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A STRESS ANALYSIS
OF THICK-WALL
ORTHOTROPIC
SPHERICAL VESSELS

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NOMENCLATURE

r = Radius to general location, in.

R = Radius to boundry, in.

σ = Stress, psi

φ = Angle, Radian

ζ = Radial displacement, in.

ξ = Strain, in/in

E = Modulus of elasticity, psi

μ = Poisson's ratio, general

α = Tangential poisson's ratio

B = Radial poisson's ratio

N = A constant

C = A constant

ρ = Pressure, psi

Subscripts

1, 2 = Constants

t = Tangential direction

r = Radial direction

o = Outside

i = Inside

INTRODUCTION

The development of high-strength, high modulus fibers of glass, carbon, boron, and other materials has led to applications of these fibers in which substantial weight reductions result as contrasted to metallic construction. The high strength, low density characteristics are of great importance in aircraft and space vehicle designs, resulting in the use of large numbers of filament wound pressure vessels. Low weight of fiber structures is also attractive to designers of deep-submergence marine vehicles. Spherical shapes are often utilized since the largest possible contained volume exists in relation to the surrounding material, and full structural utilization of the fiber is possible.

This paper is concerned with the determination of radial and tangential stresses present in spherical vessels in which the wall thickness is not necessarily small as compared to the inner radius. Solutions of this problem¹ may be found in the literature for vessels constructed of isotropic materials. The analysis presented here assumes that the material properties in the radial and tangential directions may be different, as would be the case in a filament-wound sphere. If the winding process is such that the fiber is placed on essentially great-circles, and such that near equal angular dispersion of the fiber exists at any point, the material will have constant properties in any tangential direction. Thus, two only characterizing directions (radial and tangential) suffice in defining the strength and elastic properties of the semi-orthotropic material.

¹"Engineering Design", Faupel, p. 246 and p. 421.

DERIVATION OF THE STRESS EQUATIONS

Consider an elemental "Cork-stopper" shaped piece of the spherical shell (Figure 1) as it exists in force equilibrium (Figure 2).

Figure 1.

Figure 2.

Radial force equilibrium requires that:

$$-\pi (rd\phi)^2 \sigma_r + \pi [(r+dr) d\phi]^2 (\sigma_r + d\sigma_r) - 2\pi r (dr) \sigma_t (d\phi)^2 = 0 \quad (1)$$

Discarding second order terms and simplifying:

$$2\sigma_t - 2\sigma_r - r \frac{d\sigma_r}{dr} = 0 \quad (2)$$

The tangential strain is the ratio of radial movement to radial location, and by Hooke's Law is

$$\xi_t = \frac{\zeta}{r} = \frac{\sigma_t}{E_t} - \mu_{tt} \frac{\sigma_t}{E_t} - \mu_{tr} \frac{\sigma_r}{E_r} \quad (3)$$

Radial strain is the derivative of radial movement with respect to radius, and by Hooke's Law is:

$$\xi_r = \frac{d\zeta}{dr} = \frac{\sigma_r}{E_r} - 2\mu_{rt} \frac{\sigma_t}{E_t} \quad (4)$$

Elastic energy stored in any elastic material is independent of the order in which loads or stresses are applied. This fact may be used to establish that:

$$\mu_{rt} = \frac{E_t}{E_r} \mu_{tr} \quad (5)$$

For convenience, let

$$\alpha = \mu_{tt}$$

$$B = \mu_{tr}$$

From (5),

$$\mu_{rt} = B \frac{E_t}{E_r} \quad (6)$$

Equations (3) and (4) may now be written as:

$$\frac{\zeta}{r} = (1 - \alpha) \frac{\sigma_t}{E_t} - B \frac{\sigma_r}{E_r} \quad (7)$$

$$\frac{d\zeta}{dr} = -2B \frac{\sigma_t}{E_r} + \frac{\sigma_r}{E_r} \quad (8)$$

Solving (7) and (8) for the stresses:

$$\sigma_t = \frac{\frac{\zeta}{r} + B \frac{d\zeta}{dr}}{\frac{1 - \alpha}{E_t} - \frac{2B^2}{E_r^2}} \quad (9)$$

$$\sigma_r = \frac{(1 - \alpha) \frac{E_r}{E_t} \frac{d\zeta}{dr} + 2B \frac{\zeta}{r}}{\frac{1 - \alpha}{E_t} - \frac{2B^2}{E_r^2}} \quad (10)$$

Substituting (9) and (10) in (2) the following differential equation, relating radial displacement to radius, is obtained:

$$r^2 \frac{d^2 \zeta}{dr^2} + 2r \frac{d\zeta}{dr} - 2 \frac{(1-B)}{(1-\alpha)} \frac{E_t}{E_r} \zeta = 0 \quad (11)$$

