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Coupled Partial Differential Equations for
Trinucleon "Radial" Wave Functions*

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The differential equations for the triton are derived by
procedures that can be used also for other values of $TJ\pi$.

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I. INTRODUCTION AND RESULTS

In a previous paper,¹ it was shown that the wave function for a zero-momentum three-nucleon state with quantum numbers $TJ\pi$ takes the form

$$\Psi^{TJ\pi} = \sum_{\text{possible } S, L, R, Y} g_{RY}^{SL} \Psi_{RY}^{TSLJ\pi}, \quad (1)$$

where the radial functions g_{RY}^{SL} are functions of the symmetric scalar variables in the three-body problem, and the functions $\Psi_{RY}^{TSLJ\pi}$ are the orthonormal vector harmonics for the three-body system. More explicitly, with

$$\underline{\lambda} = 6^{-1/2}(\underline{r}_1 + \underline{r}_2 - 2\underline{r}_3) \quad (2)$$

$$\underline{p} = 2^{-1/2}(\underline{r}_1 - \underline{r}_2)$$

and with r , s , and φ defined by

$$r = \lambda^2 + p^2$$

$$rs \cos \varphi = p^2 - \lambda^2 \quad 0 \leq s \leq 1 \quad (3)$$

$$rs \sin \varphi = 2p \cdot \underline{\lambda} \quad -\pi \leq \varphi \leq \pi,$$

it was shown that the symmetric scalar variables are r , s , and $\cos 3\varphi$, and that the vector harmonics are of the form

$$\Psi_{RY}^{TSLJ\pi} = \left\{ (TS)_{\underline{R}}^{\underline{\lambda}} \otimes f_{Y\underline{R}}^L \right\}_A^J. \quad (4)$$

In (4), f_Y^L is a function of $\underline{p}$ and $\underline{\lambda}$ that is a spherical tensor of order L

and symmetric under interchange of particle coordinates; it is vector-coupled with resultant J to the spin part of the isospin-spin function $(TS)\widetilde{R}$, which belongs to the representation $\widetilde{R}$ of $S_3^{\tau\sigma}$. The function S_R determines the representation of S_3^{τ} to which the space part of $\psi_{RY}^{TSLJ\pi}$ belongs; the subscript R takes on the values S, M, N, A , with

$$S_S = 1$$

$$S_M = (\cos \varphi, \sin \varphi)$$

(5)

$$S_N = S_A (\sin \varphi, -\cos \varphi)$$

$$S_A = (1 - \cos^2 3\varphi)^{-1/2} \sin 3\varphi.$$

The functions S_S and S_A belong to the symmetric and antisymmetric representations of S_3^{τ} , respectively, and S_M and S_N both belong to the mixed representation of S_3^{τ} in standard form (see I). The representation $\widetilde{R}$ is adjoint to R :

$$\widetilde{S} = A, \quad \widetilde{A} = S, \quad \widetilde{M} = \widetilde{N} = M. \quad (6)$$

The symbol $\otimes$ and the subscript A in (4) indicate that the product of $(TS)\widetilde{R}$ and S_R is to be formed in such a way as to belong to the antisymmetric representation of $S_3^{\tau\sigma}$. It was shown in I that the vector harmonics can be chosen orthonormal:

$$\sum_{i,j} \int d\omega \psi_{R'Y'}^{T'S'L'J'\pi'^*} \psi_{RY}^{TSLJ\pi} = \delta_{T,T'} \delta_{S,S'} \delta_{L,L'} \delta_{\pi,\pi'} \delta_{R,R'} \delta_{Y,Y'}, \quad (7)$$

where m values are ignored throughout, and the integral $d\omega$ stands for

multiplication by $3/32$ and integration over the solid angle of $\underline{p}$ and the azimuthal angle of $\underline{\lambda}$ about $\underline{p}$. It follows from (7) and (1) that

$$\sum_{\tau_i \sigma_i} \int d\omega \psi_{RY}^{T'SLJ'\pi'^*} \Psi^{TJ\pi} = \delta_{T,T'} \delta_{J,J'} \delta_{\pi,\pi'} g_{RY}^{SL} \quad (8)$$

and, hence, that the radial equations for the three-nucleon system are

$$\sum_{\tau_i \sigma_i} \int d\omega \psi_{RY}^{TSLJ\pi^*} (H - E) \Psi = 0 \quad (9)$$

Only radial functions g_{RY}^{SL} belonging to a particular $TJ\pi$ are coupled.

In this paper, the radial equations for $TJ\pi = \frac{1}{2} \frac{1}{2} +$, corresponding to the bound state of the trinucleon, are derived, that is, the equations

$$\sum_{\tau_i \sigma_i} \int d\omega \psi_{RY}^{1/2 SL 1/2 +*} (H - E) \Psi^{1/2 1/2 +} = 0 \quad (10)$$

As noted in I, there are sixteen independent vector harmonics with $TJ\pi = \frac{1}{2} \frac{1}{2} +$, and, hence, (10) gives sixteen coupled partial differential equations in three variables.

In order to express the radial equations in the simplest form it is convenient to extract factors of S_A from some of the representation functions and spherical tensors and define

$$S_R = K_R Q_R$$

(11)

$$f_Y^{L\pi} = K_Y^{L\pi} e_Y^{L\pi}$$

with K equal to 1 or S_A , and $K^2 = 1$. Then (1) can be

written

$$\Psi^{TJ\pi} = \sum_{S,L,R,Y} \left\{ (TS)_{\tilde{R}}^{\otimes}, \quad Q_R e_Y^{L\pi} h_{RY}^{SL} \right\}, \quad (12)$$

with

$$h_{RY}^{SL} = K_R K_Y^{L\pi} g_{RY}^{SL}, \quad (13)$$

and, with

$$\chi_{RY}^{TSLJ\pi} = K_R K_Y^{L\pi} \Psi_{RY}^{TSLJ\pi}, \quad (14)$$

(7) and (10) become

$$\sum_{i,\sigma_i} \int d\omega \chi_{R'Y'}^{T'S'L'J'\pi'^*} \chi_{RY}^{TSLJ\pi} = \delta_{T,T'} \delta_{S,S'} \delta_{L,L'} \delta_{J,J'} \delta_{\pi,\pi'} \delta_{R,R'} \delta_{Y,Y'} \quad (15)$$

$$\sum_{i,\sigma_i} \int d\omega \chi_{RY}^{TSLJ\pi*} (H - E) \Psi^{TJ\pi} = 0 \quad (16)$$

Note that χ belongs to either the symmetric or antisymmetric representation of S_3^{tor} ; only $h\chi$ belongs always to the antisymmetric representation. However, since S_A is discontinuous, it is convenient for purposes of differentiation to incorporate it with g to form h .

