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LARGE-DEFORMATION PLASTIC RESPONSE
OF HEXAGONS TO PRESSURE PULSES

by

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ABSTRACT

An analytical study is performed to determine the dynamic plastic response of a hexagonal subassembly duct to an internal pressure pulse of arbitrary shape. The analysis includes plastic distortion of the cross-section and large-deformation geometric effects that result in redistribution of the internal forces between bending and membrane stresses in the hexagon wall. This report contains a description of the analytical model, details of the calculational procedure and computer programs used to obtain numerical results, an approximate closed-form solution valid in the small-deformation range, and some typical results.

I. INTRODUCTION

Reactor subassembly ducts may be exposed to internal pressure pulses from fuel-coolant interactions or fission-gas releases. The resulting deformation may be of the order of the wall thickness or larger, and plastic yielding of the wall may occur. For a duct with a hexagonal cross section, large-deformation geometric effects result in redistribution of the loading between the bending and membrane reactions, so that the duct can withstand a static pressure higher than the yield limit of an undeformed hexagon. However, the application of a pressure pulse results in dynamic deformation, which may, because of inertia effects, significantly exceed the deformation caused by a static pressure having the same peak value. Conversely, a pressure pulse may produce less deformation than a statically applied pressure if the duration of the pulse is sufficiently short.

Accordingly, an analytical study has been performed to (1) develop a computational procedure for determining the dynamic plastic response of a hexagon to an internal pressure pulse; (2) determine the effect of pulse shape on the permanent deformation of the hexagon; (3) develop procedures for correlating permanent deformation with easily measured parameters associated with the pulse shape; (4) determine characteristic response times

for plastic deformation of a hexagon; and (5) determine the relative magnitudes of dynamic and static deformation produced by a given peak pressure.

A constraint on the goals of the investigation is that the developed computational methods and correlation procedures should be simple and easily applied. The aim is to provide the reactor engineer with a convenient tool for performing parameter studies, optimization analyses, safety studies, etc., where numerical analyses using, for instance, large finite-element programs may be excessively time-consuming and expensive. Moreover, the extension of general-purpose finite-element programs to cover large, dynamic plastic deformations is still in the development stage.

A presentation of the analysis and results of the study is divided between two reports. This report contains a description of the analytical model for the deforming hexagon, details of the calculational procedure and computer program used to obtain numerical results, an approximate closed-form solution valid in the small-deformation range, and some typical results of the numerical computations. The second report¹ will describe the parameter studies performed to accomplish the goals of the program and present the resulting correlation procedures and simplified computational methods for relating dynamic plastic deformation of a hexagon to pressure pulse shape.

The remainder of this report is organized as follows: Section II contains a statement of the problem, including, in particular, the analytical model for the large deformation of a hexagon. Section III describes the method of numerical solution of the governing sets of differential equations, and gives some typical results. An approximate closed-form solution for rectangular pulses, valid in the small deformation range, is derived in Sec. IV. The solution of the governing equations for static plastic deformation is obtained in Sec. V; this solution is needed for the parameter study comparing dynamic and static deformation produced by the same peak pressure. The appendix contains details of the numerical computation procedure and a listing of DYPLHX, the computer program used to obtain dynamic plastic-deformation results.

The approach to the large-deformation analysis is similar to that used by Symonds and Mentel,² Jones,³ and Symonds and Jones⁴ for the finite plastic deflections of beams.

II. STATEMENT OF PROBLEM

A cross section of an internally pressurized hexagonal subassembly duct of wall thickness H and side width L is shown in Fig. 1. The duct is assumed to be very long compared to its cross-sectional dimensions, and axial variation of the deflection shape is neglected. Consequently, the problem is reduced to finding the dynamic plastic deformation of a hexagonal frame loaded by a time-dependent pressure $P(t)$. Because of symmetry, only the half-side AB need be considered.

Fig. 1. Hexagon Subjected to Internal Pressure Pulse. ANL Neg. No. 113-4090.

A. Yield Condition

The hexagon is assumed to be made of a rigid, perfectly plastic material having yield stress σ_y . Elastic deformation is neglected, compared with the plastic deformation; strain hardening is neglected, compared with the large-deformation geometric effects.

Let N and M be the resultant membrane force per unit axial length and bending moment per unit axial length arising from the stress distribution across the wall thickness at a location on the side of the hexagon. Figure 2 shows the stress distributions and resultant N and M for pure stretching, pure bending, and combined stretching and bending. For pure stretching and pure bending, we have

$$N_0 = \sigma_y H, \quad M_0 = \frac{\sigma_y H^2}{4}, \quad (1)$$

and for the combined case,

$$N = 2\sigma_y y, \quad M = \pm \sigma_y \left(\frac{H^2}{4} - y^2 \right). \quad (2)$$

Fig. 2. Plastic Stress Distributions. (a) Pure Stretching; (b) pure bending; (c) combined stretching and bending. ANL Neg. No. 113-4230 Rev. 1.

Fig. 3. Yield Condition. ANL
Neg. No. 113-4049.

The sign for M is chosen according to the sign of y and the sign convention for the bending moment. Combining Eqs. 1 and 2 gives

$$\frac{|M|}{M_0} + \frac{N^2}{N_0^2} - 1 = 0 \quad (3)$$

for the yield condition, shown in Fig. 3.

B. Flow Rule

The plastic-flow rule states that, during plastic flow, the generalized strain-rate vector has the direction of the exterior normal to the yield condition at the considered generalized stress point.⁵ Let Δ

and θ be the stretch and angle change at a point on the side of the hexagon where plastic deformation is taking place, and let N and M be the membrane force and bending moment (per unit axial length) at this location. Then, writing Eq. 3 as $f(N, M) = 0$, we have

$$\frac{d\Delta}{dt} \bigg| \frac{d\theta}{dt} = \frac{\partial f}{\partial N} \bigg| \frac{\partial f}{\partial M}. \quad (4)$$

For the first quadrant of Fig. 3, which is the relevant one for this problem, the flow rule becomes, using Eqs. 3 and 4,

$$\frac{d\Delta}{dt} = \frac{2M_0 N}{N_0^2} \frac{d\theta}{dt}, \quad (5)$$

or, using Eqs. 1,

$$\frac{d\Delta}{dt} = \frac{N}{2\sigma_y} \frac{d\theta}{dt} \quad (6)$$

for $N < N_0$.

C. Limit Load of Undistorted Hexagon

Let P_y be the limit load of the undistorted hexagon, i.e., the pressure at which the hexagon is in static equilibrium but plastic deformation is incipient. Plastic hinges are produced at the corners and centers of sides, where the yield condition is satisfied, and intermediate positions along the sides correspond to interior points of the yield condition (Fig. 3).

Fig. 4. Free-body Diagram of Hexagon Half-side at Initial Yield. ANL Neg. No. 113-4231.

A free-body diagram of the half-side AB is shown in Fig. 4. From the equations of equilibrium in the horizontal direction, we have that the membrane forces at A and B are equal in magnitude (N_y) and opposite in direction. By Eq. 3, the bending moments at A and B must be equal in magnitude (M_y), and be directed as shown in Fig. 4 to resist the angle change. The symmetry of the problem requires that there be no vertical reaction at the midspan B and no reaction parallel to the dashed line at the corner A; this last condition implies that

$$N_y = \sqrt{3} F. \quad (7)$$

The equilibrium equation for the vertical direction becomes, using Eq. 7,

$$N_y = \frac{\sqrt{3}}{2} P_y L, \quad (8)$$

and moment equilibrium requires

$$M_y = \frac{P_y L^2}{16}. \quad (9)$$

Substitution from Eqs. 8 and 9 into the yield condition (Eq. 3) results in a quadratic equation for P_y ; its solution is

$$P_y = \frac{8\alpha^2 \sigma_y}{1 + \sqrt{1 + 48\alpha^2}}, \quad (10)$$

where the geometric parameter α is defined by

$$\alpha = \frac{H}{L}. \quad (11)$$

A plot of P_y/σ_y as a function of H/L is shown in Fig. 5.

The bending moment distribution in the half-side is given by

$$M(x) = M_y \left(1 - \frac{8x^2}{L^2} \right), \quad 0 \leq x \leq \frac{L}{2}, \quad (12)$$

Fig. 5. Limit Load as Function of H/L.
ANL Neg. No. 113-4236.

where x is measured from point B. Consequently, the magnitude of M is equal to M_y at points A and B and less than M_y at interior points.

D. First Plastic-deformation Mode

The simplest deformation mode that occurs during dynamic distortion has plastic hinges at the corners and midpoints of the sides of the hexagon connected by rigid sections of width $L/2$. A free-body diagram of the typical half-side AB is shown in Fig. 6. The stretches at the plastic hinges are Δ_A and Δ_B , and the angular rotation occurring at each hinge is θ . Because of symmetry, (1) the force resultants at A and B must be perpendicular to the dashed lines, and (2) the velocities and displacements of points A and B must be along the dashed lines. Therefore,

$$N_A = \sqrt{3} F \quad (13)$$

and

$$\left[V_P + \left(\frac{L}{4} + \Delta_B \right) \Omega \right] \tan \theta = \frac{d\Delta_B}{dt} - V_T \quad (14)$$

and

$$\left[V_P - \left(\frac{L}{4} + \Delta_A \right) \Omega \right] \tan \left(\frac{\pi}{6} - \theta \right) = \frac{d\Delta_A}{dt} + V_T$$

Fig. 6

Free-body Diagram of Hexagon Half-side for First Plastic-deformation Mode.
ANL Neg. No. 113-4050 Rev. 1.

where $\Omega(t)$ is the angular velocity, and $V_P(t)$ and $V_T(t)$ are the velocity components of the center of mass of the half-side perpendicular and parallel, respectively, to the instantaneous position of the half-side.

The equations of motion of the half-side of the hexagon are, using Eq. 13,

$$\left. \begin{aligned} \frac{\rho H L^3}{96} \frac{d\Omega}{dt} &= -M_A - M_B - \frac{N_B L}{4} \sin \theta + \frac{N_A L}{4} \left(\frac{\cos \theta}{\sqrt{3}} - \sin \theta \right), \\ \frac{\rho H L}{2} \frac{dV_P}{dt} &= \frac{P L}{2} - N_B \sin \theta + N_A \left(\sin \theta - \frac{\cos \theta}{\sqrt{3}} \right), \\ \text{and} \\ \frac{\rho H L}{2} \frac{dV_T}{dt} &= N_A \left(\cos \theta + \frac{\sin \theta}{\sqrt{3}} \right) - N_B \cos \theta, \end{aligned} \right\} \quad (15)$$

where ρ is the density of the duct material.

The yield condition and flow rule for the plastic hinge at A are, from Eqs. 3 and 6,

$$\left. \begin{aligned} \frac{M_A}{M_0} + \left(\frac{N_A}{N_0} \right)^2 &= 1 \\ \text{and} \\ \frac{d\Delta_A}{dt} &= \frac{N_A \Omega}{2\sigma_y} \end{aligned} \right\}, \quad (16)$$

and at the hinge at B they are

$$\left. \begin{aligned} \frac{M_B}{M_0} + \left(\frac{N_B}{N_0} \right)^2 &= 1 \\ \text{and} \\ \frac{d\Delta_B}{dt} &= \frac{N_B \Omega}{2\sigma_y} \end{aligned} \right\}. \quad (17)$$

The angular rotation θ is found from

$$\frac{d\theta}{dt} = \Omega, \quad (18)$$

and the displacements at the corner and midspan are given by

$$\left. \begin{aligned} U_A &= 2(\Delta_A + \Delta_B) \cos \theta - L(1 - \cos \theta) \\ U_B &= \frac{1}{2} \sqrt{3} U_A + (\frac{1}{2} L + \Delta_A + \Delta_B) \sin \theta \end{aligned} \right\}, \quad (19)$$

where U_A and U_B are measured positive outward along the dashed lines in Fig. 6.

Equations 13-19 give 13 relations for determining the unknown functions of time N_A , N_B , M_A , M_B , F , Ω , V_P , V_T , Δ_A , Δ_B , θ , U_A , and U_B for a prescribed loading $P(t)$. This set of equations was solved numerically; some details of the solution are given in Sec. III and the appendix.

The bending-moment distribution in the vicinity of point B, the center of the side, is given by

$$\begin{aligned} M(x,t) &= M_B + \frac{4x^2(4x - 3L)}{L^3} (M_A + M_B) \\ &\quad + \frac{4x^2(x - L)}{L^2} N_B \sin \theta + \frac{2x^2(L - 2x)}{L^2} N_A \left(\frac{\cos \theta}{\sqrt{3}} - \sin \theta \right), \end{aligned} \quad (20)$$

where x is measured from point B along the distorted side. The first derivative of M at B is zero, and the second derivative at B is

$$\left. \frac{\partial^2 M}{\partial x^2} \right|_{x=0} = -\frac{24}{L^2} (M_A + M_B) - \frac{8}{L} N_B \sin \theta + \frac{4}{L} N_A \left(\frac{\cos \theta}{\sqrt{3}} - \sin \theta \right). \quad (21)$$

Therefore, the bending moment has a relative maximum at B if its second derivative there is negative. For some pulse shapes, the computed values of N_A , N_B , M_A , and M_B are such that the second derivative of M at B becomes zero at some time while P is increasing. In order not to violate the yield condition at a point near B, which would occur if the second derivative changed sign, a hinge band begins to form at B and spreads out from there as P increases. Plastic deformation involving a hinge band at B and a hinge at A will be called the second plastic-deformation mode and will be discussed next. By Eq. 21, the transition from the first to the second deformation mode is signaled by

$$-6(M_A + M_B) - 2N_B L \sin \theta + N_A L \left(\frac{\cos \theta}{\sqrt{3}} - \sin \theta \right) = 0. \quad (22)$$

E. Second Plastic-deformation Mode

The second deformation mode has a plastic hinge at the corners and a hinge band at the center of each side; there are moving hinges at the ends of the hinge bands, and a rigid section of wall connects each moving hinge with the nearest corner hinge. A free-body diagram of the typical half-side AB is shown in Fig. 7. The half-width of the hinge band is $\epsilon(t)L/2$, and the stretches at the moving hinge and corner hinge are Δ_ϵ and Δ_A , respectively.

Fig. 7. Free-body Diagram of Hexagon Half-side for Second Plastic-deformation Mode. ANL Neg. No. 113-4232.

