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MASTER

DIAGONALIZATION OF INCLUSIVE SUM RULES

AND

INCLUSIVE BOOTSTRAP*

by

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Se-Yuen Mak and Chung-I Tan

Department of Physics
Brown University
Providence, Rhode Island 02912

ABSTRACT

We introduce the technique of multi-dimensional Laplace transforms under which the inclusive-like sum rules for discontinuities of multiparticle amplitudes can be diagonalized. The resulting equations are simple constraints on our generalized partial-wave amplitudes, and they can be used to impose unitarity conditions for bootstrap models. A simple example is given to illustrate the non-linear nature of this "inclusive bootstrap" program.

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I. Introduction

The equivalence between the inclusive-like sum rules for discontinuities of multiparticle amplitudes¹⁻⁴ and the basic discontinuity equations of Cahill and Stapp⁵ has renewed the hope of a practical enforcement of unitarity constraints at high energies. We have already shown that unitary models can be directly constructed from these linear sum rules under certain factorization approximations thus illustrating their potential usefulness as a new starting point for strong interaction dynamics.⁶ Veneziano has noted that, in any approximate bootstrap scheme, these sum rules are non-linear relations in the "coupling constants", leading to the possibility of an "inclusive bootstrap".¹

We introduce a technique of handling these sum rules so that their non-linear characteristics can easily be exploited. Using multi-dimensional Laplace transforms we show that inclusive sum rules can be "diagonalized" in a set of "generalized" complex angular momenta. They then become exact constraints on "partial-wave" amplitudes of multiparticle discontinuities. Being statements of unitarity, they can be used as a starting point for dynamical approximations.

Throughout this discussion, we assume that there is only one type of scalar particle, and we avoid the kinematic complications of transverse momenta by discussing only the integrated inclusive cross sections. We use the normalization that, for $a + b \rightarrow 1 + 2 + 3 + \dots + n + \text{anything}$, the discontinuity

$D_n(p_a, p_b; p_1, p_2, \dots, p_n)$ is related to the cross section by

$$\frac{d\sigma}{dy_1 dy_2 \dots dy_n} = \Delta^{-\frac{1}{2}}(s, m^2, m^2) D_n(p_a, p_b; p_1, p_2, \dots, p_n) \quad , \quad (1)$$

where $\Delta(x, y, z) = x^2 + y^2 + z^2 - 2(xy + yz + zx)$, and each momentum $p_i = (E_i, \mathbf{p}_i)$ is specified by a rapidity variable y_i , e.g.

$$p_a = m(\cosh y_a, \sinh y_a), p_b = m(\cosh y_b, \sinh y_b), p_i''/E_i = \tanh y_i, \quad (2)$$

The energy and the longitudinal momentum sum rules can then be written as

$$\int_{y_a}^{y_b} dy_n \begin{Bmatrix} \cosh y_n \\ \sinh y_n \end{Bmatrix} \frac{d\sigma}{dy_1 dy_2 \dots dy_n} = \begin{Bmatrix} \cosh y_a + \cosh y_b - \sum_{i=1}^{n-1} \cosh y_i \\ \sinh y_a + \sinh y_b - \sum_{i=1}^{n-1} \sinh y_i \end{Bmatrix} \frac{d\sigma}{dy_1 dy_2 \dots dy_{n-1}}. \quad (3)$$

However, it is conceptually more appealing to express them in terms of discontinuities, so that they become⁷

$$\int_{y_a}^{y_b} dy_n \cosh y_n D_n(y_a, y_b; y_1, y_2 \dots y_n) = (\cosh y_a + \cosh y_b - \sum_{i=1}^{n-1} \cosh y_i) \times D_{n-1}(y_a, y_b; y_1, \dots, y_{n-1}), \quad (4)$$

$$\int_{y_a}^{y_b} dy_n \sinh y_n D_n(y_a, y_b; y_1, y_2 \dots y_n) = (\sinh y_a + \sinh y_b - \sum_{i=1}^{n-1} \sinh y_i) \times D_{n-1}(y_a, y_b; y_1, \dots, y_{n-1}). \quad (5)$$

Equations (4) and (5) (or Eq. (3)) are, strictly speaking, correct only in the limit $\langle q_i^{1,2} \rangle \rightarrow 0$. This assumption is for notational convenience only, and, as we will point out later, it does not impair the generality of our approach. To exploit the consequences of (4) and (5), we next introduce multi-dimensional Laplace transforms for D_n , $n = 0, 1, 2, \dots$.