Equations (5) and (11) give

$$K_S = K_M = I, \quad K_N = K_A = S_A \quad (17)$$

$$Q_S = Q_A = I, \quad Q_M = (\cos \varphi, \sin \varphi), \quad Q_N = (\sin \varphi, -\cos \varphi),$$

while the results of I can be expressed in the notation of (11)

as

$$e^{0+} = \{0+\} M^{0+}$$

$$e^{1+} = \{1+\} M^{1+}/r \quad (18)$$

$$e_j^{2+} = \sum_{i=1}^3 \{2+\}_i M_{ij}^{2+}/r, \quad j = 1, 2, 3,$$

where

$$\{0+\} = 1$$

$$\{1+\} = \{\underline{\lambda}, \underline{p}\}^1$$

(19)

$$\{2+\}_1 = \{\underline{\lambda}, \underline{\lambda}\}^2$$

$$\{2+\}_2 = \{\underline{p}, \underline{p}\}^2$$

$$\{2+\}_3 = \{\underline{\lambda}, \underline{p}\}^2,$$

and

$$M^{0+} = 2/3^{1/2} \pi \quad M^{1+} = (4/\pi) (2/1 - s^2)^{1/2}$$

$$M^{2+} = \frac{2(10/3)^{1/2}}{\pi(1-s^2)^{1/2}} \begin{pmatrix} \frac{3^{1/2}(1+s \cos \varphi)}{(1-s^2)^{1/2}} & \frac{s + \cos \varphi}{(1-s^2)^{1/2}} & \sin \varphi \\ \frac{3^{1/2}(1-s \cos \varphi)}{(1-s^2)^{1/2}} & \frac{s - \cos \varphi}{(1-s^2)^{1/2}} & -\sin \varphi \\ \frac{-2 \cdot 3^{1/2} s \sin \varphi}{(1-s^2)^{1/2}} & \frac{2 \sin \varphi}{(1-s^2)^{1/2}} & 2 \cos \varphi \end{pmatrix} \quad (20)$$

and, as in I, the spherical tensor components of the vector $\underline{A}$ are taken

to be

$$A_0 = -iA_z \quad (21)$$

$$A_{\pm 1} = 2^{-1/2} (\pm iA_x - A_y) .$$

The factors

$$K^{0+} = K_1^{2+} = K_2^{2+} = 1 \quad (22)$$

$$K^{1+} = K_3^{2+} = S_A$$

have been extracted.

Consider first the kinetic energy T:

$$-T = \sum_i^3 \nabla_i^2 = \frac{1}{3} \nabla_{\underline{R}}^2 + \nabla_{\underline{p}}^2 + \nabla_{\underline{\lambda}}^2 \quad (23)$$

(units $\hbar = 2m = 1$), where

$$\underline{R} = \frac{1}{3} \sum_i^3 \underline{r}_i \quad (24)$$

is the center-of-mass coordinate; $\underline{R}$ does not appear in the internal wave function $\Psi^{TJ\pi}$, and it is only necessary to evaluate $t\Psi$, with

$$t = \nabla_{\underline{p}}^2 + \nabla_{\underline{\lambda}}^2 \quad (25)$$

Clearly,

$$t\Psi^{TJ\pi} = \sum_{SLRY} \left\{ (TS)_{\underline{R}} \otimes, t(Q_R e_Y^{L\pi} h_{RY}^{SL}) \right\} . \quad (26)$$

Since t is a symmetric scalar operator, it can change neither L nor the representation of S_3^L to which Q_R belongs. Hence,

$$t(Q_R e_Y^{L\pi} h_{RY}^{SL}) = \sum_{\beta} \sum_{\substack{R' \\ \underline{R}' = \underline{R}}} Q_{R'} e_{\beta}^{L\pi} (R' \beta | t | R_Y)^{L\pi} h_{RY}^{SL} , \quad (27)$$

where allowance has been made for the fact that t can couple the representations M and N . Equations (26) and (27) give

$$\sum_{i, \sigma_i} \int d\omega \chi_{R\gamma}^{TSLJ\pi*} t \psi^{TJ\pi} = \sum_{R'\beta} (R\gamma | t | R'\beta) L_{R'\beta}^{T\pi SL} \quad (28)$$

The operator matrices $(R\gamma | t | R'\beta) L_{R'\beta}^{T\pi}$ are shown in section II to be

$$(R | t | R')^{0+} = 4 \begin{pmatrix} k & 0 & 0 & 0 \\ 0 & k - \frac{1}{rs^2} & \frac{2}{rs^2} \frac{\partial}{\partial \varphi} & 0 \\ 0 & -\frac{2}{rs^2} \frac{\partial}{\partial \varphi} & k - \frac{1}{rs^2} & 0 \\ 0 & 0 & 0 & k \end{pmatrix} \quad (29)$$

$$(R | t | R')^{1+} = 4 \begin{pmatrix} k - \frac{1}{r(1-s^2)} & 0 & 0 & 0 \\ 0 & k - \frac{1}{rs^2(1-s^2)} & \frac{2}{rs^2} \frac{\partial}{\partial \varphi} & 0 \\ 0 & -\frac{2}{rs^2} \frac{\partial}{\partial \varphi} & k - \frac{1}{rs^2(1-s^2)} & 0 \\ 0 & 0 & 0 & k - \frac{1}{r(1-s^2)} \end{pmatrix} \quad (30)$$

$$\begin{aligned}
 (R_Y | t | R'_\beta)^{2+} = & \left(\begin{array}{cccccc}
 k - \frac{(1+2s^2)}{rs^2(1-s^2)} & \frac{2}{rs^2} \frac{\partial}{\partial \varphi} & -\frac{3^{\frac{1}{2}}s}{r(1-s^2)} & 0 & 0 & 0 \\
 -\frac{2}{rs^2} \frac{\partial}{\partial \varphi} & k - \frac{(1+2s^2)}{rs^2(1-s^2)} & 0 & -\frac{3^{\frac{1}{2}}s}{r(1-s^2)} & 0 & 0 \\
 -\frac{3^{\frac{1}{2}}s}{r(1-s^2)} & 0 & k - \frac{(2-s^2)}{rs^2(1-s^2)} & \frac{2}{rs^2} \frac{\partial}{\partial \varphi} & \frac{2(1-s^2)^{\frac{1}{2}}}{rs^2} \frac{\partial}{\partial \varphi} & \frac{2(1-s^2)^{\frac{1}{2}}}{rs^2} \\
 0 & -\frac{3^{\frac{1}{2}}s}{r(1-s^2)} & -\frac{2}{rs^2} \frac{\partial}{\partial \varphi} & k - \frac{(2-s^2)}{rs^2(1-s^2)} & -\frac{2(1-s^2)^{\frac{1}{2}}}{rs^2} & \frac{2(1-s^2)^{\frac{1}{2}}}{rs^2} \frac{\partial}{\partial \varphi} \\
 0 & 0 & -\frac{2(1-s^2)^{\frac{1}{2}}}{rs^2} \frac{\partial}{\partial \varphi} & -\frac{2(1-s^2)^{\frac{1}{2}}}{rs^2} & k - \frac{(2-s^2)}{rs^2(1-s^2)} & \frac{2}{rs^2} \frac{\partial}{\partial \varphi} \\
 0 & 0 & \frac{2(1-s^2)^{\frac{1}{2}}}{rs^2} & -\frac{2(1-s^2)^{\frac{1}{2}}}{rs^2} \frac{\partial}{\partial \varphi} & -\frac{2}{rs^2} \frac{\partial}{\partial \varphi} & k - \frac{(2-s^2)}{rs^2(1-s^2)}
 \end{array} \right) \quad (31)
 \end{aligned}$$

where the order of the representations is S, M, N, A for $0+$ and $1+$, and the order for $2+$ is $M1, N1, M2, N2, M3, N3$. The operator k in (29)* - (31) is essentially the scalar part of t :

$$k = r \frac{\partial^2}{\partial r^2} + 3 \frac{\partial}{\partial r} + \frac{1-s^2}{r} \frac{\partial^2}{\partial s^2} + \frac{1-3s^2}{rs} \frac{\partial}{\partial s} + \frac{1}{rs^2} \frac{\partial^2}{\partial \varphi^2} \quad (32)$$

