THIS DOCUMENT CONFIRMED AS
UNCLASSIFIED
DIVISION OF CLASSIFICATION
BY GH Kahn lamh
DATE 9/22/71

A DISTORTION ENERGY FAILURE THEORY

FOR BDX-613-136

ORTHOTROPIC MATERIALS

BY

G. W. FORMAN

JUNE, 1970

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

PATENT REVIEW APPROVES RELEASE. PROCEDURES
GOVERNING PATENT REVIEW AND RELEASE ARE ON
FILE IN RECEIVING SECTION.

R1452

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

pld

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

INTRODUCTION

Development of new materials which possess orthogonal strength and elastic characteristics has preceded the derivation of an adequate and reliable method of predicting failure of these materials when exposed to complex three-dimensional stresses. Curiously, wood, perhaps the oldest orthotropic material, has been used for centuries as a structural material, but rarely in applications in which three-dimensional point-stress loading is present. Methods of failure prediction for simple tension, compression, and bending loads have satisfied designers needs.

Cellular materials of plastic, honeycomb materials of metal, paper, etc., and filament-wound fiber materials are increasingly becoming available as outstanding materials of construction. Three-dimensional stress states are often present where fluid pressures are separated by walls of orthotropic materials. Typical are fiber-wound pressure vessels and the shells of aircraft, space vehicles, and deep-diving submarines.

A general state of stress, including shear stresses and principle stress which are not coincident with material geometric axes, poses a complex problem to one attempting to evaluate material integrity. Tsai¹ has considered this topic, and demonstrated the validity of a modified distortion-energy theory for one-dimensional stress systems. Hill's² modification of distortion energy theory lacks rationale.

¹ "Strength Characteristics Of Composite Materials", S. W. Tsai, NASA CR-224.

² "The Mathematical Theory Of Plasticity", R. Hill, Oxford University Press, London, 1950.

This study assumes that distortion energy poses a rational basis of predicting inelastic action for an orthotropic material exposed to principle stresses coincident with the material axes. A usable algebraic method of predicting failure is proposed.

DERIVATION OF FAILURE CRITERION

Consider an elemental unit cube of material as pictured in Figure 1, with the geometric axes of the material coincident with orthogonal axes X, Y, Z, and loaded by principle stresses σ_x , σ_y , and σ_z .

Figure 1.

Hooke's Law of Elastic Deformation is:

$$\epsilon_x = \frac{\sigma_x}{E_x} - \mu_{xy} \frac{\sigma_y}{E_y} - \mu_{xz} \frac{\sigma_z}{E_z} \quad (1a)$$

$$\epsilon_y = -\mu_{yx} \frac{\sigma_x}{E_x} + \frac{\sigma_y}{E_y} - \mu_{yz} \frac{\sigma_z}{E_z} \quad (1b)$$

$$\epsilon_z = -\mu_{zx} \frac{\sigma_x}{E_x} - \mu_{zy} \frac{\sigma_y}{E_y} + \frac{\sigma_z}{E_z} \quad (1c)$$

Nine elastic parameters, i.e., six μ 's and three E's are required to define the material characteristics.

If no inelastic action exists, the elastic energy stored must be independent of the order of application of the three stresses. Six sequences are possible, and the work done is:

$$U_{xyz} = \frac{1}{2} \left(\frac{\sigma_x^2}{E_x} + \frac{\sigma_y^2}{E_y} + \frac{\sigma_z^2}{E_z} \right) - \frac{\sigma_x \sigma_y}{E_y} \mu_{xy} - \frac{\sigma_x \sigma_z}{E_z} \mu_{xz} - \frac{\sigma_y \sigma_z}{E_z} \mu_{yz} \quad (2a)$$

$$U_{xzy} = \frac{1}{2} \left(\frac{\sigma_x^2}{E_x} + \frac{\sigma_y^2}{E_y} + \frac{\sigma_z^2}{E_z} \right) - \frac{\sigma_x \sigma_z}{E_z} \mu_{xz} - \frac{\sigma_x \sigma_y}{E_y} \mu_{xy} - \frac{\sigma_z \sigma_y}{E_y} \mu_{zy} \quad (2b)$$