As for the first mode, the symmetry of the problem requires that (1) the force resultants at A and B must be perpendicular to the dashed lines, and (2) the velocities V_A and V_B and the displacements U_A and U_B must be along the dashed lines. The condition on the force resultant at the corner gives

$$N_A = \sqrt{3} F, \quad (23)$$

and the condition on the velocity resultant at the corner gives

$$\left. \begin{aligned} V_A \cos \frac{\pi}{6} &= \left\{ V_P - \left[\frac{L}{4} (1 - \epsilon) + \Delta_A \right] \Omega \right\} \cos \theta + \left[\frac{L}{4} \frac{d\epsilon}{dt} - V_T - \frac{d\Delta_A}{dt} \right] \sin \theta \\ \text{and} \\ \left\{ V_P - \left[\frac{L}{4} (1 - \epsilon) + \Delta_A \right] \Omega \right\} \tan \left(\frac{\pi}{6} - \theta \right) + \left[\frac{L}{4} \frac{d\epsilon}{dt} - V_T - \frac{d\Delta_A}{dt} \right] &= 0 \end{aligned} \right\}. \quad (24)$$

Although the yield condition is satisfied along the entire hinge band, all plastic deformation associated with the band occurs at the moving hinge at its end. Since the velocity at B is vertical by the second symmetry condition above, the horizontal velocity component at the end of the band is just the band growth rate $(L/2)(d\epsilon/dt)$. Let the vertical velocity component at the end of the band be V_ϵ (positive downward); the velocity components at

the end of the band are then related to motion of the rigid portion of the half-side through

$$\left. \begin{aligned} \frac{L}{2} \frac{d\epsilon}{dt} &= \left\{ V_P + \left[\frac{L}{4} (1 - \epsilon) + \Delta_B \right] \Omega \right\} \sin \theta + \left(\frac{L}{4} \frac{d\epsilon}{dt} + V_T - \frac{d\Delta_B}{dt} \right) \cos \theta \\ \text{and} \\ V_\epsilon &= \left\{ V_P + \left[\frac{L}{4} (1 - \epsilon) + \Delta_B \right] \Omega \right\} \cos \theta - \left(\frac{L}{4} \frac{d\epsilon}{dt} + V_T - \frac{d\Delta_B}{dt} \right) \sin \theta \end{aligned} \right\}. \quad (25)$$

The equation of motion for the hinge band region is

$$\rho H \frac{dV(x,t)}{dt} = P(t), \quad 0 \leq x \leq \frac{\epsilon L}{2}; \quad (26)$$

this can be integrated to give

$$\rho H V(x,t) = \int P dt + G(x), \quad 0 \leq x \leq \frac{\epsilon L}{2}, \quad (27)$$

where $G(x)$ is a function to be determined from the motion. In particular, letting $x = 0$ gives

$$\rho H V_B = \int P dt + G(0), \quad (28)$$

and letting $x = (\epsilon L/2)$ gives

$$\rho H V_\epsilon = \int P dt + G\left(\frac{\epsilon L}{2}\right). \quad (29)$$

The equations of motion of the rigid portion of the half-side are, using Eq. 23,

$$\left. \begin{aligned} \frac{\rho H L^3 (1 - \epsilon)^3}{96} \frac{d\Omega}{dt} &= -M_A - M_B + \frac{L(1 - \epsilon)}{4} \left[-N_B \sin \theta + N_A \left(\frac{\cos \theta}{\sqrt{3}} - \sin \theta \right) \right], \\ \frac{\rho H L (1 - \epsilon)}{2} \frac{dV_P}{dt} &= \frac{P L (1 - \epsilon)}{2} - N_B \sin \theta - N_A \left(\frac{\cos \theta}{\sqrt{3}} - \sin \theta \right), \\ \text{and} \\ \frac{\rho H L (1 - \epsilon)}{2} \frac{dV_T}{dt} &= N_A \left(\cos \theta + \frac{\sin \theta}{\sqrt{3}} \right) - N_B \cos \theta. \end{aligned} \right\} \quad (30)$$

The yield condition and flow rule for the plastic hinge at the corner are, from Eqs. 3 and 6,

$$\left. \begin{aligned} \frac{M_A}{M_0} + \left(\frac{N_A}{N_0} \right)^2 &= 1 \\ \frac{d\Delta_A}{dt} &= \frac{N_A \Omega}{2\sigma_y} \end{aligned} \right\}, \quad (31)$$

and at the hinge at the end of the band they are

$$\left. \begin{aligned} \frac{M_B}{M_0} + \left(\frac{N_B}{N_0} \right)^2 &= 1 \\ \frac{d\Delta_{\epsilon}}{dt} &= \frac{N_B \Omega}{2\sigma_y} \end{aligned} \right\}. \quad (32)$$

The angular rotation, displacement at the corner, and displacement at midspan are given by

$$\frac{d\theta}{dt} = \Omega, \quad \frac{dU_A}{dt} = V_A, \quad \text{and} \quad \frac{dU_B}{dt} = V_B, \quad (33)$$

where V_A and V_B are found from Eqs. 24 and 28.

The bending-movement distribution near the hinge band end of the rigid portion of the half-side is

$$\begin{aligned} M = M_B + \left(x - \frac{\epsilon L}{2} \right)^2 &\left\{ \frac{4(4x + \epsilon L - 3L)}{L^3(1 - \epsilon)^3} (M_A + M_B) \right. \\ &\left. + \frac{2}{L^2(1 - \epsilon)^2} \left[(2x + \epsilon L - 2L) N_B \sin \theta + (L - 2x) N_A \left(\frac{\cos \theta}{\sqrt{3}} - \sin \theta \right) \right] \right\}. \end{aligned} \quad (34)$$

The derivative $\partial M / \partial x$ is zero at $x = \epsilon L / 2$. Let

$$Q = \frac{\partial^2 M}{\partial x^2} \quad \text{at} \quad x = \frac{\epsilon L}{2}. \quad (35)$$

The yield condition would be violated in the vicinity of the end of the band if Q were positive. If the hinge band is increasing in size, we have

$$Q = 0 \quad \text{for} \quad \frac{d\epsilon}{dt} \geq 0, \quad (36)$$

and if it is decreasing in size,

$$Q < 0 \quad \text{for} \quad \frac{d\epsilon}{dt} < 0. \quad (37)$$

Differentiating Eq. 34 twice and evaluating the result at $x = \epsilon L/2$ give

$$Q = -\frac{24}{L^2(1-\epsilon)^2} (M_A + M_B) + \frac{4}{L(1-\epsilon)} \left[-2N_B \sin \theta + N_A \left(\frac{\cos \theta}{\sqrt{3}} - \sin \theta \right) \right]. \quad (38)$$

For ϵ increasing (which occurs, in general, for P increasing), Eqs. 23-25, 28-33, and 36 are 18 relations for determining the unknown functions N_A , N_B , M_A , M_B , F , Ω , V_P , V_T , V_A , V_B , V_ϵ , Δ_A , Δ_ϵ , θ , U_A , U_B , ϵ and G for a prescribed loading $P(t)$ such that the motion is in the second deformation mode. If the hinge band is decreasing in size, Eq. 36 does not apply; however, $G(x)$ is now a known function, since ϵ is retreating through the same values it passed through as it was increasing. Consequently, there is one less unknown function, and one less equation is needed.

F. Other Plastic-deformation Modes

The two deformation modes described above each result in the hexagon being distorted into an irregular 12-sided figure. Severe loadings could produce transitions to higher deformation modes with 24 sides, 48 sides, and so on, until an approximately circular shape is attained. Such gross distortions of the cross section are beyond the scope of this study, and only the first and second deformation modes are treated.

G. Initial Conditions

At the onset of plastic yielding, all velocities and deformations are zero. If the pressure passes gradually through the yield load, the initial conditions on the membrane forces and bending moments are given by Eqs. 8 and 9.

An initial instantaneous jump to a pressure P_i can produce either first- or second-deformation-mode response. For first-mode response, the initial condition on the membrane forces is determined by solving the simultaneous equations

$$\left. \begin{aligned} & \frac{N_A - N_B}{N_B} \left[\frac{2\sqrt{3} \alpha (N_A + N_B)}{N_0} + 1 \right] + \frac{2\alpha N_A}{\sqrt{3} N_0} = \frac{P_i}{\sigma_y} \\ & \frac{N_A - N_B}{N_B} + 6\alpha^2 \left(2 - \frac{N_A^2 + N_B^2}{N_0^2} \right) - 2\sqrt{3} \alpha \frac{N_A}{N_0} = 0 \end{aligned} \right\}, \quad (39)$$

where N_0 and α are given by Eqs. 1 and 11. The initial condition on the bending moments is then found from Eqs. 16 and 17.

For second-mode response, the initial conditions on the membrane forces are

$$\left. \begin{aligned} & \frac{N_A}{N_0} = (1 + C_1) C_2 \\ & \frac{N_B}{N_0} = (1 - C_1) C_2 \end{aligned} \right\}, \quad (40)$$

and the initial hinge-bandwidth is

$$\epsilon = 1 - \frac{\alpha(1 + C_1) C_2}{\sqrt{3} C_1}, \quad (41)$$

where C_1 and C_2 are defined by

$$\left. \begin{aligned} & C_1 = \frac{2 \frac{P_i}{\sigma_y}}{4 - \frac{P_i}{\sigma_y} + \sqrt{\left(4 - \frac{P_i}{\sigma_y}\right)^2 + 112 \frac{P_i}{\sigma_y}}} \\ & C_2 = \left[\frac{9C_1}{9C_1(1 + C_1^2) + (1 + C_1)^2} \right]^{1/2} \end{aligned} \right\}. \quad (42)$$

Again, the initial values of M_A and M_B are found from the yield condition.

Let P_h be the initial jump pressure which marks the transition between first- and second-mode responses. This transition pressure can be determined either by finding the value of P_i such that the values of N_A , N_B , M_A , and M_B calculated from Eqs. 39, 16, and 17 satisfy Eq. 22 for $\theta = 0$, or by finding the value of P_i such that $\epsilon = 0$ in Eqs. 40. The result is

$$\frac{P_h}{\sigma_y} = \frac{4C(1+7C)}{1+C}, \quad (43)$$

where C is related to α through

$$\alpha^2 = \frac{C}{3} + \frac{3C^2(1+C^2)}{(1+C)^2}. \quad (44)$$

Figure 8 shows P_h/σ_y as a function of the geometric parameter $\alpha = H/L$. Comparing Figs. 5 and 8, we see that P_h is approximately three times P_y for a given α .

Fig. 8

Initial-jump Transition Pressure between First- and Second-mode Responses. ANL Neg. No. 113-4233.

III. NUMERICAL SOLUTION AND TYPICAL RESULTS

A. Discussion

The governing sets of coupled nonlinear differential equations were solved on a computer using the library subroutine DIFSUB,⁶ which employs the Bulirsch-Stoer extrapolation method. Two major difficulties were encountered in the numerical solutions.

First, the system of equations is ill-conditioned at the initial yield point. This difficulty was overcome by expanding the solution in power series for the initial time step. Four sets of power series are required, depending on the type of initial loading and mode response. The types of initial motion are: (a) first deformation mode with P having a gradual transition through the yield load, (b) first deformation mode with P having an initial instantaneous jump through the yield load, (c) second deformation mode with P having an initial instantaneous jump followed by increasing P and ϵ , and (d) second deformation mode with P having an initial instantaneous jump followed by decreasing P and ϵ . (A gradual transition through the yield pressure always initiates first deformation mode behavior.) The four sets of power series are given in the appendix.

The second major difficulty encountered was that the computing time per problem was considerably longer than expected, as compared to the similar numerical solution of Ref. 7, for instance, and the solution was unstable for some problems. The source of this behavior was traced to the differences $N_A - N_B$ and $\Delta_A - \Delta_B$ being small for small values of θ . For example, the equation for dV_T/dt of Eqs. 15 strongly depends on $N_A - N_B$ if θ is small, so that small errors in computing N_A and N_B are magnified into large errors in computing V_T . This sensitivity of the solution slows the numerical convergence and increases the likelihood of solution instability and divergence. To overcome this difficulty, new dependent variables were defined by

$$\left. \begin{aligned} \mu_1 &= \frac{N_A + N_B}{2N_0}, \quad \mu_2 = \frac{N_A - N_B}{2N_0}, \\ \gamma_1 &= \frac{\Delta_A + \Delta_B}{L}, \quad \gamma_2 = \frac{\Delta_A - \Delta_B}{L}; \end{aligned} \right\} \quad (45)$$

Δ_ϵ replaces Δ_B in the above equations for the second deformation mode. The use of these average and difference quantities, rather than the membrane forces and stretches, and the elimination of negligible terms resulted in more tractable sets of differential equations, and, consequently, improved convergence and shorter running time.

B. Typical Results

The results of detailed studies to determine the effects of loading and structural parameters on dynamic plastic deformation of a hexagon are presented in Ref. 1. Only results for a few typical problems will be presented here to illustrate the time-dependent behavior of some of the variables.

In each of the examples, $H/L = 0.056$, which corresponds to the proportions of the FFTF reactor subassembly ducts. The pressure pulse has the form

$$P(t) = P_m e^{-t/t_1}. \quad (46)$$

The values of P_m and t_1 for the three example problems are:

$$\left. \begin{array}{l} \text{Problem 1. } P_m = 2P_y = 0.02421\sigma_y, t_1 = 0.01t_0. \\ \text{Problem 2. } P_m = 2P_y = 0.02421\sigma_y, t_1 = 2t_0. \\ \text{Problem 3. } P_m = 2\alpha\sigma_y = 0.112\sigma_y, t_1 = 0.01t_0. \end{array} \right\} \quad (47)$$

Here the time constant t_0 is defined by

$$t_0 = L \sqrt{\frac{\rho}{\sigma_y}}. \quad (48)$$

We can divide the possible values of the peak pressure P_m into three ranges, "low," "medium," and "high" intensity, where "low" corresponds to $P_m < P_y$, "medium" corresponds to $P_y \leq P_m < P_h$, and "high" corresponds to $P_m > P_h$. Therefore, in the low-intensity range, no plastic deformation occurs; in the medium range, first-mode response is initiated; and in the high range, second-mode response is initiated. Problem 1 is then a medium-intensity, short-duration pulse; Problem 2 is a medium-intensity, long-duration pulse; and Problem 3 is a high-intensity, short-duration pulse. A high-intensity, long-duration pulse would produce distortions of the hexagon that are beyond the scope of this study.

Figures 9-12 show P , N_B , Ω , V_P , V_T , U_A , and U_B for Problems 1 and 2, and Figs. 13 and 14 show P , N_B , ϵ , Ω , V_P , U_A , and U_B for Problem 3. All of the variables shown in the figures are nondimensional.

Comparison of the results of Problems 1 and 2, which are for the same peak pressure but different pulse duration, indicates significant qualitative and quantitative differences in the responses. For the shorter-duration pulse, the final permanent deformation at the center of the side is approximately $1.88H \times 10^{-4}$, which is in the small-deformation range; for the longer-duration pulse, $U_B = 1.61H$ at the end of the motion, which is

in the large-deformation range. The final deformations at the corners are $7.6H \times 10^{-6}$ for Problem 1 and $-0.25H$ for Problem 2, the negative sign indicating that the corners move inwards. Thus the corner deflection is small compared to the midspan deflection for Problem 1, but it is proportionately larger for Problem 2. For the longer-duration pulse, the corners move out initially and then later move in as the large-deformation membrane effects become more important. The curves for V_T and N_B have different shapes for the two pulses. Note that, in Problem 2, the motion stops at a time when the applied pressure is still greater than the yield pressure of the undeformed hexagon, which shows that the inclusion of the large-deformation effects results in a strengthening of the hexagon.

Fig. 9

Example Problem 1: P/σ_y , N_B/N_0 , U_A/H , and U_B/H as Functions of t/t_0 . ANL Neg. No. 113-4244.