II. Multi-Dimensional Laplace Transform

Lorentz invariance guarantees that D_n only depends on rapidity differences; and, for D_n , there are $n+1$ independent variables. Energy-momentum conservation requires

$$y_a \leq y_i \leq y_b, \quad (6)$$

and the complete phase space can further be divided into $n!$ regions, each of which corresponds to a partition of the line interval $[y_a, y_b]$, characterized by

$$y_a \leq y_{\lambda_1} \leq y_{\lambda_2} \leq \dots \leq y_{\lambda_n} \leq y_b. \quad (7)$$

Let us first concentrate on the region $\{\lambda_1 = 1\}$. We can choose the independent variables to be

$$\xi_1 = y_1 - y_a, \quad \xi_2 = y_2 - y_1, \quad \dots, \quad \xi_n = y_n - y_{n-1}, \quad \xi_{n+1} = y_b - y_n, \quad (8)$$

and, in this region, we denote

$$D_n(y_a, y_b; y_1, y_2, \dots, y_n) \equiv D_n(\xi_1, \xi_2, \dots, \xi_{n+1}) \quad (9)$$

Since all $\xi_i \geq 0$, we can define a multi-Laplace transform

$$\tilde{D}_n(\ell_1, \ell_2, \dots, \ell_{n+1}) \equiv \int_0^\infty d\xi_1 e^{-\ell_1 \xi_1} \dots \int_0^\infty d\xi_{n+1} e^{-\ell_{n+1} \xi_{n+1}} D_n(\xi_1, \dots, \xi_{n+1}), \quad (10)$$

and the inversion formula is simply

$$D_n(\xi_1, \dots, \xi_{n+1}) = \int_{c-i\infty}^{c+i\infty} \frac{d\ell_1}{2\pi i} \dots \int_{c-i\infty}^{c+i\infty} \frac{d\ell_{n+1}}{2\pi i} e^{\ell_1 \xi_1} \dots e^{\ell_{n+1} \xi_{n+1}} \tilde{D}_n(\ell_1, \dots, \ell_{n+1}). \quad (11)$$

The integration contour in (11) is chosen to be to the right of all singularities of the integrand.

The complex parameter ℓ_1 is essentially the complex angular momentum variable conjugate to the energy variable $\cosh \xi_1$. Leading asymptotic behavior of D_n in ξ_1 is then controlled by the singularity of $\tilde{D}_n$ furthest to the right in the ℓ_1 -plane. In fact, leading singularities in ℓ_1 are Regge singularities, and the structure differs from that of the ordinary Regge complex plane only in their "daughter" sequences. For this reason, we refer to our ℓ -planes as if they were the conventional J-planes. We assume that D_n 's are sufficiently well behaved at both limits so that (10) is defined.⁸ This is certainly true for models we are interested in, such as the Regge pole models.⁹

Although we have worked within a particular partition of $[y_a, y_b]$, the partial-wave amplitudes $\tilde{D}_n(\ell_1, \dots, \ell_{n+1})$ is actually independent of this choice because of identical particles. For models with several different species of particles, additional labels for inequivalent partitions of $[y_a, y_b]$ are also necessary for our partial-wave amplitudes. We make a further notational simplification, for $n = 0, 1, 2$, by introducing

$$A(\ell) = \tilde{D}_0(\ell) = \int_0^\infty d\xi e^{-\ell \xi} D_0(\xi), \quad (12)$$

$$B(\ell_a, \ell_b) = \tilde{D}_1(\ell_a, \ell_b) = \int_0^\infty d\xi_a \int_0^\infty d\xi_b e^{-\ell_a \xi_a} e^{-\ell_b \xi_b} D_1(\xi_a, \xi_b), \quad (13)$$

$$C(\ell_a, \ell_c, \ell_b) = \tilde{D}_2(\ell_a, \ell_c, \ell_b) = \int_0^\infty d\xi_a \int_0^\infty d\xi_c \int_0^\infty d\xi_b e^{-\ell_a \xi_a} e^{-\ell_c \xi_c} e^{-\ell_b \xi_b} D_2(\xi_a, \xi_c, \xi_b). \quad (14)$$