The potential term in the Hamiltonian is taken to consist of central and tensor parts. The spin-orbit term is neglected, since it is assumed that the relative energies involved in the three-nucleon

bound state are low; its contribution can presumably be calculated by perturbation theory. If the relative energies should turn out to be large (> 150 MeV), the spin orbit term could also be included in the differential equations. The potential energy is

$$V = V_C + V_T$$

$$V_C = \Phi V(1, 2) \quad V_T = \Phi W(1, 2)$$

$$\Phi = 1 + P^{123} + P^{132} \quad (33)$$

$$V(1, 2) = \sum_{t=0}^1 \sum_{s=0}^1 V^{\dagger s}(r_{12}^2) P^{\dagger s}(1, 2)$$

$$W(1, 2) = \sum_{t=0}^1 W^{\dagger}(r_{12}^2) S_{12} P^{\dagger}(1, 2),$$

with

$$P^{\dagger s} = P^{\dagger} P^s$$

$$P^{\dagger=0} = \frac{1}{4}(1 - \tau_1 \cdot \tau_2) \quad P^{s=0} = \frac{1}{4}(1 - \sigma_1 \cdot \sigma_2)$$

$$P^{\dagger=1} = \frac{1}{4}(3 + \tau_1 \cdot \tau_2) \quad P^{s=1} = \frac{1}{4}(3 + \sigma_1 \cdot \sigma_2)$$

(34)

$$S_{12} = \left\{ \mathcal{J}^{(2)}(1, 2), \quad \mathcal{L}^{(2)}(1, 2) \right\}^0$$

$$\mathcal{J}^{(2)}(1, 2) = \left\{ \underline{g}(1), \quad \underline{g}(2) \right\}^2$$

$$\mathcal{L}^{(2)}(1, 2) = \frac{1}{r_{12}^2} \left\{ \underline{r}_1 - \underline{r}_2, \quad \underline{r}_1 - \underline{r}_2 \right\}^2.$$

The potential terms are most simply evaluated by using

$$\tilde{r}_{12} = 2^{1/2} \underline{p} \quad (35)$$

to calculate $(V(1, 2) + W(1, 2))\Psi$. Then

$$V\Psi = \mathcal{P}(V(1, 2) + W(1, 2))\Psi. \quad (36)$$

For a central component of V ,

$$V^{\dagger s}(r_{12}^2)P^{\dagger s}(1, 2)\Psi^{TJ\pi} \quad (37)$$

$$= V^{\dagger s}(r_{12}^2) \sum_{SLR_Y} \left\{ P^{\dagger s}(1, 2) \left((TS)_{\tilde{R}} \otimes S_R \right)_A, K_{R_Y}^{L\pi} h_{R_Y}^{SL} \right\}^J,$$

and

$$P^{\dagger s}(1, 2) \left((TS)_{\tilde{R}} \otimes S_R \right)_A = \sum_{R'} \left((TS)_{\tilde{R}'} \otimes S_{R'} \right) K_{R'} K_R (R' | P^{\dagger s} | R)^{TS}, \quad (38)$$

where $(R' | P^{\dagger s} | R)^{TS}$ need not belong to the symmetric representation of S_3^L . Then (37) and (38) give

$$\sum_{i\sigma_i} \int d\omega \chi_{R_Y}^{TSLJ\pi*} V_C \Psi^{TJ\pi} = \sum_{R'} (R | V | R')^{TS} h_{R'Y}^{SL} \quad (39)$$

with

$$(R | V | R')^{TS} = \sum_{ts} \mathcal{P} V^{\dagger s}(r(1 + s \cos \varphi)) (R | P^{\dagger s} | R')^{TS} \quad (40)$$

In order to present $(R | V | R')^{TS}$ in concise form, it is convenient to define

$$P_{\varphi} f(r, s, \varphi) = f(r, s, \varphi) + f(r, s, \varphi + \frac{2\pi}{3}) + f(r, s, \varphi + \frac{4\pi}{3}) \quad (41)$$

and

$$V^{0I} \pm V^{I0} = V^{e\pm} \quad V^{00} \pm V^{II} = V^{o\pm} \quad (42)$$

Then it is shown in section III that

$$(R|V|R)^{\frac{1}{2}\frac{1}{2}} = \frac{1}{2} P \begin{pmatrix} V^{e+} & V^{e-} \cos \varphi & V^{e-} \sin \varphi & 0 \\ V^{e-} \cos \varphi & V^{e+} \cos^2 \varphi + V^{o+} \sin^2 \varphi & (V^{e+} - V^{o+}) \cos \varphi \sin \varphi & V^{o-} \sin \varphi \\ V^{e-} \sin \varphi & (V^{e+} - V^{o+}) \cos \varphi \sin \varphi & V^{e+} \sin^2 \varphi + V^{o+} \cos^2 \varphi & -V^{o-} \cos \varphi \\ 0 & V^{o-} \sin \varphi & -V^{o-} \cos \varphi & V^{o+} \end{pmatrix} \quad (43)$$

$$(R|V|R)^{\frac{1}{2}\frac{3}{2}} = P_{\varphi} \begin{pmatrix} V^{0I} \cos^2 \varphi + V^{II} \sin^2 \varphi & (V^{0I} - V^{II}) \cos \varphi \sin \varphi \\ (V^{0I} - V^{II}) \cos \varphi \sin \varphi & V^{0I} \sin^2 \varphi + V^{II} \cos^2 \varphi \end{pmatrix}, \quad (44)$$

where the order for $TS = \frac{1}{2} \frac{1}{2}$ is S, M, N, A and for $TS = \frac{1}{2} \frac{3}{2}$ is M, N.

For a tensor component of V,

$$\begin{aligned} & W^{\dagger}(r_{12}^2) S_{12} P^{\dagger}(1,2) \Psi^{TJ\pi} \\ &= W^{\dagger}(r_{12}^2) \sum_{SLR\gamma} \left\{ \lambda^{(2)}(1,2), \lambda^{(2)}(1,2) \right\}^0 \left\{ P^{\dagger}(1,2) ((TS) \sim \otimes S_R)_{A,K_R} e^{L\pi} \gamma_{R\gamma}^{SL} \right\}^J \end{aligned}$$

$$= W^\dagger(r_{12}^2) \sum_{SS'LL'R_Y} (-)^{J+L+S'} \left[(2S' + 1)(2L' + 1)/5 \right]^{1/2} \left\{ \begin{matrix} 2 & L & L' \\ J & S & S' \end{matrix} \right\}. \quad (45)$$

$$\cdot \left\{ \left\{ \lambda^{(2)}(1, 2), P^\dagger(1, 2) ((TS)_{\tilde{R}} \otimes S_R)_A \right\}^{S'}, \left\{ \mathcal{L}^{(2)}(1, 2), K_R e_Y^{L\pi} h_{RY}^{SL} \right\}^{L'} \right\}^J,$$

where the $6j$ symbol arises from the recoupling. As in (38),

$$(-)^{S'+\frac{1}{2}} (2S' + 1)^{1/2} \left\{ \lambda^{(2)}(1, 2), P^\dagger(1, 2) ((TS)_{\tilde{R}} \otimes S_R)_A \right\}^{S'} = \quad (46)$$