Fig. 10

Example Problem 1: $V_P t_0/H$, $V_T t_0/H$, and Ωt_0 as Functions of t/t_0 . ANL Neg. No. 113-4228.

Fig. 11. Example Problem 2: P/σ_y , N_B/N_0 , U_A/H , and U_B/H as Functions of t/t_0 . ANL Neg. No. 113-4238.

Fig. 12. Example Problem 2: V_{Pt_0}/H , V_{Tt_0}/H , and Ωt_0 as Functions of t/t_0 . ANL Neg. No. 113-4239.

Fig. 13. Example Problem 3: P/σ_y , N_B/N_0 , U_A/H , U_B/H , and ϵ as Functions of t/t_0 . ANL Neg. No. 113-4229 Rev. 1.

Fig. 14. Example Problem 3: V_{Pt_0}/H and Ωt_0 as Functions of t/t_0 . ANL Neg. No. 113-4240.

The results for the high-intensity, short-duration pulse (Problem 3) are not much different, qualitatively, than those for Problem 1, except for irregularities in the vicinity of the transition between the initial second-deformation-mode response ($\epsilon > 0$) and the subsequent first-mode response ($\epsilon = 0$). Although the peak load is more than nine times the yield pressure, the final deformations are small ($U_B \approx 0.015H$ and $U_A \approx 0.0006H$) because of the short pulse duration.

IV. APPROXIMATE SMALL-DEFORMATION SOLUTION

Many terms in the governing sets of differential equations* derived in Sec. II are negligible if the pulse duration is such that the displacements of the sides of the hexagon are small compared to its wall thickness. An approximate, closed-form solution can then be obtained for rectangular pulses. The solution for rectangular pulses is useful because, as is shown in Ref. 1, pulses of other shapes are equivalent to rectangular pulses in producing final plastic deformation; this is accomplished by employing the same correlation procedures as described in Ref. 8 for small-deformation response of other structural configurations.

A. Approximations and Pressure Pulse

Results of the numerical solution to the governing sets of differential equations for the deforming hexagon show that the following approximations can reasonably be made if the deformation is small:

$$\left. \begin{aligned} \theta \simeq 0, N_A \simeq N_B, V_T \simeq 0, U_A \simeq 0, \Delta_A \ll L, \Delta_B \ll L, \frac{d\Delta_A}{dt} \simeq 0, \\ \frac{d\Delta_B}{dt} \simeq 0. \end{aligned} \right\} \quad (49)$$

Let the applied loading be a rectangular pulse of intensity P_r and duration t_r ; i.e.,

$$\left. \begin{aligned} P(t) &= P_r, \quad 0 \leq t \leq t_r, \\ &= 0, \quad t > t_r. \end{aligned} \right\} \quad (50)$$

The response to the initial jump of the rectangular pulse can be in either the first or second deformation modes, depending on the magnitude of P_r . The approximate differential equations and their solutions are treated next for each type of initial response.

B. Initial First-mode Response

Making the approximation of Eqs. 49 in the governing equations for first-mode response derived in Sec. II results in the following set of approximate equations:

*These sets of equations will be referred to as the "exact" equations in what follows.

$$\left. \begin{aligned}
 v_p^* - \frac{L}{4} \Omega^* &= 0, \\
 \frac{\rho H L^3}{96} \frac{d\Omega^*}{dt} &= -2M_B^* + \frac{N_B^* L}{4\sqrt{3}}, \\
 \frac{\rho H L}{2} \frac{dv_p^*}{dt} &= \frac{PL}{2} - \frac{N_B^*}{\sqrt{3}}, \\
 M_B^* &= M_A^* = M_0 \left[1 - \left(\frac{N_B^*}{N_0} \right)^2 \right], \\
 \frac{d\theta^*}{dt} &= \Omega^*,
 \end{aligned} \right\} \quad (51)$$

and

$$U_B^* = \frac{L}{2} \theta^*.$$

The asterisks indicate that the solution is an approximation to the numerical solution of the "exact" set of differential equations.

For the loading of Eqs. 50, the solution to Eqs. 51 for the interval $0 \leq t \leq t_r$ is

$$\left. \begin{aligned}
 N_B^* &= 3\sqrt{3}\alpha N_0 f_0, \\
 v_P^* &= \frac{H f_1 t}{t_0^2}, \\
 \Omega^* &= \frac{4\alpha f_1 t}{t_0^2}, \\
 \theta^* &= \frac{2\alpha f_1 t^2}{t_0^2},
 \end{aligned} \right\} \quad (52)$$

and

$$U_B^* = \frac{H f_1 t^2}{t_0^2},$$

where

$$\left. \begin{aligned} f_0 &= \frac{1 + \frac{P_r}{12\alpha^2\sigma_y}}{1 + \sqrt{1 + 27\alpha^2 + \frac{9P_r}{4\sigma_y}}}, \\ f_1 &= \frac{P_r}{\alpha^2\sigma_y} - 6f_0, \end{aligned} \right\} \quad (53)$$

and α and t_0 are defined by Eqs. 11 and 48, respectively. The solution given by Eqs. 52 can be readily verified by substitution into Eqs. 51.

The solution for the remainder of the motion $t_r \leq t \leq t_f$ is

$$\left. \begin{aligned} N_B^* &= \frac{\sqrt{3}}{2} \alpha N_0 f_2, \\ V_P^* &= \frac{H}{t_0^2} [-f_2(t - t_r) + f_1 t_r], \\ \Omega^* &= \frac{4\alpha}{t_0^2} [-f_2(t - t_r) + f_1 t_r], \\ \theta^* &= \frac{2\alpha}{t_0^2} [-f_2(t - t_r)^2 + 2f_1 t_r(t - t_r) + f_1 t_r^2], \end{aligned} \right\} \quad (54)$$

and

$$U_B^* = \frac{H}{t_0^2} [-f_2(t - t_r)^2 + 2f_1 t_r(t - t_r) + f_1 t_r^2],$$

where

$$f_2 = \frac{6}{1 + \sqrt{1 + 27\alpha^2}}. \quad (55)$$

The solution for this interval can be checked by substitution into Eqs. 51 and verifying the continuity of the velocities and displacements at t_r , using Eqs. 52.

The time t_f^* when the deformation stops is when $V_P^* = 0$. From Eqs. 54, we have

$$t_f^* = t_r \left(1 + \frac{f_1}{f_2} \right). \quad (56)$$

Substituting this result into the equation for U_B^* in Eqs. 54 gives

$$U_B^*(t_f^*) = \frac{Hf_1 t_r^2}{t_0^2} \left(1 + \frac{f_1}{f_2} \right) \quad (57)$$

for the final plastic deformation at midspan. As $\alpha \rightarrow 0$,

$$U_B^*(t_f^*) \rightarrow 3H \left(\frac{P_r t_r}{P_y t_0} \right)^2 \left(1 - \frac{P_y}{P_r} \right). \quad (58)$$

Figure 15 compares the numerical solution to the exact equations and the approximate solution given by Eqs. 52-57 for the problem specified by $\alpha = 0.056$, $P_r = 0.4\alpha\sigma_y = 0.0224\sigma_y$, $t_r = 0.01t_0$. The approximation is quite good.

Fig. 15

Comparison between Exact and Approximate Solutions for Example Problem with Initial First-mode Response. ANL Neg. No. 113-4245.

C. Initial Second-mode Response

The use of the approximations of Eqs. 49 in the governing equations for second-mode response results in the following set of approximate equations:

$$V_P^* - \frac{L}{4} (1 - \epsilon^*) \Omega^* = 0,$$

$$V_\epsilon^* = 2V_P^*,$$

(59)
(Contd.)

$$\left. \begin{aligned}
 \rho H V_B^* &= \int P \, dt + G^*(0), \\
 \rho H V_\epsilon^* &= \int P \, dt + G^*\left(\frac{\epsilon^* L}{2}\right), \\
 \frac{\rho H L^3}{96} (1 - \epsilon^*)^3 \frac{d\Omega^*}{dt} &= -2M_B^* + \frac{N_B^* L}{4\sqrt{3}} (1 - \epsilon^*), \\
 \frac{\rho H L}{2} (1 - \epsilon^*) \frac{dV_P^*}{dt} &= \frac{P L}{2} (1 - \epsilon^*) - \frac{N_B^*}{\sqrt{3}}, \\
 M_B^* &= M_A^* = M_0 \left[1 - \left(\frac{N_B^*}{N_0} \right)^2 \right], \\
 \frac{d\theta^*}{dt} &= \Omega^*, \\
 \text{and} \\
 \frac{dU_B^*}{dt} &= V_B^*.
 \end{aligned} \right\} \quad \begin{array}{l} \text{(Contd.)} \\ (59) \end{array}$$

For $d\epsilon^*/dt \geq 0$, the approximate hinge-band growth equation derived from Eqs. 36, 38, and 49 is

$$12\sqrt{3}M_B^* - N_B^* L(1 - \epsilon^*) = 0. \quad (60)$$

Assuming P_r is large enough that second-mode response is initiated, the hinge band has a constant width ϵ_0^* in the interval $0 \leq t \leq t_r$. From Eqs. 59 and 60, the approximate solution in this interval is

$$\left. \begin{aligned}
 N_B^* &= 3N_0 \left(\frac{P_r}{4\sigma_y + 9P_r} \right)^{1/2}, \\
 \epsilon^* &= \epsilon_0^* = 1 - \frac{4\alpha\sigma_y N_B^*}{\sqrt{3}P_r N_0}, \\
 V_P^* &= \frac{HP_r t}{2\alpha^2\sigma_y t_0^2}, \\
 V_\epsilon^* &= \frac{HP_r t}{\alpha^2\sigma_y t_0^2},
 \end{aligned} \right\} \quad \begin{array}{l} (61) \\ \text{(Contd.)} \end{array}$$

$$\left. \begin{aligned}
 \Omega^* &= \frac{2P_r t}{\alpha \sigma_y (1 - \epsilon_0^*) t_0^2}, \\
 \theta^* &= \frac{P_r t^2}{\alpha \sigma_y (1 - \epsilon_0^*) t_0^2}, \\
 G^*(x) &= 0, \\
 V_B^* &= V_\epsilon^*, \\
 U_B^* &= \frac{HP_r t^2}{2\alpha^2 \sigma_y t_0^2}.
 \end{aligned} \right\} \quad \begin{array}{l} \text{(Contd.)} \\ (61) \end{array}$$

and

The approximate transition pressure P_h^* between initial first- and second-mode responses to an instantaneous jump is the value of P_r for which $\epsilon_0^* = 0$. From Eqs. 61, P_h^* is given by

$$P_h^* = \frac{24\alpha^2 \sigma_y}{1 + \sqrt{1 + 108\alpha^2}}. \quad (62)$$

Figure 8 compares P_h^* with P_h , the solutions of Eqs. 43 and 44. The match is quite good for the usual range of values of α .

After the end of the pulse, the second-mode response continues as ϵ decreases from ϵ_0 to zero. Let t_ϵ^* be the time at which the second-mode response ends. The solution to Eqs. 59 in the interval $t_r \leq t \leq t_\epsilon^*$ is, using the continuity conditions on the velocities and displacements at t_r ,

$$\left. \begin{aligned}
 N_B^* &= 0, \\
 1 - \epsilon^* &= \left[(1 - \epsilon_0^*)^2 + \frac{48\alpha^2 \sigma_y}{P_r} \left(\frac{t}{t_r} - 1 \right) \right]^{1/2}, \\
 \Omega^* &= \frac{2P_r t_r}{\alpha \sigma_y (1 - \epsilon^*) t_0^2}, \\
 V_P^* &= \frac{HP_r t_r}{2\alpha^2 \sigma_y t_0^2}, \\
 V_\epsilon^* &= V_B^* = \frac{HP_r t_r}{\alpha^2 \sigma_y t_0^2},
 \end{aligned} \right\} \quad \begin{array}{l} (63) \\ \text{(Contd.)} \end{array}$$

$$\theta^* = \frac{P_r t_r^2}{48 \alpha^3 \sigma_y^2 t_0^2} \left[4(1 - \epsilon^*) + 9(1 - \epsilon_0^*) \frac{P_r}{\sigma_y} \right], \quad \left. \begin{array}{l} \text{(Contd.)} \\ (63) \end{array} \right\}$$

and

$$U_B^* = \frac{H P_r t_r (2t - t_r)}{2 \alpha^2 \sigma_y t_0^2}.$$

The time when the second-mode response ends is given by

$$t_{\epsilon}^* = t_r \left(1 + \frac{P_r}{48 \alpha^2 \sigma_y} - \frac{\sigma_y}{4 \sigma_y + 9 P_r} \right). \quad (64)$$

In the interval $t_{\epsilon}^* \leq t \leq t_f$, the motion is in the first deformation mode, so that Eqs. 51 apply (except for the equation for U_B^* , which is found from $dU_B^*/dt = V_B^*$). Applying the continuity conditions on the velocities and displacements at t_{ϵ}^* , the solution in this interval is

$$\left. \begin{aligned} N_B^* &= \frac{\sqrt{3}}{2} \alpha N_0 f_2, \\ V_P^* &= \frac{H}{t_0^2} \left[-f_2(t - t_{\epsilon}^*) + \frac{P_r t_r}{2 \alpha^2 \sigma_y} \right], \\ \Omega^* &= \frac{4 \alpha}{t_0^2} \left[-f_2(t - t_{\epsilon}^*) + \frac{P_r t_r}{2 \alpha^2 \sigma_y} \right], \\ V_B^* &= 2 V_P^*, \\ \theta^* &= \frac{2}{t_0^2} \left[-\alpha f_2(t - t_{\epsilon}^*)^2 + \frac{P_r t_r}{\alpha \sigma_y} (t - t_{\epsilon}^*) \right] + \theta|_{t_{\epsilon}^*}, \\ U_B^* &= \frac{H}{t_0^2} \left[-f_2(t - t_{\epsilon}^*)^2 + \frac{P_r t_r}{\alpha^2 \sigma_y} (t - t_{\epsilon}^*) \right] + U_B^*|_{t_{\epsilon}^*}, \end{aligned} \right\} \quad (65)$$

and

where f_2 is defined by Eq. 55.

The time t_f^* when the motion stops is found from $V_P^*(t_f^*) = 0$; it is given by

$$t_f^* = t_{\epsilon}^* + \frac{P_r t_r}{2 \alpha^2 \sigma_y f_2}. \quad (66)$$

The final plastic deformation at the midpoint of a side is then

$$U_B^*(t_f^*) = \frac{HP_r^2 t_r^2}{4\alpha^4 \sigma_y^2 t_0^2 t_2} + U_B^*|_{t_e^*}, \quad (67)$$

where U_B^* at t_e^* is computed from Eqs. 63. As $\alpha \rightarrow 0$,

$$U_B^*(t_f^*) \rightarrow H \left(\frac{P_r t_r}{P_y t_0} \right)^2 \left(\frac{5}{3} + \frac{P_y}{P_r} \right). \quad (68)$$

Figures 16 and 17 compare the numerical solutions to the exact equations and the approximate solutions given by Eqs. 61-67 for the problem specified by $\alpha = 0.056$, $P_r = \alpha \sigma_y = 0.056 \sigma_y$, and $t_r = 0.01 t_0$. Again, the curves are quite close.