$$U_{yxz} = \frac{1}{2} \left(\frac{\sigma_x^2}{E_x} + \frac{\sigma_y^2}{E_y} + \frac{\sigma_z^2}{E_z} \right) - \frac{\sigma_y \sigma_x}{E_x} \mu_{yx} - \frac{\sigma_y \sigma_z}{E_z} \mu_{yz} - \frac{\sigma_x \sigma_z}{E_z} \mu_{xz} \quad (2c)$$

$$U_{yzx} = \frac{1}{2} \left(\frac{\sigma_x^2}{E_x} + \frac{\sigma_y^2}{E_y} + \frac{\sigma_z^2}{E_z} \right) - \frac{\sigma_y \sigma_z}{E_z} \mu_{yz} - \frac{\sigma_y \sigma_x}{E_x} \mu_{yx} - \frac{\sigma_z \sigma_x}{E_x} \mu_{zx} \quad (2d)$$

$$U_{zxy} = \frac{1}{2} \left(\frac{\sigma_x^2}{E_x} + \frac{\sigma_y^2}{E_y} + \frac{\sigma_z^2}{E_z} \right) - \frac{\sigma_z \sigma_x}{E_x} \mu_{zx} - \frac{\sigma_z \sigma_y}{E_y} \mu_{zy} - \frac{\sigma_x \sigma_y}{E_y} \mu_{xy} \quad (2e)$$

$$U_{zyx} = \frac{1}{2} \left(\frac{\sigma_x^2}{E_x} + \frac{\sigma_y^2}{E_y} + \frac{\sigma_z^2}{E_z} \right) - \frac{\sigma_z \sigma_y}{E_y} \mu_{zy} - \frac{\sigma_z \sigma_x}{E_x} \mu_{zx} - \frac{\sigma_y \sigma_x}{E_x} \mu_{yx} \quad (2f)$$

Since the energies are all equal, the six quantities below must all be equal.

$$\frac{\mu_{xy}}{E_y} + \frac{\mu_{xz}}{E_z} + \frac{\mu_{yz}}{E_z} \quad (3a)$$

$$\frac{\mu_{xz}}{E_z} + \frac{\mu_{xy}}{E_y} + \frac{\mu_{zy}}{E_y} \quad (3b)$$

$$\frac{\mu_{yx}}{E_x} + \frac{\mu_{yz}}{E_z} + \frac{\mu_{xz}}{E_z} \quad (3c)$$

$$\frac{\mu_{yz}}{E_z} + \frac{\mu_{yx}}{E_x} + \frac{\mu_{zx}}{E_x} \quad (3d)$$

$$\frac{\mu_{zx}}{E_x} + \frac{\mu_{zy}}{E_y} + \frac{\mu_{xy}}{E_y} \quad (3e)$$

$$\frac{\mu_{zy}}{E_y} + \frac{\mu_{zx}}{E_x} + \frac{\mu_{yx}}{E_x} \quad (3f)$$

This requires that

$$\frac{\mu_{yz}}{E_z} = \frac{\mu_{zy}}{E_y} \quad (4a)$$

$$\frac{\mu_{xz}}{E_z} = \frac{\mu_{zx}}{E_x} \quad (4b)$$

$$\frac{\mu_{xy}}{E_y} = \frac{\mu_{yx}}{E_x} \quad (4c)$$

It is apparent that the nine elastic parameters are not independent. Six independent material constants suffice, as could be expected from the fact that each of energy equations (2) contains only six constants. Solving equations (4) for three of the Poisson's ratios,

$$\mu_{yx} = \frac{E_x}{E_y} \mu_{xy} \quad (5a)$$

$$\mu_{zx} = \frac{E_x}{E_z} \mu_{xz} \quad (5b)$$

$$\mu_{zy} = \frac{E_y}{E_z} \mu_{yz} \quad (5c)$$

Strain equations (1), including substitution of (5) are:

$$\epsilon_x = \frac{\sigma_x}{E_x} - \mu_{xy} \frac{\sigma_y}{E_y} - \mu_{xz} \frac{\sigma_z}{E_z} \quad (6a)$$

$$\epsilon_y = -\mu_{xy} \frac{\sigma_x}{E_y} + \frac{\sigma_y}{E_y} - \mu_{yz} \frac{\sigma_z}{E_z} \quad (6b)$$

$$\epsilon_z = -\mu_{xz} \frac{\sigma_x}{E_z} - \mu_{yz} \frac{\sigma_y}{E_z} + \frac{\sigma_z}{E_z} \quad (6c)$$

The energy of distortion may be found by subtracting from the total strain energy that energy corresponding to work done due to an equal change in volume but associated with no change in shape from that of a true cube. The increase in volume under stresses σ_x , σ_y , and σ_z is (neglecting second order effects).

$$\Delta V = \epsilon_x + \epsilon_y + \epsilon_z \quad (7)$$

An equal change in volume would exist under stresses σ_x^* , σ_y^* , σ_z^* , without distortion, if the "star" stress system is chosen so as to produce identical strains in all three directions, of magnitude ϵ^* , where

$$\epsilon^* = \frac{1}{3} (\epsilon_x + \epsilon_y + \epsilon_z) \quad (8)$$

The "star" stress system is defined by:

$$\epsilon^* = \frac{\sigma_x^*}{E_x} - \mu_{xy} \frac{\sigma_y^*}{E_y} - \mu_{xz} \frac{\sigma_z^*}{E_z} \quad (9a)$$

$$\epsilon^* = -\mu_{xy} \frac{\sigma_x^*}{E_y} + \frac{\sigma_y^*}{E_y} - \mu_{yz} \frac{\sigma_z^*}{E_z} \quad (9b)$$

$$\epsilon^* = -\mu_{xz} \frac{\sigma_x^*}{E_z} - \mu_{yz} \frac{\sigma_y^*}{E_z} + \frac{\sigma_z^*}{E_z} \quad (9c)$$

The energy associated with a change in volume is:

$$U_{\Delta V} = \frac{1}{2} (\sigma_x^* + \sigma_y^* + \sigma_z^*) \varepsilon^* \quad (10)$$

Solving equations (9) for the "star" stresses:

$$\sigma_x^* = \varepsilon^* \frac{\left(\frac{1}{E_y} - \frac{\mu_{yz}^2}{E_z} + \frac{\mu_{xy}}{E_y} + \frac{\mu_{xz} \mu_{yz}}{E_z} + \frac{\mu_{xy} \mu_{yz}}{E_y} + \frac{\mu_{xz}}{E_y} \right)}{\left(\frac{1}{E_x E_y} - \frac{\mu_{yz}^2}{E_x E_z} - \frac{\mu_{xy}^2}{E_y^2} - \frac{2 \mu_{xy} \mu_{xz} \mu_{yz}}{E_y E_z} - \frac{\mu_{xz}^2}{E_y E_z} \right)} \quad (11a)$$

$$\sigma_y^* = \varepsilon^* \frac{\left(\frac{1}{E_x} - \frac{\mu_{xz}^2}{E_z} + \frac{\mu_{xy}}{E_y} + \frac{\mu_{xz} \mu_{yz}}{E_z} + \frac{\mu_{xy} \mu_{xz}}{E_y} + \frac{\mu_{yz}}{E_x} \right)}{\left(\frac{1}{E_x E_y} - \frac{\mu_{yz}^2}{E_x E_z} - \frac{\mu_{xy}^2}{E_y^2} - \frac{2 \mu_{xy} \mu_{xz} \mu_{yz}}{E_y E_z} - \frac{\mu_{xz}^2}{E_y E_z} \right)} \quad (11b)$$