III. Diagonalization of Inclusive Sum Rules

Before we show how inclusive sum rules (4) and (5) can be diagonalized, we

shall first rewrite them in a more convenient form. Multiplying the sum and the difference of (4) and (5) by e^{-y_a} and e^{y_a} respectively, we obtain for $n = 0, 1, \dots$,

$$X_n^{(+)}(y_a, y_b; y_1, \dots, y_n) \equiv (1 + e^{\frac{+}{-}(y_b - y_a)} - \prod_{i=1}^n e^{\frac{+}{-}(y_i - y_a)}) D_n(y_a, y_b; y_1, \dots, y_n) - \int_{y_a}^{y_b} dy_{n+1} e^{\frac{+}{-}(y_{n+1} - y_a)} D_{n+1}(y_a, y_b; y_1, \dots, y_{n+1}) = 0 \quad (15)$$

This amounts to specifying rapidities in the rest frame of the particle a.

More generally, we introduce relative rapidities

$$y_{ai} \equiv y_i - y_a, y_{ij} \equiv y_j - y_i, y_{jb} \equiv y_b - y_j, y_{ab} \equiv y_b - y_a \quad (16)$$

With our convention (6), we find that

$$y_{ai} \geq 0, y_{jb} \geq 0, y_{ab} \geq 0, \quad (17)$$

and y_{ij} can be positive or negative depending on the particular partition of $[y_a, y_b]$ in which one is working. Since there are only a few partitions, for $n = 1, 2$, Eq. (15) can be written explicitly as

$$X_0^{+} (y_{ab}) = (1 + e^{\frac{+}{-}y_{ab}}) D_0(y_{ab}) - \int_0^{y_{ab}} dy_{a1} e^{\frac{+}{-}y_{a1}} D_1(y_{a1}, y_{1b}) = 0, \quad (18)$$

$$X_1^{(+)}(y_{a1}, y_{1b}) = (1 + e^{\frac{+}{-}y_{ab}} - e^{\frac{+}{-}y_{a1}}) D_1(y_{a1}, y_{1b})$$

$$- \int_0^{y_{a1}} dy_{21} e^{\frac{+}{-}y_{a2}} D_2(y_{a2}, y_{21}, y_{1b}) - \int_0^{y_{1b}} dy_{12} e^{\frac{+}{-}(y_{a1} + y_{12})}$$

$$D_2(y_{a1}, y_{12}, y_{2b}) = 0 \quad (19)$$

Alternatively, we can also specify rapidities in the rest frame of the particle 1. We find that the sum rules (15), for $n = 1$, can be written in a more symmetric form

$$Y_1^{(+)}(y_{a1}, y_{1b}) = e^{\bar{y}_{a1}} X_1^{(+)}(y_{a1}, y_{1b}) = (e^{\bar{y}_{a1}} + e^{\bar{y}_{1b}} - 1) \times D_1(y_{a1}, y_{1b}) - \int_{y_a}^{y_b} dy_2 e^{\bar{y}_{21}} D_2 = 0 \quad (20)$$

To perform the multi-Laplace transform on the inclusive sum rules, $X_n^{(+)}(y_a, y_b; y_1 \dots y_n) = 0$, $n = 0, 1, 2, \dots$, we need first to specify a particular ordering of $\{y_i\}$. However, for $n = 0$ and 1, there is no such freedom. Applying

$$\int_0^\infty dy_{ab} e^{-ly_{ab}}$$

to (18), we obtain

$$\begin{aligned} \tilde{X}_0^{(+)}(l) &= \int_0^\infty dy_{ab} e^{-ly_{ab}} X_0^{(+)}(y_{ab}) \\ &= A(l) + A(l+1) - \int_0^\infty dy_{ab} e^{-ly_{ab}} \int_0^{y_{ab}} dy_{a1} e^{\bar{y}_{a1}} D_1(y_{a1}, y_{1b}) = 0 \end{aligned} \quad (21)$$