$$= \sum_{R'} ((TS')_{\tilde{R}'} \otimes S_{R'})_A K_{R'} K_R (S'R' | \lambda^2 P^\dagger | SR)^T$$

and

$$(-)^L (2L' + 1/5)^{1/2} \left\{ \mathcal{L}^{(2)}(1, 2), e_Y^{L\pi} \right\}^{L'} = \sum_{\beta} e_{\beta}^{L'\pi} (L'\beta | \mathcal{L}^2 | LY)^{\pi}, \quad (47)$$

so that

$$\sum_{\tau_i \sigma_i} \int d\omega \chi_{RY}^{TSLJ\pi*} V_T \Psi^{TJ\pi} = \sum_{S'L'R'\beta} (SLR_Y | W | S'L'R'\beta)^{TJ\pi} h_{R'\beta}^{S'L'}, \quad (48)$$

where

$$(SLR_Y | W | S'L'R'\beta) = \sum_t \mathcal{P} W^\dagger(r(1 + s \cos \varphi)) (-)^{J-\frac{1}{2}} \left\{ \begin{matrix} 2 & L' & L \\ J & S & S' \end{matrix} \right\} \quad (49)$$

$$\cdot (SR | \lambda^2 P^\dagger | S'R')^T (LY | \mathcal{L}^2 | L'\beta)^{\pi}.$$

It is shown in section III that

$$\sum_t W^\dagger \cdot (SR | \lambda^2 P^\dagger | S'R')^T = (SR | X | S'R')^T 5^{1/2} (2S + 1)(2S' + 1) \left\{ \begin{matrix} 2 & 1 & 1 \\ \frac{1}{2} & S & S' \end{matrix} \right\}^{1/2}, \quad (50)$$

where

$$\left(\frac{1}{2}R|X|\frac{3}{2}R'\right)^{1/2} = \begin{pmatrix} w^0 \cos \varphi & w^0 \sin \varphi \\ w^0 \cos^2 \varphi - w^1 \sin^2 \varphi & (w^0 + w^1) \cos \varphi \sin \varphi \\ (w^0 + w^1) \cos \varphi \sin \varphi & w^0 \sin^2 \varphi - w^1 \cos^2 \varphi \\ w^1 \sin \varphi & -w^1 \cos \varphi \end{pmatrix} \quad (51)$$

$$\left(\frac{3}{2}R|X|\frac{3}{2}R'\right)^{1/2} = \begin{pmatrix} w^0 \cos^2 \varphi + w^1 \sin^2 \varphi & (w^0 - w^1) \cos \varphi \sin \varphi \\ (w^0 - w^1) \cos \varphi \sin \varphi & w^0 \sin^2 \varphi + w^1 \cos^2 \varphi \end{pmatrix},$$

where the order for $S = \frac{1}{2}$ is S, M, N, A and for $S = \frac{3}{2}$ is M, N. Then

(49) becomes

$$(SLR|Y|S'L'R'\beta) = P_\varphi (SR|X|S'R')^T (SLY|Y|S'L'\beta)^{J\pi} \quad (52)$$

with

$$(SLY|Y|S'L'\beta)^{J\pi} = (-)^{J-\frac{1}{2}} 5^{1/2} (2S+1)(2S'+1) \begin{Bmatrix} 2 & 1 & 1 \\ 1 & S & S' \end{Bmatrix} \begin{Bmatrix} 2 & L' & L \\ J & S & S' \end{Bmatrix} (LY|\alpha^2|L'\beta)^{\pi/2} \quad (53)$$

This latter is most easily presented in terms of

$$\alpha = \frac{s + \cos \varphi}{1 + s \cos \varphi} \quad \beta = (1 - s^2)^{1/2} \frac{\sin \varphi}{1 + s \cos \varphi} \quad (54)$$

$$\alpha^2 + \beta^2 = 1,$$

and is shown in section III to be

$$(\text{SLY}|\text{Y}| \text{S}'\text{L}'\beta) = \frac{1}{3 \cdot 5^{1/2}} \begin{pmatrix} 0 & 0 & 0 & 1 & -3^{1/2}\alpha & -3^{1/2}\beta \\ 0 & 0 & 1 & 0 & -3^{1/2}\beta & 3^{1/2}\alpha \\ 0 & 1 & 1 & 0 & 3^{1/2}\beta & -3^{1/2}\alpha \\ 1 & 0 & 0 & 1 & 3^{1/2}\alpha & 3^{1/2}\beta \\ -3^{1/2}\alpha & -3^{1/2}\beta & 3^{1/2}\beta & 3^{1/2}\alpha & -1 & 0 \\ -3^{1/2}\beta & 3^{1/2}\alpha & -3^{1/2}\alpha & 3^{1/2}\beta & 0 & -1 \end{pmatrix} \quad (55)$$

where the order of SLY is $\frac{1}{2}0, \frac{1}{2}1, \frac{3}{2}1, \frac{3}{2}21, \frac{3}{2}22, \frac{3}{2}23$.

Finally, with (28), (39), and (48), the differential equations

(16) are

$$\begin{aligned} & - \sum_{R'\beta} (\text{RY}|\text{t}|\text{R}'\beta)^{\text{L}\pi} h_{\text{R}'\beta}^{\text{SL}} + \sum_{\text{R}'} (\text{R}|\text{V}|\text{R}')^{\text{TS}} h_{\text{R}'\text{Y}}^{\text{SL}} \\ & + \sum_{\text{S}'\text{L}'\text{R}'\beta} (\text{SLRY}|\text{W}|\text{S}'\text{L}'\text{R}'\beta) h_{\text{R}'\beta}^{\text{S}'\text{L}'} - E h_{\text{RY}}^{\text{SL}} = 0 \end{aligned} \quad (55)$$

For $\text{TJ}\pi = \frac{1}{2} \frac{1}{2} +$, there are sixteen equations (55), one for each value of the set SLRY.

II. KINETIC ENERGY

In order to evaluate the kinetic energy matrix elements, let

$$\begin{aligned} u(f_1, f_2) &= \nabla_{\underline{\lambda}} f_1 \cdot \nabla_{\underline{\lambda}} f_2 + \nabla_{\underline{p}} f_1 \cdot \nabla_{\underline{p}} f_2 = u(f_2, f_1) \\ t(f) &= (\nabla_{\underline{\lambda}}^2 + \nabla_{\underline{p}}^2) f \end{aligned} \quad (56)$$

Then Equations (3) give

$$\begin{aligned}
 u(r, s) &= u(r, \varphi) = u(s, \varphi) = 0 \\
 u(r, r) &= 4r \quad u(s, s) = \frac{4}{r}(1 - s^2) \quad u(\varphi, \varphi) = \frac{4}{rs^2} \\
 t(r) &= 12 \quad t(s) = \frac{4}{rs}(1 - 3s^2) \quad t(\varphi) = 0
 \end{aligned} \tag{57}$$