$$\sigma_z^* = \varepsilon^* \frac{\left(\frac{E_z}{E_x E_y} - \frac{E_z \mu_{xy}^2}{E_y^2} + \frac{\mu_{xz}}{E_y} + \frac{\mu_{xy} \mu_{xz}}{E_y} + \frac{\mu_{xy} \mu_{yz}}{E_y} + \frac{\mu_{yz}}{E_x} \right)}{\left(\frac{1}{E_x E_y} - \frac{\mu_{yz}^2}{E_x E_z} - \frac{\mu_{xy}^2}{E_y^2} - \frac{2 \mu_{xy} \mu_{xz} \mu_{yz}}{E_y E_z} - \frac{\mu_{xz}^2}{E_y E_z} \right)} \quad (11c)$$

Adding equations (11):

$$\begin{aligned} \epsilon^* &= \left[\begin{aligned} &\frac{1}{E_x} + \frac{1}{E_y} + \frac{E_z}{E_x E_y} - \frac{\mu_{yz}^2}{E_z} - \frac{\mu_{xz}^2}{E_z} \\ &- \frac{E_z}{E_y^2} \mu_{xy}^2 + 2 \frac{\mu_{xy}}{E_y} + \frac{\mu_{xz}}{E_y} + 2 \frac{\mu_{xz} \mu_{yz}}{E_z} \\ &+ \frac{\mu_{xy} \mu_{xz}}{E_y} + 2 \frac{\mu_{xy} \mu_{yz}}{E_y} \\ &+ \frac{\mu_{xy} \mu_{xz}}{E_y} + 2 \frac{\mu_{yz}}{E_x} + \frac{\mu_{xz}}{E_y} \end{aligned} \right] \quad (12) \\ \sigma_x^* + \sigma_y^* + \sigma_z^* &= \left[\begin{aligned} &\frac{1}{E_x E_y} - \frac{\mu_{yz}^2}{E_x E_z} - \frac{\mu_{xy}^2}{E_y^2} \\ &- 2 \frac{\mu_{xy} \mu_{xz} \mu_{yz}}{E_y E_z} - \frac{\mu_{xz}^2}{E_y E_z} \end{aligned} \right] \end{aligned}$$

ϵ^* is found from (8) and (6):

$$\epsilon^* = \frac{1}{3} \left[\begin{aligned} &+ \sigma_x \left(\frac{1}{E_x} - \frac{\mu_{xy}}{E_y} - \frac{\mu_{xz}}{E_z} \right) \\ &+ \sigma_y \left(\frac{1}{E_y} - \frac{\mu_{xy}}{E_y} - \frac{\mu_{yz}}{E_z} \right) \\ &+ \sigma_z \left(\frac{1}{E_z} - \frac{\mu_{xz}}{E_z} - \frac{\mu_{yz}}{E_z} \right) \end{aligned} \right] \quad (13)$$

The energy associated with a change in volume, from equations (10), (12), and (13) is:

$$\left[\begin{aligned} & + \sigma_x \left(\frac{1}{E_x} - \frac{\mu_{xy}}{E_y} - \frac{\mu_{xz}}{E_z} \right) \\ & + \sigma_y \left(\frac{1}{E_y} - \frac{\mu_{xy}}{E_y} - \frac{\mu_{yz}}{E_z} \right) \\ & + \sigma_z \left(\frac{1}{E_z} - \frac{\mu_{xz}}{E_z} - \frac{\mu_{yz}}{E_z} \right) \end{aligned} \right] \cdot 2 \left[\begin{aligned} & \frac{1}{E_x} + \frac{1}{E_y} + \frac{E_z}{E_x E_y} - \frac{\mu_{yz}^2}{E_z} - \frac{\mu_{xz}^2}{E_z} \\ & - \frac{E_z}{E_y^2} \mu_{xy}^2 + 2 \frac{\mu_{xy}}{E_y} + \frac{\mu_{xz}}{E_y} + 2 \frac{\mu_{xz} \mu_{yz}}{E_z} \\ & + \frac{\mu_{xy} \mu_{xz}}{E_y} + 2 \frac{\mu_{xy} \mu_{yz}}{E_y} \\ & + \frac{\mu_{xy} \mu_{xz}}{E_y} + 2 \frac{\mu_{yz}}{E_x} + \frac{\mu_{xz}}{E_y} \end{aligned} \right]$$