Interchanging the order of y_{ab}, y_{a1} integrations, and changing the integration variable from y_{ab} to $y_{1b} = y_{ab} - y_{a1}$, we arrive at the following equations,

$$X_0^{(+)}(l) = A(l) + A(l+1) - B(l+1, l) = 0 \quad (22)$$

Similarly, applying

$$\int_0^\infty dy_{a1} e^{-l_a y_{a1}} \int_0^\infty dy_{1b} e^{-l_b y_{1b}}$$

to Eq. (19), we obtain

$$\begin{aligned} \tilde{X}_1^{(+)}(\ell_a, \ell_b) &= B(\ell_a, \ell_b) + B(\ell_a + 1, \ell_b + 1) - B(\ell_a + 1, \ell_b) \\ &\quad - C(\ell_a + 1, \ell_a, \ell_b) - C(\ell_a + 1, \ell_b + 1, \ell_b) = 0 \end{aligned} \quad (23)$$

Alternatively, if (20) is used, we have

$$\begin{aligned} \tilde{Y}_1^{(+)}(\ell_a, \ell_b) &= \tilde{X}_1^{(+)}(\ell_a + 1, \ell_b) = B(\ell_a + 1, \ell_b) + B(\ell_a, \ell_b + 1) - B(\ell_a, \ell_b) \\ &\quad - C(\ell_a, \ell_a + 1, \ell_b) - C(\ell_a, \ell_b + 1, \ell_b) = 0 \end{aligned} \quad (23)'$$

In a analogous manner, we can obtain, for general n ,

$$\tilde{X}_n^{(+)}(\ell_1, \dots, \ell_{n+1}) = 0,$$

relating $\tilde{D}_n$ to $\tilde{D}_{n+1}$. We shall not write this general form here.

We can now remove our unphysical restriction on transverse momenta of all produced particles. We simply carry out the same analysis with $\vec{q}_1^\perp$ kept as an additional label. In terms of normalizations for $D_n(p_a, p_b; p_1 \dots p_n)$, used in Ref. 6, we find that, for $n = 0$ and $n = 1$,

$$\tilde{D}_0(\ell) + \tilde{D}_0(\ell + 1) - [(2)(2\pi)^3]^{-1} \int d\vec{q}_1^\perp (1 + \vec{q}_1^{\perp 2}/m^2)^{1/2} \tilde{D}_1(\ell + 1, \vec{q}_1^\perp, \ell) = 0, \quad (24)$$

$$\tilde{D}_1(\ell_a + 1, \vec{q}_1^\perp, \ell_b) + \tilde{D}_1(\ell_a, \vec{q}_1^\perp, \ell_b + 1) - (1 + \vec{q}_1^{\perp 2}/m^2)^{1/2} \tilde{D}_1(\ell_a, \vec{q}_1^\perp, \ell_b)$$

$$- [(2)(2\pi)^3]^{-1} \int d\vec{q}_2 (1 + \vec{q}_2^{\perp 2}/m^2)^{1/2} [\tilde{D}_2(\ell_a, \vec{q}_2^\perp, \ell_a + 1, \vec{q}_1^\perp, \ell_b)$$

$$+ \tilde{D}_2(\ell_a, \vec{q}_1^\perp, \ell_b + 1, \vec{q}_2^\perp, \ell_b)] = 0 \quad (25)$$

If D_n 's are strongly damped in $|\vec{q}_1^\perp|$, then (24) and (25) can easily be reduced to (22) and (23)'. In general,¹⁰ a Fourier analysis in the transverse angle ϕ_{12} , $\cos\phi_{12} \equiv \hat{q}_1^\perp \cdot \hat{q}_2^\perp$, can be made for $\tilde{D}_2$, so that each Fourier component will depend only on the magnitudes $|\vec{q}_1^\perp|$ and $|\vec{q}_2^\perp|$. However, for our subsequent discussions, we shall return to Eqs. (22) and (23)'.