Fig. 16. Comparison between Exact and Approximate Solutions for Example Problem with Initial Second-mode Response. ANL Neg. No. 113-4237.

Fig. 17. Additional Results for Comparison Problem of Fig. 16: V_{pt_0}/H and Ωt_0 as Functions of t/t_0 . ANL Neg. No. 113-4247.

The final deformation results for the deformation at midspan are computed from Eqs. 57 and 67 and plotted in Fig. 18, in the nondimensional form $U_B^*(t_f^*)t_0^2/Ht_r^2$, as a function of P_r/σ_y for $\alpha = 0.056$. Also shown are the corresponding results for the numerical solutions of the "exact" equations with $t_r = 0.01 t_0$.

Fig. 18

$U_B(t_f)t_0^2/Ht_f^2$ as a Function of P_r/σ_y for $\alpha = 0.056$. Comparison of exact and approximate solutions in small deformation range. ANL Neg. No. 113-4242 Rev. 2.

The results of Figs. 15-18 and other computations indicate that the approximate solution for rectangular pulses is a good match in the small-deformation range to the numerical solutions to the exact equations. Reference 1 contains procedures to determine, for a given α , the range of values of t_r for which the resulting deformation is small.

V. STATIC PLASTIC DEFORMATION

The response of the hexagon is essentially static if the internal pressure is very slowly applied. The acceleration terms in the equations of motion (Eqs. 15) are then negligible. From Eqs. 15 and the yield conditions of Eqs. 16 and 17, we then have

$$\left. \begin{aligned} 4M_0 \left[2 - \left(\frac{N_A}{N_0} \right)^2 - \left(\frac{N_B}{N_0} \right)^2 \right] + N_B L \sin \theta + N_A L \left(\sin \theta - \frac{\cos \theta}{\sqrt{3}} \right) &= 0, \\ P_s L - 2N_B \sin \theta + 2N_A \left(\frac{\cos \theta}{\sqrt{3}} - \sin \theta \right) &= 0, \\ \text{and} \\ N_A \left(\cos \theta + \frac{\sin \theta}{\sqrt{3}} \right) - N_B \cos \theta &= 0, \end{aligned} \right\} \quad (69)$$

where P_s is the static pressure. The solution of Eqs. 69 for N_A and N_B in terms of θ and P_s gives

$$\left. \begin{aligned} N_A &= \frac{\sqrt{3}}{2} P_s L \cos \theta, \\ \text{and} \\ N_B &= \frac{1}{2} P_s L (\sqrt{3} \cos \theta + \sin \theta), \end{aligned} \right\} \quad (70)$$

and the relationship between θ and P_s is

$$\begin{aligned} (8\alpha^2 \sigma_y^2 + 2\sigma_y P_s - P_s^2) \tan^2 \theta + 2\sqrt{3} P_s (2\sigma_y - P_s) \tan \theta + 8\alpha^2 \sigma_y^2 - \\ 2\sigma_y P_s - 6P_s^2 = 0. \end{aligned} \quad (71)$$

Note from Eqs. 10 and 71 that $P_s = P_y$ at $\theta = 0$, which agrees with the limit-load analysis in Sec. II. Equations 70 and 71 are valid for $N_B < N_0$ and $P_s \geq P_y$. At $N_B = N_0$, a different deformation mode is initiated. The solution to Eq. 71 is shown in Fig. 19 for various values of $\alpha = H/L$.

The plastic stretch in the half-side must be determined in order to compute the static corner and midspan deflections. From the flow rules given in Eqs. 16 and 17, we have that

$$\frac{d(\Delta_A + \Delta_B)}{d\theta} = \frac{N_A + N_B}{2\sigma_y}, \quad (72)$$

or, using Eqs. 70,

$$\frac{d(\Delta_A + \Delta_B)}{d\theta} = \frac{P_s L}{4\sigma_y} (2\sqrt{3} \cos \theta + \sin \theta). \quad (73)$$

Fig. 19

Angular Deformation θ as a Function of Static Load P_s/σ_y for Various Values of α . ANL Neg. No. 113-4243.

Since P_s is known from Eq. 71 or Fig. 19 as a function of θ , Eq. 73 is easily integrated numerically. The deflections U_A and U_B are then found from Eqs. 19; the results are shown in Figs. 20 and 21 for various values of α . We see that the static deflection at the corner may be either inward or outward, depending on the combination of α and P_s/P_y .

The substitution of Eqs. 70 into Eq. 21 gives

$$\frac{\partial^2 M}{\partial x^2} = -P_s, \quad (74)$$

so that the bending movement always has a relative maximum at $x = 0$. Consequently, the second deformation mode cannot occur for a static load.

Fig. 20. Corner Displacement U_A/H as a Function of Static Load P_s/P_y for Various Values of α . ANL Neg. No. 113-4234.

Fig. 21. Midside Displacement U_B/H as a Function of Static Load P_s/P_y for Various Values of α . ANL Neg. No. 113-4249 Rev. 1.

APPENDIX

DYPLHX Program

This appendix gives some detailed information on the computer program DYPLHX, which was used to calculate the results presented in this report and Ref. 1. It contains the nondimensionalization of the variables, the sets of nondimensional equations solved, the initial power-series solutions, and a listing of DYPLHX with its subroutines.

1. Nondimensional Variables

Dimensionless quantities were used in the numerical computations for convenience. They include the geometric parameter α , defined in Eq. 11; dimensionless membrane force and stretches μ_1 , μ_2 , γ_1 , and γ_2 , defined in Eqs. 45; and dimensionless time τ , pressure Ψ , and angular velocity ω , defined by

$$\tau = \frac{t}{t_0}, \quad \Psi = \frac{PL}{\sigma_y H}, \quad \omega = \Omega t_0, \quad (\text{A.1})$$

where t_0 is defined in Eq. 48. The velocities are nondimensionalized by a factor H/t_0 and the displacements by H ; e.g.,

$$v_p = \frac{V_P t_0}{H}, \quad u_B = \frac{U_B}{H}. \quad (\text{A.2})$$

The function G of Eq. 27 has the nondimensional counterpart g , defined by

$$g = \frac{GL}{\sigma_y H t_0}. \quad (\text{A.3})$$

2. Deformation-mode Equations in Nondimensional Variables

The governing equations for the first plastic-deformation mode are, in terms of nondimensional variables,

$$\left. \begin{aligned} \frac{d\omega}{d\tau} &= 24 \left[2\alpha(\mu_1^2 + \mu_2^2 - 1) + \mu_1 \left(\frac{\cos \theta}{\sqrt{3}} - 2 \sin \theta \right) + \mu_2 \frac{\cos \theta}{\sqrt{3}} \right], \\ \alpha \frac{dv_P}{d\tau} &= \Psi - 2\mu_1 \frac{\cos \theta}{\sqrt{3}} + 2\mu_2 \left(2 \sin \theta - \frac{\cos \theta}{\sqrt{3}} \right), \\ \alpha \frac{dv_T}{d\tau} &= 2\mu_1 \frac{\sin \theta}{\sqrt{3}} + 2\mu_2 \left(2 \cos \theta + \frac{\sin \theta}{\sqrt{3}} \right), \\ \frac{d\gamma_1}{d\tau} &= \alpha \mu_1 \omega, \end{aligned} \right\} \quad (\text{A.4})$$

(Contd.)

$$\left. \begin{aligned}
 \frac{d\gamma_2}{d\tau} &= \alpha\mu_2\omega, \\
 \left[2\alpha v_P + \left(\frac{1}{2} + \gamma_1 - \gamma_2 \right) \omega \right] \tan \theta &= \frac{d\gamma_1}{d\tau} - \frac{d\gamma_2}{d\tau} - 2\alpha v_T, \\
 \left[2\alpha v_P - \left(\frac{1}{2} + \gamma_1 + \gamma_2 \right) \omega \right] \tan \left(\frac{\pi}{6} - \theta \right) &= \frac{d\gamma_1}{d\tau} + \frac{d\gamma_2}{d\tau} + 2\alpha v_T,
 \end{aligned} \right\} \begin{array}{l} \text{(Contd.)} \\ \text{(A.4)} \end{array}$$

and

$$\omega = \frac{d\theta}{d\tau}.$$

Similarly, the governing equations for the second deformation mode become

$$\left. \begin{aligned}
 (1 - \epsilon)^3 \frac{d\omega}{d\tau} &= 24 \left[2\alpha(\mu_1^2 + \mu_2^2 - 1) + \mu_1(1 - \epsilon) \left(\frac{\cos \theta}{\sqrt{3}} - 2 \sin \theta \right) + \mu_2(1 - \epsilon) \frac{\cos \theta}{\sqrt{3}} \right], \\
 \alpha(1 - \epsilon) \frac{dv_P}{d\tau} &= (1 - \epsilon) \Psi - 2\mu_1 \frac{\cos \theta}{\sqrt{3}} + 2\mu_2 \left(2 \sin \theta - \frac{\cos \theta}{\sqrt{3}} \right), \\
 \alpha(1 - \epsilon) \frac{dv_T}{d\tau} &= 2\mu_1 \frac{\sin \theta}{\sqrt{3}} + 2\mu_2 \left(2 \cos \theta + \frac{\sin \theta}{\sqrt{3}} \right), \\
 \frac{d\gamma_1}{d\tau} &= \alpha\omega\mu_1, \\
 \frac{d\gamma_2}{d\tau} &= \alpha\omega\mu_2, \\
 \left\{ 2\alpha v_P + \left[\frac{1}{2}(1 - \epsilon) + \gamma_1 - \gamma_2 \right] \omega \right\} \sin \theta &= \left(\frac{d\gamma_1}{d\tau} - \frac{d\gamma_2}{d\tau} - 2\alpha v_T - \frac{1}{2} \frac{d\epsilon}{d\tau} \right) \cos \theta + \frac{d\epsilon}{d\tau}, \\
 \left\{ 2\alpha v_P - \left[\frac{1}{2}(1 - \epsilon) + \gamma_1 + \gamma_2 \right] \omega \right\} \tan \left(\frac{\pi}{6} - \theta \right) &= \frac{d\gamma_1}{d\tau} + \frac{d\gamma_2}{d\tau} + 2\alpha v_T - \frac{1}{2} \frac{d\epsilon}{d\tau}, \\
 2\alpha v_\epsilon &= \left\{ 2\alpha v_P + \left[\frac{1}{2}(1 - \epsilon) + \gamma_1 - \gamma_2 \right] \omega \right\} \cos \theta + \left[\frac{d\gamma_1}{d\tau} - \frac{d\gamma_2}{d\tau} - 2\alpha v_T - \frac{1}{2} \frac{d\epsilon}{d\tau} \right] \sin \theta, \\
 \text{and} \\
 \alpha v_\epsilon &= \int \Psi d\tau + g.
 \end{aligned} \right\} \text{(A.5)}$$

For $d\epsilon/d\tau > 0$, the hinge-band growth equation to be solved is

$$(1 - \epsilon) \left[\mu_1 \left(\frac{\cos \theta}{\sqrt{3}} - 3 \sin \theta \right) + \mu_2 \left(\frac{\cos \theta}{\sqrt{3}} + \sin \theta \right) \right] = 3\alpha(1 - \mu_1^2 - \mu_2^2). \quad (\text{A.6})$$

Preliminary calculations indicated that the terms in $d\gamma_2/d\tau$ and $d\epsilon/d\tau$ in the velocity compatibility equations in Eqs. A.4 and A.5 could be dropped with little effect on the results and improved numerical behavior of the solution.

3. Initial Power-series Solutions

Let each dependent variable be expanded in a power series at time τ_y when the yield load is first exceeded; e.g.,

$$\mu_1 = \mu_1(\tau_y) + (\tau - \tau_y) \mu_1' + \frac{1}{2}(\tau - \tau_y)^2 \mu_1'' + \dots, \quad (\text{A.7})$$

where

$$\mu_1' = \left. \frac{d\mu_1}{d\tau} \right|_{\tau_y}, \quad \mu_1'' = \left. \frac{d^2\mu_1}{d\tau^2} \right|_{\tau_y}, \quad \dots \quad (\text{A.8})$$

For a gradual transition through the yield load, the motion starts in the first deformation mode. The power-series coefficients found from Eqs. A.4 are

$$\left. \begin{aligned} \mu_2(\tau_y) &= \theta(\tau_y) = \theta' = \theta'' = v_P(\tau_y) = v_P' = v_T(\tau_y) = v_T' \\ &= \gamma_1(\tau_y) = \gamma_1' = \gamma_1'' = \gamma_2(\tau_y) = \gamma_2' = \gamma_2'' = \gamma_2''' = 0, \\ \mu_1 &= \frac{4\sqrt{3}\alpha}{1 + b_1}, \\ \mu_1' &= \frac{\sqrt{3}(5 - b_1)}{2b_2} \Psi', \\ \mu_2' &= \frac{6\alpha b_1 \mu_1}{b_2} \Psi', \\ \theta''' &= \frac{48b_1}{b_2} \Psi', \\ v_P'' &= \frac{12b_1^2}{\alpha b_2} \Psi', \end{aligned} \right\} \quad \begin{array}{l} (\text{A.9}) \\ (\text{Contd.}) \end{array}$$

$$v_T'' = \frac{4}{\alpha} \mu_2',$$

$$\gamma_1''' = 8\mu_2',$$

$$\mu_1'' = \frac{1}{b_2 \mu_1} \left\{ \frac{\sqrt{3}}{2} (5 - b_1) \mu_1 \Psi'' + [3b_2 - 192\alpha\mu_1(\sqrt{3}b_1 + \alpha\mu_1)][(\mu_1')^2 + (\mu_2')^2] - 6\sqrt{3}\mu_1'\Psi' \right\},$$

$$\mu_2'' = \frac{1}{b_2 \mu_1} \left\{ 6\alpha b_1 \mu_1^2 \Psi'' + [192\alpha^2 \mu_1^2 - 3b_2][(\mu_1')^2 + (\mu_2')^2] + 6\sqrt{3}\mu_1'\Psi' \right\},$$

$$\theta''' = \frac{48}{b_2} \left\{ b_1 \Psi''' + 32\alpha[(\mu_1')^2 + (\mu_2')^2] - 12\sqrt{3}\alpha\mu_1'\Psi' \right\},$$

$$v_P''' = \frac{12b_1}{b_2} \left\{ \frac{b_1}{\alpha} + 32[(\mu_1')^2 + (\mu_2')^2] \right\},$$

$$v_T''' = \frac{4}{\alpha} \mu_2'',$$

$$\gamma_1'''' = \frac{48\alpha}{b_2} \left\{ b_1 \mu_1 \Psi''' + 32\alpha\mu_1[(\mu_1')^2 + (\mu_2')^2] + 3\mu_1'\Psi' \right\},$$

and

$$\gamma_2'''' = \frac{24}{\mu_1} (\mu_2'')^2,$$

(Contd.)
(A.9)

where b_1 and b_2 are defined by

$$\left. \begin{aligned} b_1 &= \sqrt{1 + 48\alpha^2} \\ \text{and} \\ b_2 &= 18 + 624\alpha^2 - 2b_1 \end{aligned} \right\}. \quad (A.10)$$