Also, it follows that

$$t(\{L\pi\}_i) = 0, \tag{58}$$

since

$$\{L\pi\}_i \propto \{\lambda^L \gamma^L(\hat{\lambda}), \rho^{L-L} \gamma^{L-L}(\hat{\rho})\}^L \quad l = 0, 1, \dots, L \tag{59}$$

for $\pi = (-)^L$ and

$$\{L\pi\}_i \propto \{\lambda^{L+1} \gamma^{L+1}(\hat{\lambda}), \rho^{L-L} \gamma^{L-L}(\hat{\rho})\}^L \quad l = 0, 1, \dots, L-1 \tag{60}$$

for $\pi = (-)^{L-1}$, while

$$\nabla_{\lambda}^2 \lambda^L \gamma^L(\hat{\lambda}) = 0. \tag{61}$$

Now

$$t(Q_S eh) = Q_S t(eh)$$

$$t(Q_A eh) = Q_A t(eh)$$

$$t(Q_M eh) = Q_M t(eh) + eh t(Q_M) + 2u(Q_M, eh)$$

$$t(Q_N eh) = Q_N t(eh) + eh t(Q_N) + 2u(Q_N, eh).$$

(62)

Since Q_M and Q_N depend only on φ , and have the form given in (17),

$$u(Q_M, eh) = \frac{dQ_M}{d\varphi} u(\varphi, eh) = -Q_N u(\varphi, eh)$$

$$u(Q_N, eh) = Q_M u(\varphi, eh) \quad (63)$$

$$t(Q_M) = \frac{dQ_M}{d\varphi} t(\varphi) + \frac{d^2 Q_M}{d\varphi^2} u(\varphi, \varphi) = -\frac{4}{rs^2} Q_M$$

$$t(Q_N) = -\frac{4}{rs^2} Q_N .$$

Now define $t_{\beta\gamma}^{L\pi}$ and $u_{\beta\gamma}^{L\pi}$ by

$$t(e_Y^{L\pi} h) = \sum_{\beta} e_{\beta}^{L\pi} t_{\beta\gamma}^{L\pi} h \quad (64)$$

$$u(\varphi, e_Y^{L\pi} h) = \sum_{\beta} e_{\beta}^{L\pi} u_{\beta\gamma}^{L\pi} h .$$

Then (27), (62), (63), and (64) give

$$(S\beta | t | S_Y)^{L\pi} = (A\beta | t | A_Y)^{L\pi} = t_{\beta\gamma}^{L\pi}$$

$$(M\beta | t | M_Y)^{L\pi} = (N\beta | t | N_Y)^{L\pi} = t_{\beta\gamma}^{L\pi} - \frac{4}{rs^2} \delta_{\beta,\gamma} \quad (65)$$

$$-(N\beta | t | M_Y)^{L\pi} = (M\beta | t | N_Y)^{L\pi} = 2u_{\beta\gamma}^{L\pi} .$$

Equations (56) give

$$\begin{aligned} t(e_Y^{L\pi} h) &= e_Y^{L\pi} t(h) + h t(e_Y^{L\pi}) + 2u(h, e_Y^{L\pi}) \\ &= e_Y^{L\pi} t(h) + 2 \left[u(r, e_Y^{L\pi}) \frac{\partial}{\partial r} + u(s, e_Y^{L\pi}) \frac{\partial}{\partial s} + u(\varphi, e_Y^{L\pi}) \frac{\partial}{\partial \varphi} \right] h \\ &\quad + h t(e_Y^{L\pi}) \end{aligned}$$

$$= \left[e_Y^{L\pi} \left(4r \frac{\partial^2}{\partial r^2} + \frac{4(1-s^2)}{r} \frac{\partial^2}{\partial s^2} + \frac{4}{rs^2} \frac{\partial^2}{\partial \varphi^2} \right) + (12e_Y^{L\pi} + 2u(r, e_Y^{L\pi})) \frac{\partial}{\partial r} \right. \\ \left. + \left(\frac{4(1-3s^2)}{rs} e_Y^{L\pi} + 2u(s, e_Y^{L\pi}) \right) \frac{\partial}{\partial s} + 2u(\varphi, e_Y^{L\pi}) \frac{\partial}{\partial \varphi} + (e_Y^{L\pi}) \right] h \quad (66)$$

$$u(\varphi, e_Y^{L\pi} h) = \left[\frac{4}{rs^2} e_Y^{L\pi} \frac{\partial}{\partial \varphi} + u(\varphi, e_Y^{L\pi}) \right] h .$$

Now it can be derived from Equations (3) that

$$u(r, f) = 2(\underline{\lambda} \cdot \underline{\nabla}_{\underline{\lambda}} + \underline{\rho} \cdot \underline{\nabla}_{\underline{\rho}}) f \\ u(s, f) = \left[-\frac{2s}{r} (\underline{\lambda} \cdot \underline{\nabla}_{\underline{\lambda}} + \underline{\rho} \cdot \underline{\nabla}_{\underline{\rho}}) - \frac{2 \cos \varphi}{r} (\underline{\lambda} \cdot \underline{\nabla}_{\underline{\lambda}} - \underline{\rho} \cdot \underline{\nabla}_{\underline{\rho}}) \right. \\ \left. + \frac{2 \sin \varphi}{r} (\underline{\rho} \cdot \underline{\nabla}_{\underline{\lambda}} + \underline{\lambda} \cdot \underline{\nabla}_{\underline{\rho}}) \right] f \quad (67) \\ u(\varphi, f) = \left[\frac{2 \sin \varphi}{rs} (\underline{\lambda} \cdot \underline{\nabla}_{\underline{\lambda}} - \underline{\rho} \cdot \underline{\nabla}_{\underline{\rho}}) + \frac{2 \cos \varphi}{rs} (\underline{\rho} \cdot \underline{\nabla}_{\underline{\lambda}} + \underline{\lambda} \cdot \underline{\nabla}_{\underline{\rho}}) \right] f ,$$

whence it is seen that if f is homogeneous of degree n in $\underline{\rho}$ and $\underline{\lambda}$, then

$$u(r, f) = 2nf , \quad (68)$$

and, therefore

$$u(r, e_Y^{L\pi}) = 0 \\ u(s, e_Y^{L\pi}) = u'(s, e_Y^{L\pi}) \equiv \left[-\frac{2 \cos \varphi}{r} (\underline{\lambda} \cdot \underline{\nabla}_{\underline{\lambda}} - \underline{\rho} \cdot \underline{\nabla}_{\underline{\rho}}) \right. \\ \left. + \frac{2 \sin \varphi}{r} (\underline{\rho} \cdot \underline{\nabla}_{\underline{\lambda}} + \underline{\lambda} \cdot \underline{\nabla}_{\underline{\rho}}) \right] e_Y^{L\pi} . \quad (69)$$

Note that

$$u'(s, s) = u(s, s), \quad u'(s, \varphi) = u(s, \varphi)$$

(70)

$$u'(s, r) = u(s, r) + 4s = 4s.$$

Therefore

$$\begin{aligned} u'(s, e_Y^{L\pi}) &= \sum_i u'(s, \{L\pi\}_i) r^{-n_{L\pi}} M_{iY}^{L\pi} \\ &= \sum_i \left[\{L\pi\}_i r^{-n_{L\pi}} u'(s, M_{iY}^{L\pi}) + \{L\pi\}_i M_{iY}^{L\pi} u'(s, r^{-n_{L\pi}}) \right. \\ &\quad \left. + u'(s, \{L\pi\}_i) r^{-n_{L\pi}} M_{iY}^{L\pi} \right] \\ &= -\frac{4n_{L\pi}s}{r} e_Y^{L\pi} + r^{-n_{L\pi}} \sum_i \left[\{L\pi\}_i u'(s, M_{iY}^{L\pi}) + u'(s, \{L\pi\}_i) M_{iY}^{L\pi} \right] \\ u(\varphi, e_Y^{L\pi}) &= r^{-n_{L\pi}} \sum_i \left[\{L\pi\}_i u(\varphi, M_{iY}^{L\pi}) + u(\varphi, \{L\pi\}_i) M_{iY}^{L\pi} \right]. \end{aligned} \quad (71)$$