IV. Bootstrap Models

The necessary assumptions for the existence of our multi-dimensional Laplace transforms immediately imply that $\tilde{D}_n(l_1, \dots, l_{n+1})$ vanishes in the limit $\text{Re}(l_1) \rightarrow \infty$, so that they can be represented by a dispersion integral, e.g.

$$A(l) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dl' \frac{A(l')}{l - l'} \quad (26)$$

The integration contour can be deformed to the left, picking up singularities of $A(l')$ along the way. Let us assume that there exists several poles at $l = \alpha_i$, $\text{Re}(\alpha_i) > \text{Re}(\alpha_j)$, for $i < j$, $i, j = 0, 1, \dots, m$. We can then choose $c' < \text{Re}(\alpha_m)$, and represent $A(l)$ by

$$A(l) = \sum_{i=0}^m \frac{G_i^2}{l - \alpha_i} + \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} dl' \frac{A(l')}{l - l'} \quad (27)$$

where G_i^2 is the residue of $A(l)$ at $l = \alpha_i$. It follows from (22) that $B(l, l+1)$ also have poles at $l = \alpha_i$, and are analytic for $\text{Re}(l) > \alpha_0$. Assuming that $B(l_a, l_b)$ does not have singularities in $l_a(l_b)$ that are $l_b(l_a)$ dependent in the region $\text{Re}(l_a) > c'$ ($\text{Re}(l_b) > c'$), we can then conclude that $B(l_a, l_b)$ has poles at $l_a = \alpha_i$ and $l_b = \alpha_j$, and it can be represented by

$$\begin{aligned}
 B(l_a, l_b) &= \sum_{i=0}^m \frac{\gamma_i}{l_a - \alpha_i} + \sum_{i=0}^m \frac{\gamma_i}{l_b - \alpha_i} + \sum_{i,j=0}^m \frac{\beta_{ij}}{(l_a - \alpha_i)(l_b - \alpha_j)} \\
 &+ \int_{c'-i\infty}^{c'+i\infty} \frac{dl'_a}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{dl'_b}{2\pi i} \frac{B(l'_a, l'_b)}{(l_a - l'_a)(l_b - l'_b)} \\
 &= \sum_{i=0}^m \frac{G_i G_i^0}{l_a - \alpha_i} + \sum_{i=0}^m \frac{G_i^0 G_i}{l_b - \alpha_i} \\
 &+ \sum_{i,j=0}^{\infty} G_i \frac{1}{l_a - \alpha_i} g_{ij} \frac{1}{l_b - \alpha_j} G_j + (\text{Background integral}) .
 \end{aligned} \tag{28}$$

Performing the inversion procedure, (11), together with Eq. (1), we find that inclusive cross sections now have the desired scaling properties. In fact, what we have outlined is precisely the Regge assumption made by Mueller.¹¹ We stress that the "independence" of singularities of $B(l_a, l_b)$ is a debatable assumption. It holds in dynamical models discussed in Ref. 11, but certainly not true for the fragmentation model of Yang and his collaborators.¹² It is not our present purpose to demonstrate that Regge behavior follows from our sum rules, but rather to obtain restrictions on Regge couplings due to unitarity constraints. Therefore, the usual Regge expansion is a dynamical input for us. Together with the "independence" of singularities assumption, we also postulate the factorization of Regge residues.¹³ Applying these two assumptions to $C(l_a, l_c, l_b)$, we find

$$\begin{aligned}
 C(l_a, l_c, l_b) &= \sum_{i,j,k=0}^m G_i \frac{1}{l_a - \alpha_i} g_{ij} \frac{1}{l_c - \alpha_j} g_{jk} \frac{1}{l_b - \alpha_k} G_k \\
 &+ (\text{Double-pole terms}) + (\text{Single-pole terms}) \\
 &+ (\text{Background integral}) .
 \end{aligned} \tag{29}$$