Consider an initial instantaneous jump to a pressure P_i such that $P_y < P_i < P_h$. The response is then in the first plastic-deformation mode. Let Ψ_i be defined in terms of P_i by Eq. A.1. The initial values of μ_1 and μ_2 are found by iteration from (see Eqs. 39 and 45)

$$\left. \begin{aligned} \mu_1 &= \frac{b_4}{b_3 + \sqrt{b_3^2 + 36\alpha b_4}} \\ \text{and} \\ \mu_2 &= 3\alpha(\mu_1 - \mu_2) [2\alpha(\mu_1^2 + \mu_2^2 - 1) + (\mu_1 + \mu_2)/\sqrt{3}] \end{aligned} \right\}, \quad (\text{A.11})$$

where

$$\left. \begin{aligned} b_3 &= \frac{4}{\sqrt{3}} - 24\sqrt{3}\alpha^2 + 12\alpha\mu_2 \\ \text{and} \\ b_4 &= \Psi_1 + 12\alpha(1 - \mu_2^2) - \frac{8}{\sqrt{3}}\mu_2 - 48\sqrt{3}\alpha^2\mu_1(\mu_1^2 + \mu_2^2) \end{aligned} \right\}. \quad (\text{A.12})$$

The power-series coefficients are given by

$$\left. \begin{aligned} \theta(\tau_Y) &= \theta' = v_P(\tau_Y) = v_T(\tau_Y) = \gamma_1(\tau_Y) = \gamma_1' = \gamma_2(\tau_Y) = \gamma_2' = 0, \\ \theta'' &= \frac{b_5}{\alpha}, \\ v_P' &= \frac{b_5 b_6}{4\alpha^2}, \\ v_T' &= \frac{4\mu_2}{\alpha}, \\ \gamma_1'' &= b_5 \mu_1, \\ \gamma_2'' &= b_5 \mu_2, \\ \mu_1' &= \frac{\sqrt{3}\Psi'}{b_7} \left[1 + \frac{b_5}{4} - \alpha(\mu_1 - \mu_2)(\sqrt{3} + 12\alpha\mu_2) \right], \\ \mu_2' &= \frac{\sqrt{3}\Psi'}{b_7} \left[\frac{b_5}{4} + \sqrt{3}\alpha(\mu_1 - \mu_2) b_6 \right], \\ \theta''' &= \frac{\sqrt{3}\Psi'}{b_7} [24\alpha b_5(\mu_1 + \mu_2) + 4\sqrt{3}(b_5 + 2b_6)], \\ v_P'' &= \frac{\Psi'}{\alpha b_7} [3b_5^2 + 9b_5(1 + 8\alpha^2) + 6b_6^2], \end{aligned} \right\} \quad \begin{aligned} &(\text{A.13}) \\ &(\text{Contd.}) \end{aligned}$$

$$v_T'' = \frac{\sqrt{3}\Psi'}{b_7} [\theta'' + 4\sqrt{3}(\mu_1 - \mu_2) b_6],$$

$$\gamma_1''' = \frac{\sqrt{3}\Psi'}{b_7} [b_5^2 + 2b_5(1 + 12\alpha^2) - 2\sqrt{3}\alpha b_5(\mu_1 + \mu_2) + 8\sqrt{3}\alpha b_6\mu_1],$$

$$\gamma_2''' = \frac{\sqrt{3}\Psi'}{b_7} [b_5^2 + 24\alpha^2 b_5 - 2\sqrt{3}\alpha(\mu_1 + \mu_2) b_5 + 8\sqrt{3}\alpha b_6\mu_2],$$

$$\mu_1'' = b_8[\frac{3}{2}b_5 + 4\alpha(\mu_1 + \mu_2)(\sqrt{3} + 12\alpha\mu_2) + 8\sqrt{3}\alpha\mu_2 + \frac{20}{3}]$$

$$+ b_9[\frac{3}{2}b_5 - 4\alpha(\mu_1 - \mu_2)(\sqrt{3} + 12\alpha\mu_2) + 4],$$

$$\mu_2'' = b_9[\frac{3}{2}b_5 + 4\sqrt{3}\alpha b_6(\mu_1 - \mu_2)] - b_8[\frac{3}{2}b_5 + 4\sqrt{3}\alpha b_6(\mu_1 + \mu_2) + 2b_6 + \frac{2}{3}],$$

$$\theta''' = 96\alpha[(\mu_1')^2 + (\mu_2')^2 + \mu_1\mu_1'' + \mu_2\mu_2''] - 48\mu_1\theta'' + 8\sqrt{3}(\mu_1'' + \mu_2''),$$

$$v_P''' = \frac{1}{\alpha} \left[\Psi'' + 4\mu_2\theta'' - \frac{2}{\sqrt{3}}(\mu_1'' + \mu_2'') \right],$$

$$v_T''' = \frac{1}{\alpha} \left[4\mu_2'' + \frac{2}{\sqrt{3}}(\mu_1 + \mu_2)\theta'' \right],$$

$$\gamma_1'''' = \alpha(3\mu_1''\theta'' + 3\mu_1'\theta''' + \mu_1\theta''''),$$

and

$$\gamma_2'''' = \alpha(3\mu_2''\theta'' + 3\mu_2'\theta''' + \mu_2\theta''').$$

(Contd.)
(A.13)

In the above, b_5 , b_6 , b_7 , b_8 , and b_9 are defined by

$$b_5 = \frac{8\mu_2}{\mu_1 - \mu_2},$$

$$b_6 = 1 + 4\sqrt{3}\alpha\mu_1,$$

$$b_7 = 3b_5^2 + (10 + 72\alpha^2)b_5 + 24\alpha^2(\mu_1 - \mu_2)^2 + 6b_6^2 + 2,$$

$$b_8 = \frac{1}{b_{10}} \left\{ 3\alpha\theta'''(\mu_2' - \mu_1') - 96\alpha^2(\mu_1 - \mu_2)[(\mu_1')^2 + (\mu_2')^2] \right.$$

$$\left. + 48b_5\mu_1(\mu_1 - \mu_2) + \frac{4}{\sqrt{3}}(\mu_1 + \mu_2)\theta'' + 6b_5v_P' + \frac{3}{2}(\theta'')^2 \right\},$$

(A.14)
(Contd.)

and

$$b_9 = \frac{1}{b_{10}} \left\{ -3\alpha \theta''' (\mu_1' + \mu_2') - 16\alpha [6\alpha(\mu_1 + \mu_2) + \sqrt{3}] [(\mu_1')^2 + (\mu_2')^2] \right. \\ \left. + 48b_5\mu_1(\mu_1 + \mu_2) - \frac{4}{\sqrt{3}}(\mu_1 - \mu_2) \theta'' - 2\sqrt{3} b_5 v_T' \right. \\ \left. + \frac{2}{\sqrt{3}} \Psi'' - 6b_5 v_P' - 2\sqrt{3} b_5 (\mu_1 + \mu_2) \theta'' + 8\sqrt{3} \mu_1 \theta'' + \frac{3}{2} (\theta'')^2 \right\}, \quad \left. \begin{array}{l} \text{(Contd.)} \\ \text{(A.14)} \end{array} \right\}$$

where

$$b_{10} = 15b_5^2 + (40 + 288\alpha^2) b_5 + 64\alpha^2(\mu_1 - \mu_2)^2 + 16b_6^2 + \frac{16}{3}. \quad (\text{A.15})$$

If the initial instantaneous jump is to a pressure P_i , which exceeds P_h , the initial response is in the second plastic-deformation mode. Some of the power-series coefficients depend on whether dP/dt is positive or negative immediately after the jump. The power-series coefficients found from Eqs. A.5 and A.6 are (see also Eqs. 40-42)

$$\left. \begin{aligned} \theta(\tau_y) &= \theta' = v_P(\tau_y) = v_T(\tau_y) = \gamma_1(\tau_y) = \gamma_1' = \gamma_2(\tau_y) = \gamma_2' = 0, \\ \mu_1(\tau_y) &= \left[\frac{9a_1}{9a_1(1+a_1^2) + (1+a_1)^2} \right]^{1/2}, \\ \mu_2(\tau_y) &= a_1\mu_1, \\ \epsilon(\tau_y) &= .1 - \frac{1}{a_2}, \\ \theta'' &= \frac{8a_1a_2}{\alpha}, \\ v_P' &= \frac{1}{\alpha} \left(\Psi_i - \frac{2a_1}{\alpha} \right), \\ v_T' &= \frac{4a_2\mu_2}{\alpha}, \\ \gamma_1'' &= 2\alpha v_T', \\ \gamma_2'' &= a_1\gamma_1'', \end{aligned} \right\} \quad \begin{array}{l} \text{(A.16)} \\ \text{(Contd.)} \end{array}$$

$$\mu_1' = \frac{\Psi'(1+a_1)}{2\sqrt{3}a_2[(1+25a_1)a_5 + (1+13a_1)a_4 + 8a_1(1+a_1)]} \text{ if } \Psi' > 0,$$

$$\mu_1' = \frac{\sqrt{3}\Psi'}{4a_2a_7[1+7a_1(3+2a_1)]} \text{ if } \Psi' \leq 0,$$

$$\mu_2' = a_3\mu_1' \text{ if } \Psi' > 0,$$

$$\mu_2' = a_6\mu_1' \text{ if } \Psi' \leq 0,$$

$$\epsilon' = \frac{\alpha(3a_5+a_4)}{\sqrt{3}a_1} \mu_1' \text{ if } \Psi' > 0,$$

$$\epsilon' = \frac{(a_7-1)\mu_1'}{a_2\mu_1} \text{ if } \Psi' \leq 0,$$

$$\theta''' = 8a_2^2[12\alpha a_2\mu_1(\mu_1'+a_1\mu_2') + \sqrt{3}(\mu_1'+\mu_2')],$$

$$v_P'' = \frac{\Psi' - a_8}{\alpha},$$

$$v_T'' = \frac{4a_2}{\alpha}(a_1a_2\mu_1\epsilon' + \mu_2'),$$

$$\gamma_1''' = 2\alpha v_T'',$$

$$\gamma_2''' = \alpha a_1\mu_1\theta''' + 16a_1a_2\mu_2',$$

$$\mu_1'' = \frac{d_5d_4 - d_6d_2}{d_1d_4 - d_2d_3},$$

$$\mu_2'' = \frac{d_6d_1 - d_5d_3}{d_1d_4 - d_2d_3},$$

$$\begin{aligned} \epsilon'' = \frac{1}{a_1} & \left[6\alpha^2 a_2 \mu_1 (\mu_1'' + a_1 \mu_2'') + \frac{\alpha}{\sqrt{3}} (\mu_1'' + \mu_2'') - \frac{2}{\sqrt{3}} \alpha a_2 a_4 \mu_1' \epsilon' \right. \\ & \left. + 6\alpha^2 a_2 (1+a_3^2) (\mu_1')^2 \right] + 8a_2(a_1-3)\mu_1 \text{ if } \Psi' > 0, \end{aligned}$$

$$\begin{aligned} \epsilon'' = \frac{1}{a_1} & \left\{ 3\alpha^2 a_2 \mu_1 (\mu_1'' + a_1 \mu_2'') + \frac{\alpha}{2\sqrt{3}} (\mu_1'' + \mu_2'') - \frac{5}{8} \alpha a_8 \epsilon' + 3\alpha^2 a_2 [(\mu_1')^2 + (\mu_2')^2] \right\} \\ & + \frac{3}{2} a_2 (\epsilon')^2 - 4a_2 \mu_1 (3 - 4a_1 + 6a_1^2) \text{ if } \Psi' \leq 0, \end{aligned}$$

(Contd.)
(A.16)

(A.16)
(Contd.)

$$\theta'''' = 6a_2\epsilon'\theta''' + 8\sqrt{3}a_2^2(\mu_1'' + \mu_2'') + 16a_2^3\left\{6\alpha\mu_1(\mu_1'' + a_1\mu_2'') - \frac{3}{\alpha}a_1(\epsilon')^2\right. \quad \left. \begin{array}{l} \text{(Contd.)} \\ \text{(A.16)} \end{array} \right.$$

$$+ 6\alpha[(\mu_1')^2 + (\mu_2')^2] - \sqrt{3}\epsilon'(\mu_1' + \mu_2') - \frac{24}{\alpha}a_1\mu_1\},$$

$$v_P''' = \frac{1}{\alpha}\left[\Psi'' - 2a_2a_8\epsilon' - \frac{2}{\alpha}a_1a_2\epsilon'' - \frac{2}{\sqrt{3}}a_2(\mu_1'' + \mu_2'') + \frac{32}{\alpha}a_1^2a_2^2\mu_1\right],$$

$$v_T''' = 2a_2v_T''\epsilon' + \frac{4a_2}{\alpha}(a_1a_2\mu_1\epsilon'' + \mu_2'') + \frac{16}{\alpha^3}a_1^2a_2,$$

$$\gamma_1'''' = 2\alpha v_T''' + \frac{48}{\alpha}a_1a_2\Psi_i,$$

and

$$\gamma_2'''' = \alpha\mu_2\theta'''' + 3\alpha\mu_2'\theta''' + 24a_1a_2\mu_2''.$$

In the above, the quantities a_1 through a_8 are defined by

$$\left. \begin{aligned} a_1 &= \frac{2\alpha\Psi_i}{4 - \alpha\Psi_i + \sqrt{(4 - \alpha\Psi_i)^2 + 112\alpha\Psi_i}}, \\ a_2 &= \frac{\sqrt{3}a_1}{\alpha(1 + a_1)\mu_1}, \\ a_3 &= \frac{2a_1(2 + 12a_1 + a_1^2)}{1 - a_1^2 - 18a_1^3}, \\ a_4 &= 1 + a_3, \\ a_5 &= \frac{6a_1(1 + a_1a_3)}{1 + a_1}, \\ a_6 &= \frac{a_1(13 + 91a_1 + 6a_1^2)}{3 - a_1 - 4a_1^2 - 72a_1^3}, \\ a_7 &= \frac{a_6 - a_1}{2a_1(1 + a_1)}, \\ a_8 &= \frac{2}{\alpha}a_1a_2\epsilon' + \frac{2a_2}{\sqrt{3}}(\mu_1' + \mu_2'). \end{aligned} \right\} \quad \text{(A.17)}$$

and

If $\Psi' > 0$, the quantities d_1 through d_6 are defined by

$$\begin{aligned}
 d_1 &= 1 - 23a_1 - 18a_1^2, \\
 d_2 &= 13 + 7a_1, \\
 d_3 &= a_1(5 + 26a_1 + 3a_1^2), \\
 d_4 &= -1 + a_1^2 + 18a_1^3, \\
 d_5 &= \frac{3a_1\Psi''}{4\alpha a_2^2\mu_1} + 36a_1a_2(a_4 + a_5)\mu_1'\epsilon' - \frac{12a_1^2}{\alpha^2\mu_1}(\epsilon')^2 \\
 &\quad + \frac{54a_1}{\mu_1}(2a_5 + a_4)(\mu_1')^2 + \frac{3\sqrt{3}a_1(6a_5 + a_4)}{2\alpha\mu_1}\mu_1'\epsilon' \\
 &\quad - \frac{24\sqrt{3}a_1a_3}{\alpha\mu_1}\mu_1'\epsilon' - \frac{24a_1^2}{\alpha^2(1+a_1)}(11 + 101a_1 + 105a_1^2 + 3a_1^3), \\
 \text{and} \\
 d_6 &= \frac{\alpha}{a_1} \left[6\alpha a_2\mu_1(\mu_1'' + a_1\mu_2'') + \frac{1}{\sqrt{3}}(\mu_1'' + \mu_2'') - \frac{2a_2a_4}{\sqrt{3}}\mu_1'\epsilon' \right. \\
 &\quad \left. + 6\alpha a_2(1 + a_1^2)(\mu_1')^2 \right] + 8a_2(a_1 - 3)\mu_1.
 \end{aligned} \tag{A.18}$$