With the definitions

$$u'(s, \{L\pi\}_i) = \sum_j \{L\pi\}_j s_{ji} \quad (72)$$

$$u(\varphi, \{L\pi\}_i) = \sum_j \{L\pi\}_j \varphi_{ji},$$

and since

$$\{L\pi\}_j r^{-n_{L\pi}} = \sum_{\beta} e_{\beta}^{L\pi} (M^{L\pi})_{\beta j}^{-1} \quad (73)$$

it follows that

$$\begin{aligned} u(s, e_Y^{L\pi}) &= -\frac{4n_{L\pi}s}{r} e_Y^{L\pi} + \sum_{i,j,\beta} e_{\beta}^{L\pi} (M^{L\pi})_{\beta j}^{-1} \left[\delta_{j,i} \frac{4(1-s^2)}{r} \frac{\partial}{\partial s} + s_{ji} \right] M_{iY}^{L\pi} \\ u(\varphi, e_Y^{L\pi}) &= \sum_{i,j,\beta} e_{\beta}^{L\pi} (M^{L\pi})_{\beta j}^{-1} \left[\delta_{j,i} \frac{4}{rs^2} \frac{\partial}{\partial \varphi} + \varphi_{ji} \right] M_{iY}^{L\pi}. \end{aligned} \quad (74)$$

Now, let

$$s_{\beta\gamma}^{L\pi} = \sum_{ij} (M^{L\pi})_{\beta j}^{-1} s_{ji} M_{i\gamma}^{L\pi}$$

$$\phi_{\beta\gamma}^{L\pi} = \sum_{ij} (M^{L\pi})_{\beta j}^{-1} \varphi_{ji} M_{i\gamma}^{L\pi}$$

(75)

$$\left(\frac{\partial M^{L\pi}}{\partial s} \right)_{\beta\gamma} = \sum_j (M^{L\pi})_{\beta j}^{-1} \frac{\partial M_{j\gamma}^{L\pi}}{\partial s}$$

$$\left(\frac{\partial M^{L\pi}}{\partial \phi} \right)_{\beta\gamma} = \sum_j (M^{L\pi})_{\beta j}^{-1} \frac{\partial M_{j\gamma}^{L\pi}}{\partial \phi}$$

and $e^{L\pi}$ be the row vector $e_{\beta}^{L\pi}$. Then

$$u(s, e^{L\pi}) = e^{L\pi} \left[-\frac{4n_{L\pi}s}{r} + \frac{4(1-s^2)}{r} \frac{\partial M^{L\pi}}{\partial s} + s^{L\pi} \right]$$

(76)

$$u(\varphi, e^{L\pi}) = e^{L\pi} \left[\frac{4}{rs^2} \frac{\partial M^{L\pi}}{\partial \phi} + \phi^{L\pi} \right]$$

Also

$$\begin{aligned} t(e_Y^{L\pi}) &= \sum_i t(\{L\pi\}_i r^{-n_{L\pi}} M_{iY}^{L\pi}) \\ &= \sum_i \left[t(\{L\pi\}_i r^{-n_{L\pi}} M_{iY}^{L\pi} + 2u(\{L\pi\}_i r^{-n_{L\pi}}, M_{iY}^{L\pi}) \right. \\ &\quad \left. + \{L\pi\}_i r^{-n_{L\pi}} t(M_{iY}^{L\pi}) \right] \end{aligned}$$

(77)

$$\begin{aligned}
 &= \sum_i \left[2u(\{L\pi\}_i, r^{-n_{L\pi}}) M_{iY}^{L\pi} + e_Y^{L\pi} r^{n_{L\pi}} \dagger(r^{-n_{L\pi}}) \right. \\
 &\quad + (2u'(s, \{L\pi\}_i, r^{-n_{L\pi}}) \frac{\partial}{\partial s} + 2r^{-n_{L\pi}} u(\varphi, \{L\pi\}_i) \frac{\partial}{\partial \varphi}) M_{iY}^{L\pi} \\
 &\quad \left. + 4 \{L\pi\}_i r^{-n_{L\pi}} \left(\frac{1-s^2}{r} \frac{\partial^2}{\partial s^2} + \frac{1-3s^2}{rs} \frac{\partial}{\partial s} + \frac{1}{rs^2} \frac{\partial^2}{\partial \varphi^2} \right) M_{iY}^{L\pi} \right].
 \end{aligned}$$

Now, since $\{L\pi\}_i$ is homogeneous of degree $2n_{L\pi}$ in $\underline{\lambda}$ and $\underline{\rho}$,

$$\begin{aligned}
 \sum_i 2u(\{L\pi\}_i, r^{-n_{L\pi}}) M_{iY}^{L\pi} &= \sum_i (-2n_{L\pi}) r^{-n_{L\pi}-1} u(r, \{L\pi\}_i) M_{iY}^{L\pi} \\
 &= \sum_i (-8n_{L\pi}^2) r^{-n_{L\pi}-1} \{L\pi\}_i M_{iY}^{L\pi} = -8n_{L\pi}^2 r^{-1} e_Y^{L\pi},
 \end{aligned} \tag{78}$$

and

$$r^{n_{L\pi}} \dagger(r^{-n_{L\pi}}) = r^{n_{L\pi}} \left(4r \frac{\partial^2}{\partial r^2} + 12 \frac{\partial}{\partial r} \right) r^{-n_{L\pi}} = 4n_{L\pi}(n_{L\pi} - 2)r^{-1} \tag{79}$$

$$u'(s, \{L\pi\}_i, r^{-n_{L\pi}}) = -\frac{4n_{L\pi}s}{r} \{L\pi\}_i r^{-n_{L\pi}} + u'(s, \{L\pi\}_i) r^{-n_{L\pi}},$$

so that

$$\begin{aligned}
 \dagger(e^{L\pi}) &= e^{L\pi} \left[-\frac{4}{r} n_{L\pi}(n_{L\pi} + 2) - \frac{8n_{L\pi}s}{r} \frac{\partial M^{L\pi}}{\partial s} + 2s^{L\pi} \frac{\partial M^{L\pi}}{\partial s} \right. \\
 &\quad + 2\frac{L\pi}{\Phi} \frac{\partial M^{L\pi}}{\partial \Phi} + \frac{4(1-s^2)}{r} \frac{\partial^2 M^{L\pi}}{\partial s^2} + \frac{4(1-3s^2)}{rs} \frac{\partial M^{L\pi}}{\partial s} \\
 &\quad \left. + \frac{4}{rs^2} \frac{\partial^2 M^{L\pi}}{\partial \Phi^2} \right],
 \end{aligned} \tag{80}$$

where

$$(M^{L\pi})^{-1} \frac{\partial^2 M^{L\pi}}{\partial s^2} \equiv \frac{\partial^2 M^{L\pi}}{\partial s^2} \quad (81)$$

$$(M^{L\pi})^{-1} \frac{\partial^2 M^{L\pi}}{\partial \varphi^2} \equiv \frac{\partial^2 M^{L\pi}}{\partial \Phi^2}.$$