In Eq. (28), the first single pole term on the right hand side only contributes to the limiting distribution of b , the second term contributes to the limiting distribution of a , the third term contributes to both, and the background integral contribution is asymptotically negligible. Therefore, to have "asymptotic" consistency, we need only to equate residues at poles of $\tilde{X}_0^{(+)}(\ell) = 0$, $\tilde{X}_1^{(+)}(\ell_a, \ell_b) = 0$ in the region $\text{Re}(\ell)$, $\text{Re}(\ell_a)$, $\text{Re}(\ell_b) > c'$. This automatically leads to non-linear relations among g_{ij} , G_j , G_k^0 , α_i , etc. To illustrate the mathematical structure of the resulting equations, we assume that the background term as well as the single pole term in B and the single and the double pole terms in C can be ignored. This is not completely unrealistic if we keep enough Regge poles. We first evaluate $\tilde{X}_0^{(+)}(\ell) = 0$ at $\ell = \alpha_i + 1$, $\tilde{X}_0^{(-)}(\ell) = 0$ does not yield any new information because of the symmetry of $B(\ell_a, \ell_b)$, and find

$$G_i = \sum_{j=0}^m \frac{g_{ij} G_j}{\alpha_i - \alpha_j + 1} \quad (30)$$

This "eigenvalue" condition guarantees the "asymptotic" consistency¹⁴ of Eq. (18). The same minimal asymptotic consistency for Eq. (19) requires the vanishing of residues of $\tilde{Y}_1^{(+)}(\ell_a, \ell_b) = 0$ ($\tilde{Y}_1^{(-)}(\ell_a, \ell_b) = 0$) at $\ell_a = \alpha_i$ ($\ell_a = \alpha_i + 1$), and $\ell_b = \alpha_j + 1$ ($\ell_b = \alpha_j$). It is interesting that these conditions are automatically satisfied if Eq. (30) holds.

Very stringent conditions result if we demand further consistency at lower poles. Under the same simplifying assumptions and assuming $\alpha_i - \alpha_j \neq \pm 1$, we find that $\tilde{X}_0^{(+)}(\ell = \alpha_i) = 0$ implies

$$G_i = \sum_{j=0}^m \frac{g_{ij} G_j}{\alpha_i - \alpha_j - 1} \quad (31)$$

and $\tilde{Y}_1^{(+)}(\ell_a = \alpha_i, \ell_b = \alpha_j) = 0$ imply

$$g_{ij} = g_{ji} = \sum_{k=0}^m \frac{(\alpha_i + \alpha_j - 2\alpha_k)}{(\alpha_i - \alpha_k + 1)(\alpha_k - \alpha_j + 1)} g_{ik} g_{kj} \quad (32)$$

These are highly non-linear constraints on Regge couplings. Due to our numerous unphysical assumptions, we do not attempt to solve these equations here. It is interesting to note that Eqs. (30)-(32) are not compatible with each other if only one or two poles are kept.¹⁵

V. Additional Remarks

We would like to comment further on our approximate dynamical scheme of Section IV. First, branch point contributions should in general be included in our expansions (27)-(29). Although their inclusion will not cause any difficulty of principles, we no longer may invoke the factorization assumption. It is therefore unlikely for us to achieve a complete bootstrap including both poles and branch points. However, if their effects are used as additional inputs,¹⁶ partial bootstraps can result. Second, we discuss the effect of finite truncation in Regge singularities. There is indeed a chance of realism if the discontinuity D_n always remains finite (it may diverge) when relative rapidities $y_{ij}, y_{ai}, y_{jb} \rightarrow 0$, i.e., this behavior can be simulated by a finite sum of Regge pole terms. However, if D_n diverges, our approximations (27)-(29) are certainly inadequate. One possible way out is to keep the single-pole term in (27) and to choose G^0 in such a way to represent the average effect of this divergent behavior¹⁷ (such as triple-Regge behavior with $2\alpha(t) - \alpha(0) > 0$).

We mention in passing that one of the requirements for the existence of our multi-dimensional Laplace transforms is that D_n 's can not be too singular

as $y_{ib}, y_{ai} \rightarrow 0$, e.g., if $y_{al} \sim 0$, $D_n \propto (1 - e^{-y_{al}})^{-\epsilon}$, we require $\epsilon < 1$.