It can be shown that the complete solution of equation (11) is:

$$\zeta = C_1 r^{N_1} + C_2 r^{N_2} \quad (12)$$

Where C_1 and C_2 are arbitrary constants, and N_1 and N_2 are the two roots of:

$$N = -\frac{1}{2} \pm \frac{1}{2} \sqrt{1 + 8 \frac{(1-B)}{(1-\alpha)} \frac{E_t}{E_r}} \quad (13)$$

The stresses are now found by substitution of (12) in (9) and (10), resulting in:

$$\sigma_t = \frac{C_1 (1 + BN_1) r^{(N_1 - 1)} + C_2 (1 + BN_2) r^{(N_2 - 1)}}{\frac{1 - \alpha}{E_t} - \frac{2B^2}{E_r^2}} \quad (14)$$

$$\sigma_r = \frac{C_1 \left[(1 - \alpha) \frac{E_r}{E_t} N_1 + 2B \right] r^{(N_1 - 1)} + C_2 \left[(1 - \alpha) \frac{E_r}{E_t} + 2B \right] r^{(N_2 - 1)}}{\frac{1 - \alpha}{E_t} - \frac{2B^2}{E_r^2}} \quad (15)$$

The constants C_1 and C_2 depend on boundary conditions as follows:

$$\sigma_r \Big|_{r = R_i} = -\rho_i \quad (16)$$

$$\sigma_r \Big|_{r = R_o} = -\rho_o \quad (17)$$

Substituting,

$$-\rho_i = \frac{C_1 \left[(1 - \alpha) \frac{E_r}{E_t} N_1 + 2B \right] R_i^{(N_1 - 1)} + C_2 \left[(1 - \alpha) \frac{E_r}{E_t} N_2 + 2B \right] R_i^{(N_2 - 1)}}{\frac{1 - \alpha}{E_t} - \frac{2B^2}{E_r^2}} \quad (18)$$

$$-\rho_o = \frac{C_1 \left[(1 - \alpha) \frac{E_r}{E_t} N_1 + 2B \right] R_o^{(N_1 - 1)} + C_2 \left[(1 - \alpha) \frac{E_r}{E_t} N_2 + 2B \right] R_o^{(N_2 - 1)}}{\frac{1 - \alpha}{E_t} - \frac{2B^2}{E_r^2}} \quad (19)$$

Solving the last two equations for C_1 and C_2 gives:

$$C_1 = \frac{\left[\frac{1 - \alpha}{E_t} - \frac{2B^2}{E_r^2} \right] \left[-\rho_i + \rho_o \left(\frac{R_i}{R_o} \right)^{(N_2 - 1)} \right]}{\left[(1 - \alpha) \frac{E_r}{E_t} N_1 + 2B \right] \left[R_i^{(N_1 - 1)} - R_o^{(N_1 - 1)} \left(\frac{R_i}{R_o} \right)^{(N_2 - 1)} \right]} \quad (20)$$

$$C_2 = \frac{\left[\frac{1 - \alpha}{E_t} - \frac{2B^2}{E_r} \right] \left[-\rho_i + \rho_o \left(\frac{R_i}{R_o} \right)^{(N_1 - 1)} \right]}{\left[(1 - \alpha) \frac{E_r}{E_t} N_2 + 2B \right] \left[R_i^{(N_2 - 1)} - R_o^{(N_2 - 1)} \left(\frac{R_i}{R_o} \right)^{(N_1 - 1)} \right]} \quad (21)$$

Substituting (20) and (21) in (14) and (15) now yields equations of the stress distributions throughout the spherical shell. The resulting relationships are:

$$\sigma_t = \left\{ \begin{aligned} &+ \frac{\left[-\rho_i + \rho_o \left(\frac{R_i}{R_o} \right)^{(N_2 - 1)} \right] (1 + BN_1) \left(\frac{r}{R_i} \right)^{(N_1 - 1)}}{\left[(1 - \alpha) \frac{E_r}{E_t} N_1 + 2B \right] \left[1 - \left(\frac{R_o}{R_i} \right)^{(N_1 - N_2)} \right]} \\ &+ \frac{\left[-\rho_i + \rho_o \left(\frac{R_i}{R_o} \right)^{(N_1 - 1)} \right] (1 + BN_2) \left(\frac{r}{R_i} \right)^{(N_2 - 1)}}{\left[(1 - \alpha) \frac{E_r}{E_t} N_2 + 2B \right] \left[1 - \left(\frac{R_o}{R_i} \right)^{(N_2 - N_1)} \right]} \end{aligned} \right\} \quad (22)$$

$$\sigma_r = \left\{ \begin{aligned} &+ \left[-\rho_i + \rho_o \left(\frac{R_i}{R_o} \right)^{(N_2 - 1)} \right] \left[\frac{\left(\frac{r}{R_i} \right)^{(N_1 - 1)}}{1 - \left(\frac{R_o}{R_i} \right)^{(N_1 - N_2)}} \right] \\ &+ \left[-\rho_i + \rho_o \left(\frac{R_i}{R_o} \right)^{(N_1 - 1)} \right] \left[\frac{\left(\frac{r}{R_i} \right)^{(N_2 - 1)}}{1 - \left(\frac{R_o}{R_i} \right)^{(N_2 - N_1)}} \right] \end{aligned} \right\} \quad (23)$$

Equations (13), (22), and (23) describe the stress distributions in a semi-orthotropic spherical shell of any specified diameter and thickness as exposed to internal and/or external pressure. Values of the two poisson's ratios (α and B) and the modulus ratio E_r/E_t must be known from test information.