Thus (64) and (80) give

$$\begin{aligned} t^{L\pi} = & 4k + 2(s^{L\pi} + \frac{4(1-s^2)}{r} \frac{\partial M^{L\pi}}{\partial s} - \frac{4n_{L\pi}s}{r}) \frac{\partial}{\partial s} \\ & + 2(\Phi^{L\pi} + \frac{4}{rs^2} \frac{\partial M^{L\pi}}{\partial \Phi}) \frac{\partial}{\partial \varphi} - \frac{4}{r} n_{L\pi} (n_{L\pi} + 2) \\ & + 2s^{L\pi} \frac{\partial M^{L\pi}}{\partial s} + 2\Phi^{L\pi} \frac{\partial M^{L\pi}}{\partial \Phi} + \frac{4(1-s^2)}{r} \frac{\partial^2 M^{L\pi}}{\partial s^2} \\ & + \frac{4(1-(3+2n_{L\pi})s^2)}{rs} \frac{\partial M^{L\pi}}{\partial s} + \frac{4}{rs^2} \frac{\partial^2 M^{L\pi}}{\partial \Phi^2} \end{aligned} \quad (82)$$

and

$$u^{L\pi} = \frac{4}{rs^2} \frac{\partial}{\partial \varphi} + \Phi^{L\pi} + \frac{4}{rs^2} \frac{\partial M^{L\pi}}{\partial \Phi} \quad (93)$$

where k is given by Equation (32).

From (19), (20), (67), and (72) it follows easily that

$$s^{0+} = s^{1+} = \varphi^{0+} = \varphi^{1+} = s^{0+} = s^{1+} = \Phi^{0+} = \Phi^{1+} = 0$$

$$\frac{\partial M^{0+}}{\partial s} = \frac{\partial M^{0+}}{\partial \Phi} = \frac{\partial M^{1+}}{\partial \Phi} = \frac{\partial^2 M^{1+}}{\partial \Phi^2} = 0 \quad (84)$$

$$\frac{\partial M^{1+}}{\partial s} = \frac{s}{1-s^2} M^{1+}, \quad \frac{\partial M^{1+}}{\partial s} = \frac{s}{1-s^2}$$

$$\frac{\partial^2 M^{1+}}{\partial s^2} = \frac{1+2s^2}{(1-s^2)^2} M^{1+} \quad \frac{\partial^2 M^{1+}}{\partial s^2} = \frac{1+2s^2}{(1-s^2)^2} \quad >$$

so that

$$t^{0+} = 4k \quad t^{1+} = 4k - \frac{4}{r(1-s^2)} \quad (85)$$

$$u^{0+} = u^{1+} = \frac{4}{rs^2} \frac{\partial}{\partial \varphi}$$

Also, (19) and (67) give

$$s^{2+} = \frac{2}{r} \begin{pmatrix} -2 \cos \varphi & 0 & \sin \varphi \\ 0 & 2 \cos \varphi & \sin \varphi \\ 2 \sin \varphi & 2 \sin \varphi & 0 \end{pmatrix} \quad (86)$$

$$\varphi^{2+} = \frac{2}{rs} \begin{pmatrix} 2 \sin \varphi & 0 & \cos \varphi \\ 0 & -2 \sin \varphi & \cos \varphi \\ 2 \cos \varphi & 2 \cos \varphi & 0 \end{pmatrix} ,$$

and (20) gives

$$(M^{2+})^{-1} = \frac{\pi(1-s^2)^{1/2}}{4(10/3)^{1/2}} \begin{pmatrix} \frac{1-s \cos \varphi}{3^{1/2}(1-s^2)^{1/2}} & \frac{1+s \cos \varphi}{3^{1/2}(1-s^2)^{1/2}} & \frac{s \sin \varphi}{3^{1/2}(1-s^2)^{1/2}} \\ \frac{-s \cos \varphi}{(1-s^2)^{1/2}} & \frac{-s \cos \varphi}{(1-s^2)^{1/2}} & \frac{-\sin \varphi}{(1-s^2)^{1/2}} \\ \sin \varphi & -\sin \varphi & \cos \varphi \end{pmatrix}$$

$$\frac{\partial M^{2+}}{\partial s} = \frac{s}{1-s^2} M^{2+} + \frac{M^2}{1-s^2} \begin{pmatrix} 0 & 3^{-1/2} & 0 \\ 3^{1/2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (87)$$

$$\frac{\partial M^{2+}}{\partial \varphi} = - \frac{M^{2+}}{(1-s^2)^{1/2}} \begin{pmatrix} 0 & 0 & 3^{-1/2}s \\ 0 & 0 & -1 \\ 3^{1/2}s & 1 & 0 \end{pmatrix}$$

$$\frac{\partial^2 M^{2+}}{\partial s^2} = \frac{1+2s^2}{(1-s^2)^2} M^{2+} + \frac{4s}{(1-s^2)^2} M^{2+} \begin{pmatrix} 0 & 3^{-1/2} & 0 \\ 3^{1/2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{M^{2+}}{(1-s^2)^2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\frac{\partial^2 M^{2+}}{\partial \varphi^2} = \frac{M^{2+}}{1-s^2} \begin{pmatrix} s^2 & 3^{-1/2}s & 0 \\ -3^{1/2}s & -1 & 0 \\ 0 & 0 & -1+s^2 \end{pmatrix})$$

so that

$$s^{2+} = -\frac{4}{r} \begin{pmatrix} 0 & 3^{-1/2} & 0 \\ 3^{1/2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Phi^{2+} = \frac{4}{rs(1-s^2)^{1/2}} \begin{pmatrix} 0 & 0 & 3^{-1/2} \\ 0 & 0 & -s \\ 3^{1/2} & s & 0 \end{pmatrix}$$

$$\frac{\partial M^{2+}}{\partial s} = \frac{-s}{1-s^2} + \frac{-1}{1-s^2} \begin{pmatrix} 0 & 3^{-1/2} & 0 \\ 3^{1/2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(88)

$$\frac{\partial M^{2+}}{\partial \Phi} = -\frac{1}{(1-s^2)^{1/2}} \begin{pmatrix} 0 & 0 & 3^{-1/2}s \\ 0 & 0 & -1 \\ 3^{1/2}s & 1 & 0 \end{pmatrix}$$

$$\frac{\partial^2 M^{2+}}{\partial s^2} = \frac{1+2s^2}{(1-s^2)^2} + \frac{4s}{(1-s^2)^2} \begin{pmatrix} 0 & 3^{-1/2} & 0 \\ 3^{1/2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{(1-s^2)^2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\frac{\partial^2 M^{2+}}{\partial \Phi^2} = \frac{1}{1-s^2} \begin{pmatrix} s^2 & 3^{-1/2}s & 0 \\ -3^{1/2}s & -1 & 0 \\ 0 & 0 & -1+s^2 \end{pmatrix}$$

$$s^{2+} \frac{\partial M^{2+}}{\partial s} = - \frac{4s}{r(1-s^2)} \begin{pmatrix} 0 & 3^{-1/2} & 0 \\ 3^{1/2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{4}{r(1-s^2)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\phi^{2+} \frac{\partial M^{2+}}{\partial \phi} = - \frac{4}{rs(1-s^2)} \begin{pmatrix} s & 3^{-1/2} & 0 \\ -3^{1/2}s^2 & -s & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and finally

$$t^{2+} = 4k + \frac{8(1-s^2)^{1/2}}{rs^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \frac{\partial}{\partial \phi} - \frac{4}{r(1-s^2)} \begin{pmatrix} 3 & 3^{1/2}s & 0 \\ 3^{1/2}s & s^{-2} & 0 \\ 0 & 0 & s^{-2} \end{pmatrix} \quad (89)$$

$$u^{2+} = \frac{4}{rs^2} \frac{\partial}{\partial \phi} + \frac{4(1-s^2)^{1/2}}{rs^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Substitution of (85) and (89) into (65) gives (29), (30) and (31).