Since $D_0 < \infty$ when $y_{ab} \rightarrow 0$, $A(\ell)$ is therefore well-defined. However, if $\alpha(0) = 1$, it follows that $D_1(y_{a1}, y_{1b}) \propto (1 - e^{-y_{a1}})^{-1}$ as $y_{a1} \rightarrow 0$. Our sum rule $\tilde{X}_0^{(+)}(\ell) = 0$ then forces the triple-Pomeron coupling to vanish in the forward limit because $B(\ell, \ell + 1)$ has to be finite (i.e., it is equal to $A(\ell) + A(\ell + 1)$). This is precisely equivalent to the argument given in Ref. 2, but now in the language of the partial-wave analysis. One possible advantage of this present approach is our being able to see explicitly that other singularities, such as branch points at $J = 2\alpha(0) - 1$, can not change this conclusion because they come from divergences at the upper limit of ξ_a integration in (13).

We are fully aware that simple models without incorporating internal symmetry, baryon number, etc. are of very little real use. It is perhaps a blessing that a single- and a two-pole models are not compatible with our unitarity constraints. The inclusion of isospin, for instance, increases not only our internal trajectory multiplicity but also external particle species. This added structure may allow our sum rules to sustain a realistic bootstrap system. Results of investigations along this line will be reported elsewhere.

Despite its shortcomings, our bootstrap model of Section IV does have many interesting and desirable features. Since Eq. (30) is not too difficult to satisfy, it follows that the two-particle correlation function¹⁸ exhibits the features of short-range correlation in the relative rapidities. We find that

$$C_{12}(y_a, y_1, y_2, y_b) \equiv D_0(y_{ab})^{-1} \left\{ D_2(y_a, y_1, y_2, y_b) - D_1(y_{a1}, y_{1b}) D_1(y_{a2}, y_{2b}) D_0^{-1}(y_{ab}) \right\} \quad (33)$$

in general is non-zero whenever any one of the relative rapidities y_{ai}, y_{ij}, y_{jb} is not infinite. Inclusive sum rules, therefore, do not directly infer any

constraints on the value of correlation length!¹⁹

Our multi-dimensional Laplace transforms can also be used to study other sum rules of the type²⁻⁴

$$\int dp \frac{d\sigma}{dp} = \langle n \rangle \sigma^{\text{tot}} \quad , \quad (34)$$

$$\int dp_1 dp_2 \frac{d\sigma}{dp_1 dp_2} = \langle n(n-1) \rangle \sigma^{\text{tot}} \quad . \quad (35)$$

It follows from (12)-(14) that

$$\langle n \rangle \sim (\sigma^{\text{tot}})^{-1} \int_{c-i\infty}^{c+i\infty} \frac{dl}{2\pi i} e^{(l-1)y_{ab}} B(l, l) \sim (\text{const}) \ln s \rightarrow \infty \quad , \quad (36)$$

$$\langle n(n-1) \rangle \sim 2(\sigma^{\text{tot}})^{-1} \int_{c-i\infty}^{c+i\infty} \frac{dl}{2\pi i} e^{(l-1)y_{ab}} C(l, l, l) \sim (\text{const})(\ln s)^2 \rightarrow \infty \quad , \quad (37)$$

and

$$\langle n(n-1) \rangle - \langle n \rangle^2 \sim (\text{const}) \ln s \rightarrow \infty \quad , \quad (38)$$

with proportionality constants directly related to residues of poles of our partial-wave amplitudes.²⁰ Conversely, experimental measurements of these average quantities can yield informations on asymptotic behavior of inclusive cross sections through Eq. (11).

