamma r)}{\gamma^v r}$	$+jk \frac{J_v(\gamma r)}{\gamma^{v-1}}$	$jk\beta \frac{J_{v+1}(\gamma r)}{\gamma^{v+1}}$	$-j \left\{ v \frac{J_v(\gamma r)}{\gamma^v r} - k^2 \frac{J_{v+1}(\gamma r)}{\gamma^{v+1}} \right\}$	$je^{-jv\theta}$	
$E_z \sim$	$\frac{J_v(\gamma r)}{\gamma^{v-2}}$	0	$k \frac{J_v(\gamma r)}{\gamma^v}$	$\beta \frac{J_v(\gamma r)}{\gamma}$	$e^{-jv\theta}$	
$Z_o H_r \sim$	$-jkv \frac{J_v(\gamma r)}{\gamma^v r}$	$-j\beta \frac{J_v(\gamma r)}{\gamma^{v-1}}$	$-j \left\{ v \frac{J_v(\gamma r)}{\gamma^v r} + \beta^2 \frac{J_{v+1}(\gamma r)}{\gamma^{v+1}} \right\}$	$-jk\beta \frac{J_{v+1}(\gamma r)}{\gamma^{v+1}}$	$je^{-jv\theta}$	
$Z_o H_\theta \sim$	$-jk \frac{J_v(\gamma r)}{\gamma^{v-1}}$	$-j\beta v \frac{J_v(\gamma r)}{\gamma^v r}$	$-j \left\{ v \frac{J_v(\gamma r)}{\gamma^v r} - k^2 \frac{J_{v+1}(\gamma r)}{\gamma^{v+1}} \right\}$	$jk\beta \frac{J_{v+1}(\gamma r)}{\gamma^{v+1}}$	$e^{-jv\theta}$	
$Z_o H_z \sim$	0	$\frac{J_v(\gamma r)}{\gamma^{v-2}}$	$-\beta \frac{J_v(\gamma r)}{\gamma^v}$	$-k \frac{J_v(\gamma r)}{\gamma^v}$	$je^{-jv\theta}$	

Table 1 Transverse and Hybrid Solutions

	HM	HE	θ	z
$E_r \sim$	$j \frac{k^2 r^{v+1}}{2^{v+1} (v+1)!}$	$j \left\{ \frac{r^{v-1}}{2^v (v-1)!} + \frac{k^2 r^{v+1}}{2^{v+1} (v+1)!} \right\}$	$e^{-jv\theta}$	
$E_\theta \sim$	$j \frac{k^2 r^{v+1}}{2^{v+1} (v+1)!}$	$-j \left\{ \frac{r^{v-1}}{2^v (v-1)!} - \frac{k^2 r^{v+1}}{2^{v+1} (v+1)!} \right\}$	$je^{-jv\theta}$	
$E_z \sim$	$\frac{k r^v}{2^v v!}$	$\frac{k r^v}{2^v v!}$	$e^{-jv\theta}$	
$Z_o H_r \sim$	$-j \left\{ \frac{r^{v-1}}{2^v (v-1)!} + \frac{k^2 r^{v+1}}{2^{v+1} (v+1)!} \right\}$	$-j \frac{k^2 r^{v+1}}{2^{v+1} (v+1)!}$	$je^{-jv\theta}$	$e^{-j\beta z}$
$Z_o H_\theta \sim$	$-j \left\{ \frac{r^{v-1}}{2^v (v-1)!} - \frac{k^2 r^{v+1}}{2^{v+1} (v+1)!} \right\}$	$j \frac{k^2 r^{v+1}}{2^{v+1} (v+1)!}$	$e^{-jv\theta}$	
$Z_o H_z \sim$	$-\frac{k r^v}{2^v v!}$	$-\frac{k r^v}{2^v v!}$	$je^{-jv\theta}$	

Table 2: Hybrid Solutions for $v_{ph} = c$

APPENDIX

All abbreviations used in the main body of the report are here defined.

1. Definition of propagation constants.

$$k = \frac{\omega}{c} = \frac{2\pi}{\lambda_0} ; \quad \beta = \frac{\omega}{v_{ph}} = \frac{2\pi}{\lambda_g}$$

In region I:

$$\beta_n = \beta_0 + \frac{2\pi}{w} n$$

$$\gamma_n^2 = k^2 - \beta_n^2$$

In region II:

$$\eta_m = 2m \frac{\pi}{d} ; \quad \eta'_m = (2m-1) \frac{\pi}{d}$$

$$\Gamma_m^2 = k^2 - \eta_m^2$$

$$\Gamma'_m{}^2 = k^2 - \eta'_m{}^2$$

2. Definition of integrals.

$$\frac{1}{d} \int_{-d/2}^{+d/2} e^{\pm j\beta_n z} \cos \eta_m z dz = C_{nm}$$

$$\frac{1}{jd} \int_{-d/2}^{+d/2} e^{\pm j\beta_n z} \sin \eta_m z dz = \pm S_{nm}$$

$$\begin{aligned} C_{nm} &= \frac{1}{2} \left(\frac{\sin(\beta_n - \eta_m)d/2}{(\beta_n - \eta_m)d/2} + \frac{\sin(\beta_n + \eta_m)d/2}{(\beta_n + \eta_m)d/2} \right) \\ S_{nm} &= \frac{1}{2} \left(\frac{\sin(\beta_n - \eta_m)d/2}{(\beta_n - \eta_m)d/2} - \frac{\sin(\beta_n + \eta_m)d/2}{(\beta_n + \eta_m)d/2} \right) \end{aligned}$$

$$C'_{nm} = C_{nm}(\eta'_m)$$

$$S'_{nm} = S_{nm}(\eta'_m)$$

3. Definition of cylindrical functions

$$Z_v(\Gamma_m r) = J_v(\Gamma_m r) Y_v(\Gamma_m b) - J_v(\Gamma_m b) Y_v(\Gamma_m r)$$

$$\dot{Z}_v(\Gamma_m r) = \dot{J}_v(\Gamma_m r) Y_v(\Gamma_m b) - J_v(\Gamma_m b) \dot{Y}_v(\Gamma_m r)$$

$$B_v(\Gamma_m r) = J_v(\Gamma_m r) \dot{Y}_v(\Gamma_m b) - \dot{J}_v(\Gamma_m b) Y_v(\Gamma_m r)$$

$$\dot{B}_v(\Gamma_m r) = \dot{J}_v(\Gamma_m r) \dot{Y}_v(\Gamma_m b) - \dot{J}_v(\Gamma_m b) \dot{Y}_v(\Gamma_m r)$$

The same definitions are valid for the primed values Γ'_m .

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