The stresses as found from these equations may be used as a basis for evaluation of structural integrity, but the user should recognize any residual stresses present as a consequence of fabrication methods, as well as thermal stresses induced by temperature gradients.

The collapse of these equations for the case of isotropic material is of interest. For isotropic materials,

$$\begin{aligned}\alpha &= B \\ E_r &= E_t\end{aligned}$$

$$\begin{aligned}\text{From (13)} \quad N_1 &= +1 \\ N_2 &= -2\end{aligned}$$

And the stress distributions, by (22) and (23) become:

$$\sigma_t = \frac{\rho_i \left[2 + \left(\frac{R_o}{r} \right)^3 \right] - \rho_o \left[2 \left(\frac{R_o}{R_i} \right)^3 + \left(\frac{R_o}{r} \right)^3 \right]}{2 \left[\left(\frac{R_o}{R_i} \right)^3 - 1 \right]} \quad (24)$$

Isotropic

$$\sigma_r = \frac{-\rho_i \left[\left(\frac{R_o}{r} \right)^3 - 1 \right] - \rho_o \left[\left(\frac{R_o}{R_i} \right)^3 - \left(\frac{R_o}{r} \right)^3 \right]}{\left[\left(\frac{R_o}{R_i} \right)^3 - 1 \right]} \quad (25)$$

Isotropic

A comparison of the stress distribution as found by equations (22) and (23), and the corresponding stresses in isotropic shells per (24) and (25) is of interest in evaluation of the departure of orthotropic materials from isotropic behavior. As an example, consider a pressure bottle in which the external pressure is zero. The inner surface of the shell is the location of prime interest, since both tangential and radial stresses are maximum at

$r = R_i$. The tabulation in Table I displays ratios of the tangential stress for the orthotropic material as compared to the isotropic material stress at $r = R_i$ for a range of values of the elastic parameters and three shell radius-thickness ratios. The data of the table indicates:

- The error of assumption of an isotropic material increases with increased wall-thickness.
- The error is negligible (less than two percent) for radius-thickness ratios of 12 or more.
- For thick spherical shells (radius-thickness ratio of less than 6) an error of ten percent or more is probable if the stress analysis is based on isotropic materials.

Table I

$$\frac{\sigma_t \text{ Orthotropic}}{\sigma_t \text{ Isotropic}} \quad \text{At } r = R_i$$

α	B	$\frac{E_r}{E_t}$	For $R_i/(R_o - R_i)$		
			6	12	24
0.1	0.1	0.25	1.0250	1.0066	1.0017
0.1	0.1	0.4	1.0497	1.0132	1.0034
0.1	0.1	1.0	1.0000	1.0000	1.0000
0.1	0.25	0.625	1.0056	1.0015	1.0004
0.1	0.25	1.0	0.9972	0.9993	0.9998
0.1	0.25	2.5	0.9888	0.9971	0.9992
0.1	0.4	1.0	0.9944	0.9985	0.9996
0.1	0.4	1.6	0.9902	0.9974	0.9993
0.1	0.4	4.0	0.9860	0.9963	0.9991
0.25	0.1	0.25	1.0628	1.0167	1.0043
0.25	0.1	0.4	1.0332	1.0088	1.0023
0.25	0.1	1.0	1.0033	1.0009	1.0002
0.25	0.25	0.625	1.0100	1.0027	1.0007
0.25	0.25	1.0	1.0000	1.0000	1.0000
0.25	0.25	2.5	0.9900	0.9974	0.9993
0.25	0.4	1.0	0.9967	0.9991	0.9998
0.25	0.4	1.6	0.9916	0.9978	0.9994
0.25	0.4	4.0	0.9866	0.9965	0.9991
0.4	0.1	0.25	1.0823	1.0220	1.0020
0.4	0.1	0.4	1.0456	1.0121	1.0031
0.4	0.1	1.0	1.0084	1.0022	1.0006
0.4	0.25	0.625	1.0167	1.0044	1.0011
0.4	0.25	1.0	1.0041	1.0011	1.0003
0.4	0.25	2.5	0.9916	0.9978	0.9994
0.4	0.4	1.0	1.0000	1.0000	1.0000
0.4	0.4	1.6	0.9937	0.9984	0.9996