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Footnotes and References

1. G. Veneziano, Phys. Rev. Letters 28, 578 (1972); Phys. Letters 36B, 397 (1971).
2. T. T. Chou, and C. N. Yang, Phys. Rev. Letters 25, 1072 (1971).
3. C. DeTar, D. Freedman, and G. Veneziano, Phys. Rev. D4, 906 (1971).
4. K. H. Biebl and J. Wolf, Phys. Letters 37B, 197 (1971); E. Predazzi and G. Veneziano, Nuovo Cimento Letters 2, 749 (1971); S. H.-H. Tye (MIT Report No. CTP 239); L. S. Brown, Phys. Rev. D5, 748 (1972).
5. Cahill and Stapp, Berkeley Report No. LBL-381, 1971 (to be published).
6. C.-I Tan, "Construction of Multiperipheral Dynamical Equations and Inclusive Sum Rules" (to be published in Phys. Rev.). A factor of $(2\pi)^{+3n}$ was inadvertently left out on the left-hand side of Eq. (2.1) there.
7. We have left out the inhomogeneous terms to simplify the writings. For the dynamical models discussed in Section IV, we believe they are not important. For their dynamical effects in a different context, see Ref. 6.
8. Questions relating to the triple-Regge behavior are discussed in Section V. In particular, it emerges naturally from our analysis that the triple-Pomeron coupling $g(t)$ vanishes at $t = 0$ if $\alpha(t = 0) = 1$.
9. Generally, we need the assumption of polynomial boundedness in $|\cosh \xi_1|$.
10. Details concerning the diagonalization of sum rules with transverse momenta will be discussed elsewhere.
11. A. Mueller, Phys. Rev. D2, 2962 (1970); D. Silverman and C.-I Tan, Phys. Rev. D3, 991 (1971), and references therein.
12. J. Benecke, T. T. Chou, C. N. Yang, and E. Yen, Phys. Rev. 188, 2159 (1969).
13. Because of the $\cos \phi_{ij}$ dependence, daughter poles are in general degenerate. For our present discussion, we shall ignore the presence of daughters, i.e.,

we assume $\alpha_i - \alpha_j \neq \pm 1$.

14. This level of asymptotic consistency has also been discussed by H. Tye and G. Veneziano, Phys. Letters 38B, 30 (1972); and by L. Caneschi, Phys. Letters 37B, 288 (1971). They did not, however, explicitly write down this eigenvalue condition. At this same level, T. C. Yang (La Jolla preprint) has attempted to derive a lower bound on the separation between the leading two Regge trajectories. Unfortunately, his result, as is evident from our analysis, is incorrect. (See further discussion on the two-particle correlation function in Section V.)
15. This conclusion has also been reached independently in a direct calculation of the sum rules by A. DiGiacomo (private communication).
16. Such as the use of branch cuts in the multiperipheral models.
17. It is easy to see that threshold behavior of D_n corresponds to asymptotic behavior of $\tilde{D}_n(l_i)$ as $\text{Re}(l_i) \rightarrow +\infty$. A more definite procedure, therefore, is to perform the appropriate subtraction at $\text{Re}(l_i) = +\infty$.
18. See, for instance, K. Wilson, CLNS-131, 1970 (unpublished); Jen, Kang, Shen, and Tan, Phys. Letters 38B, 81 (1972). It is also clear that C is a useful concept only in the central region where one is free of kinematic effects.
19. That is, inclusive sum rules, at the minimal consistency level, Eq. (30), are sufficiently flexible so as not to restrict the value of $\alpha_0 - \alpha_1$, α_0, α_1 being the two leading Regge poles. This is not to be confused, however, with the more stringent requirements Eqs. (31) and (32).
20. C. Quigg, J.-M. Wang, and C. N. Yang (Stony Brook preprint). Based on their intuitive fragmentation picture, they have speculated that

$$\langle n(n-1) \rangle - \langle n \rangle^2 \sim s^\alpha \rightarrow \infty$$

with $\alpha \leq 1/2$. For $\alpha \neq 0$, this prediction is very different from ours at infinite energy. However, a mere increase in this difference with energy at present machine energies cannot be used to support their picture. We note that the coefficient of the $\ln s$ term in Eq. (38) has been shown to be "positive" in multiperipheral models where the correlation is provided by the Regge cut. See: S. Y. Mak and C.-I Tan, "Correlation Lengths, Inclusive Transverse Momentum Distributions and Multiperipheralism", Brown University Preprint No. COO-3130TA-257. The observation made by Yang, et. al. has also been noted before by L. Caneschi, Ref. TH. 1419-CERN, Oct. (1971).