If $\Psi' \leq 0$, d_1 through d_6 are defined by

$$\begin{aligned}
 d_1 &= 1 + 32a_1 + 187a_1^2 + 12a_1^3, \\
 d_2 &= 1 + 14a_1 + 19a_1^2 + 150a_1^3, \\
 d_3 &= a_1(11 + 71a_1 + 6a_1^2), \\
 d_4 &= -2 + a_1 + 3a_1^2 + 54a_1^3, \\
 d_5 &= (1 + a_1)^2 \left\{ \frac{\Psi''}{2\sqrt{3}a_2} - \frac{\sqrt{3}a_8\epsilon'}{4} - a_2^2\mu_1(1 + 49a_1)(\epsilon')^2 - \frac{3\alpha}{2a_2}\mu_1'\theta''' \right. \\
 &\quad \left. - \frac{2\sqrt{3}\alpha a_2(1 + 25a_1)}{1 + a_1}[(\mu_1')^2 + (\mu_2')^2] \right\} + \frac{24a_1^2}{\alpha^2}(3 + 25a_1 - 6a_1^2 - 4a_1^3),
 \end{aligned} \tag{A.19}$$

(Contd.)

$$d_6 = (1 + a_1)^2 \left\{ -\frac{5}{4} \alpha a_2 a_8 \mu_1 \epsilon' - 21 a_1 a_2 \mu_1 (\epsilon')^2 - \frac{3\alpha}{4a_2} \mu_1' \theta''' + \alpha v_T'' \epsilon' \right. \\ \left. - 18\alpha^2 a_2^2 \mu_1 [(\mu_1')^2 + (\mu_2')^2] \right\} + \frac{8a_1^2}{\alpha^2} (7 + 77a_1 + 55a_1^2 - 18a_1^3). \quad \left. \vphantom{\frac{8a_1^2}{\alpha^2}} \right\} \quad \begin{matrix} \text{(Contd.)} \\ \text{(A.19)} \end{matrix}$$