$U_{\Delta V} =$

$$18 \left[\begin{aligned} & \frac{1}{E_x E_y} - \frac{\mu_{yz}^2}{E_x E_z} - \frac{\mu_{xy}^2}{E_y^2} \\ & - 2 \frac{\mu_{xy} \mu_{xz} \mu_{yz}}{E_y E_z} - \frac{\mu_{xz}^2}{E_y E_z} \end{aligned} \right]$$

(14)

The Energy of Distortion is found by subtracting (14) from (2a):

$$U_{DIST} = \left[\begin{aligned} &\frac{1}{2} \left[\frac{\sigma_x^2}{E_x} + \frac{\sigma_y^2}{E_y} + \frac{\sigma_z^2}{E_z} \right] \\ &- \frac{\sigma_x \sigma_y}{E_y} \mu_{xy} \\ &- \frac{\sigma_x \sigma_z}{E_z} \mu_{xz} \\ &- \frac{\sigma_y \sigma_z}{E_z} \mu_{yz} \end{aligned} \right] - \left[\begin{aligned} &2 \left[\frac{1}{E_x} + \frac{1}{E_y} + \frac{E_z}{E_x E_y} - \frac{\mu_{yz}^2}{E_z} - \frac{\mu_{xz}^2}{E_x} \right. \\ &\quad \left. - \frac{E_z}{E_y^2} \mu_{xy}^2 + 2 \frac{\mu_{xy}}{E_y} + \frac{\mu_{xz}}{E_y} + 2 \frac{\mu_{xz} \mu_{yz}}{E_z} \right. \\ &\quad \left. + \frac{\mu_{xy} \mu_{xz}}{E_y} + 2 \frac{\mu_{xy} \mu_{yz}}{E_y} \right. \\ &\quad \left. + \frac{\mu_{xy} \mu_{xz}}{E_y} + 2 \frac{\mu_{yz}}{E_x} + \frac{\mu_{xz}}{E_y} \right] \\ &18 \left[\frac{1}{E_x E_y} - \frac{\mu_{yz}^2}{E_x E_z} - \frac{\mu_{xy}^2}{E_y^2} \right. \\ &\quad \left. - 2 \frac{\mu_{xy} \mu_{xz} \mu_{yz}}{E_y E_z} - \frac{\mu_{xz}^2}{E_y E_z} \right] \end{aligned} \right] \quad (15)$$

This method of failure prediction is based on the assumption that a limiting value of the distortion energy as described by the last equation is constant for all combinations of σ_x , σ_y , and σ_z . To apply the failure criterion, the six elastic constants must be known, plus one simple tensile strength in order to evaluate the limiting energy value.

Equation (15) reduces to the usual distortion-energy relationship presented in strength of materials texts if

$$E_x = E_y = E_z = E$$

$$\mu_{xy} = \mu_{xz} = \mu_{yz} = \mu$$

$$U_{DIST} = \frac{1+\mu}{6} \left[(\sigma_x - \sigma_y)^2 + (\sigma_x - \sigma_z)^2 + (\sigma_y - \sigma_z)^2 \right] \quad (16)$$

(ISOTROPIC)

APPLICATION TO A SEMI-ORTHOTROPIC MATERIAL

As an example of the application of equation (15), consider a material which has identical properties in two directions but different in the third. Further, assume that the applied stress is of the same magnitude in the two equal property directions but different in the third. These conditions would exist in a spherical, filament-wound vessel, such as a deep submergence marine vehicle.