4. Printout of DYPLHX Program

```

PROGRAM DYPLHX
DIMENSION F(12),DF(12),FMAX(12),RLV(12),FF(12)
COMMON/ECK/KEP
COMMON/PUL/PSIM,TO
COMMON/DIF/NOFNS,ALP,TBAND,TMAX
COMMON/CBF/JBF,BFL(500),XBL(500)
RT3=1.7320508
1 READ 100,PSIM,TO,ALP
IF(EOF,60)2,3
2 STOP
3 PRINT 101,PSIM,TO,ALP
C**** CALCULATE PY/PO
PSIY=8*ALP/(1+SQRTF(1+48*ALP**2))
IF(PSIM.GT.PSIY) GO TO 5
PRINT 102
GO TO 1
5 CONTINUE
CALL PSINV(PSIY,TY,TM,INSTJMP)
TMAX=TM
TAUY=TY
PRINT 103,PSIY,TAUY
PRINT 104
C**** INITIAL VALUES
NOFNS=0
KEP=-1
EPF=0.0
OMEGA=F(1)=0.0
VP=F(2)=0.0
VT=F(3)=0.0
AGM=F(4)=BGM=F(5)=ADEL=BDEL=0.0
THETA=F(6)=0.0
EPS=F(7)=0.0
TBAND=-1.0
JBF=0
AN=BN=RT3*PSIY/2
F(8)=AMU=AN
F(9)=BMU=0.0
AM=BM=PSIY/4/ALP
WO=WY=WR=0.0
PRINT 105,TAUY,PSIY,AN,BN,AM,BM,OMEGA,VP,VT,ADEL,BDEL,EPS,
1 THETA,WO,WY,WR,NOFNS
CALL PSISER(TY,PO,P1,P2,P3,P4)
IF(INSTJMP.EQ.0) GO TO 15
C**** INITIAL CONDITION AT INSTANTANEOUS JUMP
C=2*PO*ALP/((4-PO*ALP)+SQRTF((4-PO*ALP)**2+112*PO*ALP))
AMU=SQRTF(9*C/(9*C*(1+C**2)+(1+C)**2))
BMU=C*AMU
A=RT3*C/ALP/AMU/(1+C)
EPS=1-1/A
IF(EPS.GT.0.0) GO TO 11
EPS=0.0
AMUX=BMUX=0.0
DO 7 J=1,15
BBB=4/RT3-24*RT3*ALP**2+12*ALP*BMUX

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CCC=PO+12*ALP*(1-BMUX**2)-8*BMUX/RT3-48*RT3*ALP**2*AMUX*
1  (AMUX**2+BMUX**2)
AMU=CCC/(BBB+SQRTF(BBB**2+36*ALP*CCC))
BMU=3*ALP*(AMU-BMUX)*(2*ALP*(AMU**2+BMUX**2-1)+(AMU+BMUX)/RT3)
IF(ABSF(1-AMUX/AMU).LE.0.00001.AND.ABSF(1-BMUX/BMU)
1  .LE.0.00001) GO TO 11
AMUX=AMU
7  BMUX=BMU
PRINT 110
GO TO 1
11 AN=AMU-BMU
BN=AMU+BMU
AM=1-AN**2
BM=1-BN**2
PRINT 105,TAUY,PO,AN,BN,AM,BM,OMEGA,VP,VT,ADEL,BDEL,EPS,
1  THETA,WO,WY,WR,NOFNS
IF(EPS.GT.0.0) GO TO 18
GO TO 17
C**** INITIAL SERIES SOLUTION
C NO INITIAL JUMP
15 TH1=AGM1=BGM1=TH2=VP1=VT1=AGM2=BGM2=EP1=EP2=0.0
CSX=1+48*ALP**2
CSRT=SQRTF(CSX)
BCS=5-2*CSRT+13*CSX
AMU1=RT3*P1*(5-CSRT)/2/BCS
BMU1=6*ALP*P1*CSRT*AMU/BCS
TH3=48*CSRT*P1/BCS
VP2=12*CSX*P1/ALP/BCS
VT2=4*BMU1/ALP
AGM3=8*BMU1
BGM3=0.0
SQ=AMU1**2+BMU1**2
AMU2=RT3*(5-CSRT)*P2/2/BCS+(3/AMU-192*ALP*(RT3*CSRT+ALP*AMU)/BCS)
1  *SQ-6*RT3*P1*AMU1/BCS/AMU
BMU2=6*ALP*AMU*CSRT*P2/BCS+(192*ALP**2*AMU/BCS-3/AMU)*SQ
1  +6*RT3*AMU1*P1/BCS/AMU
TH4=48*CSRT*P2/BCS+1536*ALP*SQ/BCS-576*RT3*ALP*AMU1*P1/BCS
VP3=12*CSX*P2/ALP/BCS+384*CSRT*SQ/BCS
VT3=4*BMU2/ALP
AGM4=48*ALP*AMU*CSRT*P2/BCS+1536*ALP**2*AMU*SQ/BCS+144*ALP*AMU1
1  *P1/BCS
BGM4=24*BMU1**2/AMU
GO TO 20
C INITIAL JUMP, NO HINGE BAND
17 TH1=AGM1=BGM1=EP1=EP2=0.0
CC1=1+4*RT3*ALP*AMU
QQ=8*BMU/AN
TH2=QQ/ALP
VP1=2*CC1*BMU/ALP**2/AN
VT1=4*BMU/ALP
AGM2=AMU*QQ
BGM2=BMU*QQ
GG1=3*QQ**2+10*QQ+72*ALP**2*QQ+24*ALP**2*AN**2+6*CC1**2+2
AMU1=(1+QQ/4-RT3*ALP*AN-12*ALP**2*AN*BMU)*P1*RT3/GG1
BMU1=(QQ/4+RT3*ALP*AN*CC1)*P1*RT3/GG1
TH3=(24*ALP*BN*QQ+4*RT3*(QQ+2*CC1))*P1*RT3/GG1
VP2=(3*QQ**2+9*QQ*(1+8*ALP**2)+6*CC1**2)*P1/ALP/GG1
VT2=(TH2+4*RT3*AN*CC1)*P1*RT3/GG1
AGM3=(QQ**2+QQ*(2+24*ALP**2)+2*RT3*ALP*(4*CC1*AMU-QQ*BN))
1  *P1*RT3/GG1
BGM3=(QQ**2+24*ALP**2*QQ+2*RT3*ALP*(4*CC1*BMU-QQ*BN))*P1*RT3/GG1
GG2=15*QQ**2+QQ*(40+288*ALP**2)+64*ALP**2*AN**2+16*CC1**2+16/3
RR1=-3*ALP*TH3*(AMU1-BMU1)-96*ALP**2*AN*(AMU1**2+BMU1**2)
1  +48*AN*AMU*QQ+4*BN*TH2/RT3+6*VP1*QQ+3*TH2**2/2
RR2=-3*ALP*TH3*(AMU1+BMU1)-48*ALP*(2*ALP*BN+1/RT3)*(AMU1**2+
1  BMU1**2)+6*QQ*(8*BN*AMU-VP1)+4*TH2*(2*BMU-BN+6*AMU)/RT3
2  -6*QQ*VT1/RT3+2*P2/RT3-6*TH2*BN*QQ/RT3+3*TH2**2/2
RDF=-6*ALP*BMU1*TH3-16*ALP*(12*ALP*BMU+RT3)*(AMU1**2+BMU1**2)

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1      +2*P2/RT3+2*RT3*TH2*(4*AN-BN*QQ)+96*AMU*BMU*QQ-8*AMU*TH2/RT3
2      -3*CC1*TH2**2
AMU2=(RR1*(3*QQ/2+48*ALP**2*BN*BMU+4*RT3*ALP*(2*BMU+BN)+20/3)
1      +RR2*(3*QQ/2-48*ALP**2*AN*BMU-4*RT3*ALP*AN+4))/GG2
BMU2=(RDF*3*QQ/2-RR1*(4*RT3*ALP*BN*CC1+2*CC1+2/3)
1      +RR2*4*RT3*ALP*AN*CC1)/GG2
TH4=96*ALP*(AMU1**2+8*BMU1**2+AMU*AMU2+BMU*BMU2)
1      -48*AMU*TH2+8*RT3*(AMU2+BMU2)
VP3=(P2+4*BMU*TH2-2*(AMU2+BMU2)/RT3)/ALP
VT3=(4*BMU2+2*BN*TH2/RT3)/ALP
AGM4=ALP*(3*AMU2*TH2+3*AMU1*TH3+AMU*TH4)
BGM4=ALP*(3*BMU2*TH2+3*BMU1*TH3+BMU*TH4)
GO TO 20
C      INITIAL JUMP TO HINGE BAND
18 TH1=AGM1=BGM1=0.0
KEP=+1
JBF=2
XBL(1)=BFL(1)=BFL(2)=0.0
XBL(2)=EPS
TH2=8*C*A/ALP
VP1=(P0-2*C/ALP)/ALP
VT1=4*A*BMU/ALP
AGM2=2*ALP*VT1
BGM2=C*AGM2
IF(P1.LE.0.0) GO TO 19
C1=2*C*(2+12*C+C**2)/(1-C**2-18*C**3)
D1=1+C1
D2=6*C*(1+C*C1)/(1+C)
AMU1=P1*(1+C)/2/RT3/A/((1+25*C)*D2+(1+13*C)*D1+8*C*(1+C))
BMU1=C1*AMU1
EP1=ALP*AMU1*(3*D2+D1)/RT3/C
GO TO 21
19 C2=C*(13+91*C+6*C**2)/(3-C-4*C**2-72*C**3)
D4=(C2-C)/2/C/(1+C)
AMU1=P1*RT3/4/A/D4/(1+7*C*(3+2*C))
BMU1=C2*AMU1
EP1=AMU1*(D4-1)/AMU/A
21 TH3=96*ALP*AMU*A**3*(AMU1+C*BMU1)+8*RT3*A**2*(AMU1+BMU1)
Q=2*A*(C*EP1/ALP+(AMU1+BMU1)/RT3)
VP2=(P1-Q)/ALP
VT2=(C*A*AMU*EP1+BMU1)*4*A/ALP
AGM3=2*ALP*VT2
BGM3=ALP*BMU*TH3+16*C*A*BMU1
IF(P1.LE.0.0) GO TO 22
B1=1-23*C-18*C**2
B2=13+7*C
B3=C*(5+26*C+3*C**2)
B4=-1+C**2+18*C**3
DET=B1*B4-B2*B3
R1=3*C*P2/4/A**2/ALP/AMU-12*(C*EP1/ALP)**2/AMU+54*C*AMU1**2
1      *(2*D2+D1)/AMU+AMU1*EP1*(36*C*A*(D2+D1)+3*RT3*C*(6*D2+D1)
2      /2/ALP/AMU-24*RT3*C*C1/ALP/AMU)-(11+101*C+105*C**2+3*C**3)
3      *24*C**2/ALP**2/(1+C)
R2=((1+C)*C1-3*C*(2*D2+D1))*2*RT3*C*AMU1*EP1/ALP/AMU
1      -((1+C)*(2*D2+D1)+2*C*(1+C1**2))*9*C*AMU1**2/AMU
2      +(7+68*C+49*C**2)*4*C**2/ALP**2
AMU2=(R1*B4-R2*B2)/DET
BMU2=(R2*B1-R1*B3)/DET
EP2=6*ALP**2*A*AMU*(AMU2+C*BMU2)/C+ALP*(AMU2+BMU2)/RT3/C
1      -2*ALP*A*D1*AMU1*EP1/RT3/C+6*ALP**2*A*(1+C1**2)*AMU1**2/C
2      +8*A*(C-3)*AMU
GO TO 23
22 B1=1+32*C+187*C**2+12*C**3
B2=1+14*C+19*C**2+150*C**3
B3=C*(11+71*C+6*C**2)
B4=-2+C+3*C**2+54*C**3
DET=B1*B4-B2*B3
R1=(1+C)**2*(P2/2/RT3/A-RT3*Q*EP1/4-A**2*AMU*EP1**2*(1+49*C)

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1      -3*ALP*AMU1*TH3/2/A-2*RT3*ALP*A*(AMU1**2+BMU1**2)*(1+25*C)
2      /(1+C))+24*C**2*(3+25*C-6*C**2-4*C**3)/ALP**2
R2=(1+C)**2*(-5*ALP*AMU*A*EP1*Q/4-21*AMU*A*C*EP1**2
1      -3*ALP*AMU1*TH3/4/A+ALP*VT2*EP1-18*ALP**2*AMU*A**2
2      *(AMU1**2+BMU1**2))+8*C**2*(7+77*C+55*C**2-18*C**3)/ALP**2
AMU2=(R1*B4-R2*B2)/DET
BMU2=(R2*B1-R1*B3)/DET
EP2=3*ALP**2*AMU*A*(AMU2+C*BMU2)/C+ALP*(AMU2+BMU2)/2/RT3/C
1      -5*ALP*Q*EP1/8/C+3*A*EP1**2/2+3*ALP**2*A*(AMU1**2+BMU1**2)/C
2      +4*A*AMU*(-3+4*C-6*C**2)
23 TH4=6*A*EP1*TH3+8*RT3*A**2*(AMU2+BMU2)+A**3*(-48*C*EP1**2/ALP
1      +96*ALP*AMU*(AMU2+C*BMU2)+96*ALP*(AMU1**2+BMU1**2)
2      -16*RT3*EP1*(AMU1+BMU1)-384*C*AMU/ALP)
VP3=(P2-2*A*Q*EP1-2*C*A*EP2/ALP-2*A*(AMU2+BMU2)/RT3
1      +32*(C*A)**2*AMU/ALP)/ALP
VT3=2*A*VT2*EP1+(4*C*A*AMU*EP2+4*BMU2+16*C**2/ALP**2)*A/ALP
AGM4=2*ALP*VT3+48*C*A*P0/ALP
BGM4=ALP*BMU*TH4+3*ALP*BMU1*TH3+24*C*A*BMU2
GO TO 20
20 OM1=TH2
   OM2=TH3
   OM3=TH4
C  EVALUATE SERIES SOLUTION
   DT=0.02*TO
   IF(DT.GT.0.002) DT=0.002
   DT2=DT**2/2
   DT3=DT**3/6
   DT4=DT**4/24
   TT=T+TY+DT
   OMEGA=F(1)=OMEGA+OM1*DT+OM2*DT2+OM3*DT3
   VP=F(2)=VP+VP1*DT+VP2*DT2+VP3*DT3
   VT=F(3)=VT+VT1*DT+VT2*DT2+VT3*DT3
   AGM=F(4)=AGM+AGM1*DT+AGM2*DT2+AGM3*DT3+AGM4*DT4
   BGM=F(5)=BGM+BGM1*DT+BGM2*DT2+BGM3*DT3+BGM4*DT4
   ADEL=AGM-BGM
   BDEL=AGM+BGM
   THETA=F(6)=THETA+TH1*DT+TH2*DT2+TH3*DT3+TH4*DT4
   EPF=EPS
   EPS=EPS+EP1*DT+EP2*DT2
   AMU=AMU+AMU1*DT+AMU2*DT2
   BMU=BMU+BMU1*DT+BMU2*DT2
   AN=AMU-BMU
   BN=AMU+BMU
   AM=1-AN**2
   BM=1-BN**2
   IF(EPS.GT.0.0) GO TO 26
   WY=(RT3/2/ALP)*((ADEL+BDEL)*(1-THETA**2/2+THETA**4/24)
1      -THETA**2/2+THETA**4/24-THETA**6/720)
   WO=WY+(1+ADEL+BDEL)*(THETA-THETA**3/6+THETA**5/120)/2/ALP
   GO TO 27
26 WQ1=WY1=0.0
   WY2=VP1-TH2/4/A/ALP
   WY3=VP2+(EP1*TH2/2-TH3/4/A)/ALP
   WY4=VP3-RT3*WY2*TH2+3*TH2*(EP2/4-(AGM2+BGM2)/2)/ALP
1      +(3*EP1*TH3-TH4/A)/4/ALP
   WQ2=P0/ALP
   WQ3=P1/ALP
   WQ4=P2/ALP
   WY=WY+WY1*DT+WY2*DT2+WY3*DT3+WY4*DT4
   WO=WO+WQ1*DT+WQ2*DT2+WQ3*DT3+WQ4*DT4
27 WR=WO-WY
   CALL PSISER(T,PSI,P1,P2,P3,P4)
   PRINT 105,T,PSI,AN,BN,AM,BM,OMEGA,VP,VT,ADEL,BDEL,EPS,
1      THETA,WQ,WY,WR,NOFNS
   IF(EPS.LE.EPF) GO TO 28
   JBF=3
   XBL(3)=EPF=EPS
   CALL PULSE(TAU,T,PIO,PI1)

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      BFL(3)=(ALP*VP+(1-EPS+2*AGM-2*BGM)*OMEGA/4)/(1-THETA**2/2
1      +THETA**4/24)-PIO
28 F(7)=WY
   F(8)=WO
   F(9)=EPS
C**** INITIALIZATIONS FOR DIFSUB ROUTINE
      NEQ=9
      MORD=6
      ERV=0.001
      METH=0
      KKSTP=+1
      DTMIN=0.000001*TO
      EERR=0.00001
      NOFNS1=NDN2=0
      DO 30 I=1,NEQ
30  FMAX(I)=FF(I)=F(I)
      KDTO=KDTM=+1
      DT=0.01*TO
31  IF(T.GE.TM) KDTM=-1
      IF(T.GE.TO) KDTO=-1
      IF(T+DT.GT.TM.AND.KDTM.GT.0) 32,33
32  DT=TM-T
33  IF(T+DT.GT.TO.AND.KDTO.GT.0) 34,35
34  DT=TO-T
35  CONTINUE
C**** USE DIFSUB TO SOLVE DIFFERENTIAL EQUATIONS
      CALL DIFSUB(NEQ,T,F,DF,DT,DTMIN,EERR,MORD,METH,FMAX,RLV,KKSTP)
      IF(KKSTP) 40,39
39  PRINT 106,T
      GO TO 1
40  CALL PSISER(T,PSI,PSI1,PSI2,PSI3,PSI4)
      NDN1=NOFNS-NOFNS1
      NOFNS1=NOFNS
      NDN2=NDN1
      OMEGA=F(1)
      VP=F(2)
      VT=F(3)
      AGM=F(4)
      BGM=F(5)
      ADEL=F(4)-F(5)
      BDEL=F(4)+F(5)
      THETA=F(6)
      WY=F(7)
      WO=F(8)
      EPS=F(9)
      CTH=COSE(THETA)
      STH=SINF(THETA)
      CTHV=(THETA**2/2)*{(1-THETA**2/12+THETA**4/360)
      IF(ABSF(THETA).GT.0.2) CTHV=1-CTH
      SPF=CTH+STH/RT3
      SPFP=CTH/RT3-STH
      SF1=SPFP/SPF+STH/CTH
      SF2=SPFP/SPF-STH/CTH
      SF3=4/3/SPF**2+1/CTH**2
      SF4=4/3/SPF**2-1/CTH**2
      H1=CTH/RT3
      H2=SPFP-STH
      H3=STH/RT3
      H4=SPF+CTH
      CEP=1-EPS
      AMU=((ALP*VP-BGM*OMEGA/2)*SF1-(CEP+2*AGM)*OMEGA*SF2/4)/ALP/OMEGA
      YBMU=4*H4+2*H2*SF2+6*H1*SF1+ALP*OMEGA**2*CEP*SF2/2
1      +12*(AGM*SF1+BGM*SF2)*H1/CEP
      CBMU=4*H3+2*H1*SF2+6*H2*SF1+12*(AGM*SF1+BGM*SF2)*H2/CEP
1      +ALP*OMEGA**2*CEP*SF1/2
      XBMU=PSI*CEP*SF2+12*ALP*{(CEP+2*AGM)*SF1+2*BGM*SF2}*{(1-AMU**2)
1      /CEP**2-AMU*CBMU-ALP*VP*CEP*OMEGA*SF3
2      +CEP*OMEGA**2*{(CEP+2*AGM)*SF4+2*BGM*SF3)/4

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BMU=XBMU/YBMU
WR=WO-WY
AN=AMU-BMU
BN=AMU+BMU
AM=1-AN**2
BM=1-BN**2
EPSC=1-3*ALP*(1-AMU**2-BMU**2)/(AMU*(CTH/RT3-3*STH)+BMU
1  *(CTH/RT3+STH))
PRINT 105,T,PSI,AN,BN,AM,BM,OMEGA,VP,VT,ADEL,BDEL,EPS,
1  THETA,WO,WY,WR,NOFNS
IF(KEP.LT.0.0.AND.EPSC.GT.0.0) GO TO 55
IF(EPS.GT.EPF) 41,43
41 CALL PULSE(TAU,T,PIO,PI1)
ABFL=(ALP*VP+(1-EPS+2*AGM-2*BGM)*OMEGA/4)/CTH-PIO
IF(EPF.EQ.0.0) GO TO 42
JBF=JBF+1
XBL(JBF)=EPS
BFL(JBF)=ABFL
GO TO 43
42 JBF=2
XBL(2)=EPS
BFL(2)=ABFL
XBL(1)=0.0
CALL PULSE(TAU,TT,PI00,PI10)
BFL(1)=(ALP*FF(2)+(1+2*FF(4)-2*FF(5))*FF(1)/4)
1  /(1-FF(6)**2/2+FF(6)**4/24)-PI00
43 EPF=EPS
IF(EPS.GE.0.0) GO TO 44
EPS=EPF=F(9)=0.0
44 CONTINUE
IF(ABSF(OMEGA/FMAX(1)).LE.ERV.AND.ABSF(VP/FMAX(2)).LE.ERV)GO TO 60
IF(OMEGA.LT.0.0) GO TO 50
47 TT=T
DO 48 I=1,NEQ
48 FF(I)=F(I)
GO TO 31
50 DT=(T-TT)*FF(1)/(FF(1)-OMEGA)
T=TT
DO 51 I=1,NEQ
51 F(I)=FF(I)
GO TO 35
55 IF(T.GT.TMAX) GO TO 43
KEP=+1
EPS=F(9)=EPSC
GO TO 35
C**** FINAL OUTPUT CALCULATIONS
60 TF=T
CALL PULSE(TY,TF,PSIINO,PSIIN1)
TMEAN=PSIIN1/PSIINO
PSIE=PSIINO/2/TMEAN
PAV=PSIINO/(TF-TY)
CORR=(TF-TY)*PSIY/PSIINO
PSIMY=PSIM/PSIY
PSIEY=PSIE/PSIY
WOFII=WO/PSIINO**2
WRFII=WR/PSIINO**2
WYFII=WY/PSIINO**2
PRINT 109,PSIINO,TMEAN,PSIE,PAV,PSIEY,PSIMY,CORR,WOFII,WYFII,WRFII
GO TO 1
109 FORMAT(1H0,8X*I =*E15.7//8X*TAUMEAN =*E15.7//8X*PE/PO =*E15.7//
1  8X*PAVG/PO =*E15.7//8X*PE/PY =*E15.7//8X*PMAX/PY =*E15.7//
2  8X*(TF-TY)(PSIY)/I =*E15.7//8X*WOF/I/I =*E15.7//
3  8X*WYF/I/I =*E15.7//8X*WRF/I/I =*E15.7)
100 FORMAT(3E15.7)
101 FORMAT(1H1,17X*PROGRAM DYPLHX//
1  8X*INPUT PARAMETERS//13X*PMAX/PO =*E15.7//
2  13X*TAUO =*E15.7//13X*ALPHA =*E15.7)
102 FORMAT(1H0,18X*NO MOTION OCCURS SINCE PMAX IS LESS THAN PY*)

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103 FORMAT(1H0,12X*PY/PO ==E15.7//13X*TAUY ==E15.7)
104 FORMAT(1H0/9X*TAU*13X*P/PO*13X*N1/NO*12X*N2/NO*12X*M1/MO*
1      12X*M2/MO*/16X*OMEGA*13X*VP*15X*VT*14X*DELT1*12X*DELT2*
2      13X*EPS*/16X*THETA*13X*WO*15X*WY*14X*WO-WY*9X*NOFNS*)
105 FORMAT(1H0/6E17.7/8X,6E17.7/8X,4E17.7,I10)
106 FORMAT(1H0,7X*NO CONVERGENCE IN DIFSUB, T ==E15.7)
110 FORMAT(1H0,7X*NO CONVERGENCE FOR N1 AND N2 AT JUMP*)
END
SUBROUTINE DIFFUN(T,F,DF)
COMMON/ECK/KEP
COMMON/DIF/NOFNS,ALP,TBAND,TMAX
DIMENSION F(12),DF(12)
NOFNS=NOFNS+1
RT3=1.7320508
CALL PSISER(T,PSI,P1,P2,P3,P4)
OMEGA=F(1)
VP=F(2)
VT=F(3)
AGM=F(4)
BGM=F(5)
THETA=F(6)
WY=F(7)
WO=F(8)
EPS=F(9)
IF(THETA.GT.0.5) THETA=0.5
IF(THETA.LT.0.0) THETA=0.0
STH=SINF(THETA)
CTH=COSE(THETA)
CTHV=(THETA**2/2)*(1-THETA**2/12+THETA**4/360)
IF(ABSF(THETA).GT.0.2) CTHV=1-CTH
SPF=CTH+STH/RT3
SPFP=CTH/RT3-STH
SF1=SPFP/SPF+STH/CTH
SF2=SPFP/SPF-STH/CTH
SF3=4/3/SPF**2+1/CTH**2
SF4=4/3/SPF**2-1/CTH**2
SFP3=-8*SPFP/3/SPF**3+2*STH/CTH**3
SFP4=-8*SPFP/3/SPF**3-2*STH/CTH**3
H1=CTH/RT3
H2=SPFP-STH
H3=STH/RT3
H4=SPF+CTH
CEP=1-EPS
AMU=((ALP*VP-BGM*OMEGA/2)*SF1-(CEP+2*AGM)*OMEGA*SF2/4)/ALP/OMEGA
YBMU=4*H4+2*H2*SF2+6*H1*SF1+ALP*OMEGA**2*CEP*SF2/2
1      +12*(AGM*SF1+BGM*SF2)*H1/CEP
CBMU=4*H3+2*H1*SF2+6*H2*SF1+12*(AGM*SF1+BGM*SF2)*H2/CEP
1      +ALP*OMEGA**2*CEP*SF1/2
XBMU=PSI*CEP*SF2+12*ALP*((CEP+2*AGM)*SF1+2*BGM*SF2)*(1-AMU**2)
1      /CEP**2-AMU*CBMU-ALP*VP*CEP*OMEGA*SF3
2      +CEP*OMEGA**2*((CEP+2*AGM)*SF4+2*BGM*SF3)/4
BMU=XBMU/YBMU
VDIF=0.0
IF(EPS.GT.0.0) 10,20
10 X=1-CEP
CALL BFUN(X,BF,DBF)
XO=0.0
CALL BFUN(XO,BFO,DBFO)
VDIF=(BFO-BF)/ALP
20 VY=(VP-(CEP+2*AGM+2*BGM)*OMEGA/4/ALP)/SPF
VO=(VP+(CEP+2*AGM-2*BGM)*OMEGA/4/ALP)/CTH+VDIF
DF(1)=24*(2*ALP*(AMU**2+BMU**2-1)+CEP*(AMU*(SPFP-STH)
1      +BMU*CTH/RT3))/CEP**3
DF(2)=(PSI-2*(AMU*CTH/RT3+BMU*(SPFP-STH))/CEP)/ALP
DF(3)=2*(AMU*STH/RT3+BMU*(SPF+CTH))/ALP/CEP
DF(4)=ALP*OMEGA*AMU
DF(5)=ALP*OMEGA*BMU
DF(6)=OMEGA

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      DF(7)=VY
      DF(8)=VO
      IF(KEP.GT.0.0) GO TO 25
      DF(9)=0.0
      RETURN
25 IF(T.GT.TMAX) GO TO 30
      A1=-SF2/4/ALP
      A2=-DF(1)*(ALP*AMU+BGM*SF1/2+(CEP+2*AGM)*SF2/4)/ALP/OMEGA
1      +DF(2)*SF1/OMEGA-VP*SF4+OMEGA*((CEP+2*AGM)*SF3/2-ALP*BMU*SF1
2      +BGM*SF4-ALP*AMU*SF2)/2/ALP
      A3=YBMU
      A4=-CBMU-24*ALP*AMU*((CEP+2*AGM)*SF1+2*BGM*SF2)/CEP**2
      A5=OMEGA**2*(CEP*SF4/4+((CEP+2*AGM)*SF4+2*BGM*SF3)/4
1      -ALP*BMU*SF2/2-ALP*AMU*SF1/2)+PSI*SF2-ALP*VP*OMEGA*SF3
2      +12*(AGM*SF1+BGM*SF2)*(BMU*H1+AMU*H2)/CEP**2
3      +12*ALP*(1-AMU**2)*(CEP*SF1-2*((CEP+2*AGM)*SF1+2*BGM*SF2))
4      /CEP**3
      A6=CEP*OMEGA*DF(1)*((CEP+2*AGM)*SF4/2+BGM*SF3-ALP*BMU*SF2
1      -ALP*AMU*SF1)+P1*CEP*SF2-PSI*CEP*OMEGA*SF3
2      -ALP*CEP*(DF(2)*OMEGA*SF3+VP*DF(1)*SF3+VP*OMEGA**2*SFP3)
3      +12*OMEGA*(AGM*SF1+BGM*SF2)*(AMU*H4+BMU*H3)/CEP
4      +12*OMEGA*(AGM*SF4+BGM*SF3-ALP*AMU*SF1-ALP*BMU*SF2)
5      *(AMU*H2+BMU*H1)/CEP
6      +OMEGA*BMU*(-4*H2+2*H4*SF2+2*H2*SF3+6*H3*SF1+6*H1*SF4)
7      +OMEGA*AMU*(-4*H1+2*H3*SF2+2*H1*SF3+6*H4*SF1+6*H2*SF4)
8      +24*ALP*OMEGA*(1-AMU**2)*(ALP*AMU*SF1+ALP*BMU*SF2
9      -(CEP+2*AGM)*SF4/2-BGM*SF3)/CEP**2
A      +CEP*OMEGA**3*(ALP*AMU*SF4+ALP*BMU*SF3+(CEP+2*AGM)*SFP4/4
B      +BGM*SFP3/2)
      CP1=12*ALP*AMU+CEP*(3*H2-H1)
      CP2=12*ALP*BMU+CEP*(3*H1-H2)
      DF(9)=(A2*CP1+(A4*A2+A6)*CP2/A3+CEP*OMEGA*(AMU*(H3-3*H4)
1      +BMU*(H4-3*H3)))/(A1*CP1+(A4*A1+A5)*CP2/A3
2      +AMU*(3*H2-H1)+BMU*(3*H1-H2))
      RETURN
30 IF(EPS.GT.0.0) GO TO 35
      DF(9)=0.0
      RETURN
35 BB=4*DBF*CTH/OMEGA
      B1=1+BB
      DF(9)=(4*PSI*CTHV+4*ALP*VP*OMEGA*STH/CTH+CEP*OMEGA**2*STH/CTH
1      -8*(AMU*H1+BMU*H2)/CEP+48*ALP*(AMU**2+BMU**2-1)/CEP**2
2      +24*(AMU*H2+BMU*H1)/CEP)/OMEGA/B1
      IF(DF(9).GT.0.0) DF(9)=0.0
      END
      SUBROUTINE BFUN(X,BF,DBF)
      COMMON/CBF/JBF,BFL(500),XBL(500)
      IF(JBF.LT.2) GO TO 5
      IF(X.LT.XBL(1)) X=XBL(1)
      IF(X.GT.XBL(JBF)) X=XBL(JBF)
      DO 1 J=2,JBF
      IF(XBL(J-1).EQ.XBL(J)) GO TO 1
      IF(X.GE.XBL(J-1).AND.X.LE.XBL(J)) GO TO 2
1      CONTINUE
      STOP
2      BF=(BFL(J)*(X-XBL(J-1))+BFL(J-1)*(XBL(J)-X))/
1      (XBL(J)-XBL(J-1))
      DBF=(BFL(J)-BFL(J-1))/(XBL(J)-XBL(J-1))
      RETURN
5      BF=DBF=0.0
      END
      SUBROUTINE DIFSUB (N,X,Y,DY,H,HMIN,EPS,JM,MF,S,R,KK)
      NANCY CLARK, ANL, OCTOBER 1965
      DIFSUB PERFORMS ONE INTEGRATION STEP WITH STEP SIZE H .LE. HOD2500003
      FOR A SYSTEM OF N (NMAX NOW 20) FIRST ORDER ORDINARY DIFFERENTIALD2500004
      EQUATIONS OF THE FORM DY/DX = F(X,Y), THE RIGHT HAND SIDE OF D2500005
      WHICH MUST BE GIVEN AS A SUBROUTINE WITH THE HEADING SUBROUTINE D2500006
      DIFFUN(X,Y,DY) $ DIMENSION Y(1),DY(1). D2500007

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C      X AND Y ARE THE INITIAL VALUES. IN THE SUBROUTINE DIFSUB      D2500008
C      THE ORIGINAL VALUES OF X AND Y ARE REPLACED BY X + H0 AND Y(X+H0), D2500009
C      RESPECTIVELY. ALSO, H0 MAY BE CHANGED (AUTOMATIC STEPSIZE COR- D2500010
C      RECTION). THE ARRAY S(N) AND THE CONSTANT EPS ARE USED TO CON:ROL D2500011
C      THE ACCURACY OF THE COMPUTED VALUES. THE PROCEDURE IS LEFT IF FOR D2500012
C      ALL I = 1, 2, ..., N TWO SUCCESSIVE VALUES OF Y(I) DIFFER AT MOST D2500013
C      BY AN AMOUNT EPS*S(I). THE ARRAY R(I) CONTAINS THE ABSOLUTE ERROR D2500014
C      (THE DIFFERENCE IN TWO SUCCESSIVE VALUES OF Y(I)). D2500015
C      EPS SHOULD NOT BE SMALLER THAN 10**(-T+3), WHERE T IS THE D2500016
C      NUMBER OF DECIMAL DIGITS OF THE MACHINE REPRESENTATION. D2500017
C      THE PROGRAM TAKES THE FIRST OF THE NUMBERS H0,H0/2,H0/4,..., D2500018
C      HMIN, AS STEP SIZE H FOR WHICH NO MORE THAN 9 EXTRAPOLATION STEPS D2500019
C      ARE NEEDED TO OBTAIN A SUFFICIENTLY ACCURATE RESULT. D2500020
C      IF A STEP SIZE LESS THAN HMIN IS NEEDED TO GIVE THE REQUIRED D2500021
C      ACCURACY THE PROCEDURE IS LEFT WITH THE FINAL ESTIMATES OF Y(I) D2500022
C      FILLED IN AND THE FLAG KK NEGATIVE. OTHERWISE KK IS SET TO +1. D2500023
C      FOR THE FIRST INTEGRATION STEP IT IS ADVISABLE TO SET S(I) = D2500024
C      ABSF(Y(I)), THE MAGNITUDE OF THE INPUT VALUES FOR THE DEPENDENT D2500025
C      VARIABLES. AFTER LEAVING, THE ARRAY S(N) IS CHANGED TO S(I) = D2500026
C      MAX (S(I), ABSF(Y(I,XJ))) WHERE X .LT. XJ .LE. X+H0. D2500027
C      THIS SUBROUTINE IS A 3600 FORTRAN PROGRAM BASED ON AN ALGOL D2500028
C      PROGRAM BY R. BULIRSCH AND J. STOER, CONTAINED IN A REPORT NUMER- D2500029
C      ICAL TREATMENT OF ORDINARY DIFFERENTIAL EQUATIONS BY EXTRAPOLAT- D2500030
C      ION METHODS. A COPY OF THIS REPORT IS AVAILABLE FROM W. J. CODY. D2500031
C      SUBSCRIPTS J,K,L, AND KH ARE ONE GREATER THAN IN ALGOL VERSION. D2500032
C      BOOLEAN KONV,JBO,JBH,FIN USE THE CONVENTION TRUE=+1, FALSE=-1. D2500033
C      DIMENSION Y(20),DY(20),S(20),R(20),YA(20),YL(20),YM(20),DZ(20), D2500034
1 SA(20),D(7),DT(20,7),YG(20,8),YH(20,8),SG(20,8) D2500035
JMAX=JM+4 $ DO 2 I=1,N $ YA(I)=Y(I) $ SA(I)=S(I) D2500036
2 CONTINUE D2500037
CALL DIFFUN (X,Y,DZ) D2500038
3 JBH=-1 $ KK=+1 D2500039
4 A=H+X $ JBO=-1 $ M=1 $ JR=2 $ JS=3 $ JJ=0 D2500040
DO 32 J=1,JMAX D2500041
IF(JBO.LT.0) GOTO 6 D2500042
D(2)=16./9. $ D(4)=64./9. $ D(6)=256./9. $ GOTO 8 D2500043
6 D(2)=2.25 $ D(4)=9. $ D(6)=36. D2500044
8 KONV=+1 $ IF(J.LE.(JM/2)) KONV=-1 D2500045
IF(J.LE.(JM+1)) GOTO 10 D2500046
L=JM+1 $ D(L)=4.*D(L-2) $ FC=.7071068*FC $ GOTO 12 D2500047
10 L=J $ D(L)=FLOATF(M*M) $ FC=1.0+FLOATF(JM+1-J)/6.0 D2500048
12 M=M+M $ G=H/FLOATF(M) $ B=G+G D2500049
IF((JBH.LT.0).OR.(J.GE.(JMAX-1))) GOTO 16 D2500050
DO 14 I=1,N D2500051
YM(I)=YH(I,J) $ YL(I)=YG(I,J) $ S(I)=SG(I,J) D2500052
14 CONTINUE $ GOTO 25 D2500053
16 DO 18 I=1,N D2500054
YL(I)=YA(I) $ YM(I)=YA(I)+G*DZ(I) $ S(I)=SA(I) D2500055
18 CONTINUE D2500056
KH=M/2 $ XU=X D2500057
DO 24 K=2,M $ XU=XU+G D2500058
CALL DIFFUN (XU,YM,DY) D2500059
DO 20 I=1,N D2500060
U=YL(I)+B*DY(I) $ YL(I)=YM(I) $ YM(I)=U D2500061
U=ABSF(U) $ IF(U.GT.S(I)) S(I)=U D2500062
20 CONTINUE D2500063
IF((K.NE.KH).OR.(K.EQ.3)) GOTO 24 D2500064
JJ=1+JJ D2500065
DO 22 I=1,N D2500066
YH(I,JJ)=YM(I) $ YG(I,JJ)=YL(I) $ SG(I,JJ)=S(I) D2500067
22 CONTINUE D2500068
24 CONTINUE D2500069
25 CALL DIFFUN (A,YM,DY) D2500070
DO 30 I=1,N D2500071
V=DT(I,1) $ TA=C=DT(I,1)=(YM(I)+YL(I)+G*DY(I))*5 D2500072
IF(L.LT.2) GOTO 29 D2500073
IF(ABSF(V)*1073741824..LT.ABSF(C)) GOTO 33 D2500074
IF(MF.GT.0) GOTO 37 $ DO 28 K=2,L D2500075

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      B1=D(K)*V $ B=B1-C $ U=V
      IF(B.EQ.0) GOTO 26
      B=(C-V)/B $ U=C*B $ C=B1*B
26      V=DT(I,K) $ DT(I,K)=U $ TA=U+TA
28      CONTINUE
29      U=ABSF(TA) $ IF(U.GT.S(I)) S(I)=U
      R(I)=ABSF(Y(I)-TA) $ Y(I)=TA
      IF(R(I).GT.EPS*S(I)) KONV=-1
30      CONTINUE
      IF(KONV.GE.0) GOTO 34
      D(3)=4. $ D(5)=16. $ JBO=-JBO $ M=JR $ JR=JS $ JS=M+M
32      CONTINUE $ JBH=-JBH
33      IF(ABSF(H).LE.HMIN) GOTO 35 $ H=H/2.
      IF(ABSF(H).GE.HMIN) GOTO 4 $ H=SIGNF(HMIN,H) $ GOTO 3
34      H=FC*H $ X=A $ RETURN
35      KK=-1 $ GOTO 34
      ENTRY REDIF
      DO 36 I=1,N $ Y(I)=YA(I)
36      CONTINUE $ GOTO 3
37      DO 38 K=2,L
          TA=TA+(TA-V)/(D(K)-1.0) $ V=DT(I,K) $ DT(I,K)=TA
38      CONTINUE $ GOTO 29
      END
      SUBROUTINE PSIINV(PSIY,TY,TM,INSTJMP)
C      EXPONENTIAL PULSE
      COMMON/PUL/PSIM,TO
      PRINT 10
10      FORMAT(1H0,7X*PSI(T) IS AN EXPONENTIAL DECAY PULSE*//
1          13X*PSI = (PSIM)(EXP(-T/TO))* )
      TY=TM=0.0
      INSTJMP=1
      END
C      SUBROUTINE PSISER(T,PSI,PSI1,PSI2,PSI3,PSI4)
      EXPONENTIAL PULSE
      COMMON/PUL/PSIM,TO
      IF(T.LT.0.0) GO TO 3
      PSI=PSIM*EXP(-T/TO)
      PSI1=-PSI/TO
      PSI2=-PSI1/TO
      PSI3=-PSI2/TO
      PSI4=-PSI3/TO
      RETURN
3      PSI=PSI1=PSI2=PSI3=PSI4=0.0
      END
C      SUBROUTINE PULSE(TAU1,TAU2,PSIINO,PSIIN1)
C      EXPONENTIAL PULSE
C      PSIINO = INTEGRAL FROM TAU1 TO TAU2 OF PSI
C      PSIIN1 = INTEGRAL FROM TAU1 TO TAU2 OF (PSI)*(T-TAU1)
      COMMON/PUL/PSIM,TO
      T1=AMAX1(TAU1,0.0)
      T2=TAU2
      IF(T2.LE.T1) 1,2
1      PSIINO=PSIIN1=0.0
      RETURN
2      EX1=EXP(-T1/TO)
      EX2=EXP(-T2/TO)
      PSIINO=PSIM*TO*(EX1-EX2)
      PSIIN1=PSIM*TO*(TO*(EX1-EX2)-(T2-T1)*EX2)
      END

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D2500076
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 D2500098

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