Let S_z = Tangential, one-dimensional, strength

$$\sigma_x = \sigma_y = \sigma_z$$

$$\sigma_z = \sigma_z$$

$$E_x = E_y = E_z$$

$$E_z = E_z$$

$$\alpha = \mu_{xy}$$

$$\beta = \mu_{xz} = \mu_{yz}$$

The limiting value of Distortion Energy is found from equation (15)

by setting $\sigma_x = S_x$, $\sigma_y = \sigma_z = 0$.

$$U_{\text{LIMITING}} = \frac{S_x^2}{2 E_x} \left\{ 1 - \frac{\left[1 - \alpha - \beta \frac{E_t}{E_L} \right]^2 \left[(1 - \alpha) \frac{E_t}{E_L} + 2(1 + 2\beta) \right]}{9 \left(1 - \alpha - 2\beta^2 \frac{E_t}{E_L} \right)} \right\} \quad (17)$$

Note that three only elastic parameters remain, i.e., α , β , and $\frac{E_t}{E_L}$.

The Distortion Energy for the semi-orthotropic material for the loading under $\sigma_x = \sigma_y = \sigma_x$ and $\sigma_z = \sigma_L$ is also found from (15).

This becomes:

$$U_{\text{DIST}} = \frac{\sigma_x^2}{2 E_x} \left\{ +2(1 - \alpha) + \left(\frac{\sigma_L}{\sigma_x} \right)^2 \frac{E_t}{E_L} - 4\beta \left(\frac{\sigma_L}{\sigma_x} \right) \frac{E_t}{E_L} - \frac{\left[2(1 - \alpha - \beta \frac{E_t}{E_L}) + (1 - 2\beta) \left(\frac{\sigma_L}{\sigma_x} \right) \frac{E_t}{E_L} \right]^2 \left[(1 - \alpha) \frac{E_t}{E_L} + 2(1 + 2\beta) \right]}{9 \left(1 - \alpha - 2\beta^2 \frac{E_t}{E_L} \right)} \right\}$$

(18)

The failure value of σ_x is now found by equating the distortion energy (18) to the limiting value (17):

$$\sigma_x = S_c \left[\frac{1 - \frac{\left(1 - \alpha - \beta \frac{E_t}{E_L}\right)^2 \left[(1 - \alpha) \frac{E_t}{E_L} + 2(1 + 2\beta) \right]}{9 \left(1 - \alpha - 2\beta^2 \frac{E_t}{E_L}\right)}}{\left[\begin{aligned} &+ 2(1 - \alpha) \\ &+ \left(\frac{\sigma_r}{\sigma_x}\right)^2 \frac{E_t}{E_L} \\ &- 4\beta \frac{\sigma_r}{\sigma_x} \frac{E_t}{E_L} \end{aligned} \right]} - \frac{\left[2\left(1 - \alpha - \beta \frac{E_t}{E_L}\right) + (1 - 2\beta) \frac{\sigma_r}{\sigma_x} \frac{E_t}{E_L} \right]^2 \left[(1 - \alpha) \frac{E_t}{E_L} + 2(1 + 2\beta) \right]}{9 \left(1 - \alpha - 2\beta^2 \frac{E_t}{E_L}\right)} \right]$$

(19)

Experimental verification of this theory of failure is not yet established. Exposure of a test specimen to two equal compressive stresses (by using fluid pressure) and one tensile or compressive stress (applied mechanically) orthogonal to the compressive stresses could easily be carried out in an experimental program planned to evaluate the proposed theory.

As a point of interest, the collapse of equation (19) for an isotropic material exposed to an equal bi-axial stress field may be examined.

For this degenerate case,

$$\sigma_R = 0$$

$$\alpha = \beta = \mu$$

$$\frac{F_x}{F_R} = 1$$

And equation (19) becomes simply

$$\sigma_x = S_x ,$$

which suggests that no loss in strength results as a consequence of two-dimensional loading. This conclusion coincides with the findings of usual theories of failure.