III. POTENTIAL ENERGY

It was shown in I that

$$\sum_{\tau_i \sigma_i} ((T'S')_{R'} \otimes S_{R'})_A^* ((TS)_{\tilde{R}} \otimes S_R)_A = \delta_{TT'} \delta_{SS'} \delta_{RR'} \quad (90)$$

so that Equation (38) gives

$$(R'|P^{\dagger S}|R)^{TS} = K_R K_R \sum_{\tau_i \sigma_i} ((TS)_{\tilde{R}} \otimes S_{R'})^* P^{\dagger S}(1,2) ((TS)_{\tilde{R}} \otimes S_R)_A \quad (91)$$

Now the notation α_T^T, β_S^S , with

$$\begin{aligned} \alpha_1^{3/2} &= T^{3/2} & \alpha_0^{3/2} &= 0 \\ \alpha_1^{1/2} &= T_1^{1/2} & \alpha_0^{1/2} &= T_2^{1/2} \end{aligned} \quad (92)$$

and similarly for $\alpha \rightarrow \beta, T \rightarrow \Sigma$ (See I) gives

$$K_R ((TS)_{\tilde{R}} \otimes S_R)_A = \frac{(2S+1)^{1/2}}{2} \sum_{ts} (ts|R)^{TS} \alpha_T^T \beta_S^S \quad (93)$$

with

$$(ts|R)^{1/2 \ 1/2} = \begin{matrix} & \begin{matrix} S & M & N & A \end{matrix} \\ \begin{pmatrix} 0 & \sin \varphi & -\cos \varphi \\ 0 & -\sin \varphi & \cos \varphi \\ -1 & -\cos \varphi & -\sin \varphi \\ 1 & -\cos \varphi & -\sin \varphi \end{pmatrix} & \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \end{matrix} \begin{matrix} 00 \\ 11 \\ 01 \\ 10 \end{matrix} \quad (94)$$

$$(ts|R)^{1/2 \ 3/2} = \begin{matrix} & \begin{matrix} M & N \end{matrix} \\ \begin{pmatrix} \sin \varphi & -\cos \varphi \\ -\cos \varphi & -\sin \varphi \end{pmatrix} & \begin{pmatrix} 11 \\ 01 \end{pmatrix} \end{matrix}$$

The matrices $(ts|R)^{3/2 \ 1/2}$ and $(ts|R)^{3/2 \ 3/2}$ are not required for the three-nucleon bound state. It follows from I and (92) that

$$P^\dagger \alpha_{t'}^T = \delta_{t,t'} \alpha_{t'}^T, \quad P^S \beta_{s'}^S = \delta_{s,s'} \beta_{s'}^S, \quad (95)$$

so that

$$(R'|P^\dagger S|R)^{TS} = \frac{2S+1}{4} (ts|R')^{TS} (ts|R)^{TS} = (R|P^\dagger S|R')^{TS}, \quad (96)$$

Also, it is easily seen from the fact that S_M belongs to R_M in standard form (see I) that

$$P^{123} \varphi = \varphi + \frac{2\pi}{3}, \quad (97)$$

so that

$$P f(r, s, \varphi) = P_\varphi f(r, s, \varphi). \quad (98)$$

Substitution of (96) and (94) into (40) and use of (98) gives (43) and (44).

With (90) and (93), Equation (46) becomes

$$(S'R' | \delta^2 P^\dagger | SR)^T = (-)^{S'+\frac{1}{2}} (2S'+1)^{1/2} K_{R',K_R}$$

$$\begin{aligned} & \sum_{\tau_i \sigma_i} ((TS')_{\tilde{R}'} \otimes S_{R'})_A^* \left\{ \delta^{(2)}(1,2), P^\dagger(1,2) ((TS)_{\tilde{R}} \otimes S_R)_A \right\}^{S'} \\ &= \sum_{s, s'} (-)^{S'+\frac{1}{2}} \frac{(2S'+1)(2S+1)^{1/2}}{4} (ts|R')^{TS'} (ts|R)^{TS} \sum_{\sigma_i} \beta_{s'}^{S'+*} \\ & \quad \left\{ \delta^{(2)}(1,2), \beta_s^S \right\}^{S'}. \end{aligned} \quad (99)$$

Since $\delta^{(2)}$ is of rank 2 and $s = 0$ or $s = 1$ only, it follows that

$s = s' = 1$ only in Equation (99). Also

$$\begin{aligned} \left\{ \mathcal{L}^{(2)}(1,2), \beta_1^s \right\}^{s'} &= \left\{ \mathcal{L}^{(2)}(1,2), \left\{ \{s(1), s(2)\}^1, s(3) \right\}^s \right\}^{s'} \\ &= U(2|S, \frac{1}{2}; |S) \left\{ \left\{ \mathcal{L}^{(2)}(1,2), \{s(1), s(2)\}^1 \right\}^1, s(3) \right\}^{s'} \end{aligned} \quad (100)$$

and a standard computation gives

$$\left\{ \mathcal{L}^{(2)}(1,2), \{s(1), s(2)\}^1 \right\}^1 = -2(5/3)^{1/2} \{s(1), s(2)\}^1, \quad (101)$$

so that (50) holds, with

$$(SR|X|S'R')^T = (1|1|R')^{TS'} (1|1|R)^{TS} = (S'R'|X|SR)^T, \quad (102)$$

as given in (51).

According to I,

$$\left\{ e_{Y'}^{L'\pi'}, e_Y^{L\pi} \right\}^0 = \left(\frac{4}{3\pi^2} \right) (2L+1)^{1/2} \delta_{LL'} \delta_{\pi\pi'} \delta_{YY'}, \quad (103)$$

so that (47) gives

$$(L'\beta|\mathcal{L}^2|LY)^\pi = (-)^L \left(\frac{3\pi^2}{4 \cdot 5^{1/2}} \right) \left\{ e_\beta^{L'\pi}, \left\{ \mathcal{L}^{(2)}(1,2), e_Y^{L\pi} \right\}^{L'} \right\}^0. \quad (104)$$

A standard recoupling calculation then gives

$$(L'\beta|\mathcal{L}^2|LY)^\pi = (LY|\mathcal{L}^2|L'\beta)^\pi. \quad (105)$$

Now

$$\begin{aligned} \sum_\beta e_\beta^{L'\pi} (L'\beta|\mathcal{L}^2|LY)^\pi &= (-)^L (2L'+1/5)^{1/2} \left\{ \mathcal{L}^{(2)}(1,2), e_Y^{L\pi} \right\}^{L'} \\ &= (-)^L (2L'+1/5)^{1/2} \sum_i \left\{ \mathcal{L}^{(2)}(1,2), \{L\pi\}_i \right\}^{L'} M_{iY}^{L'\pi} \end{aligned} \quad (106)$$

$$\begin{aligned}
 &= \sum_j \{L'\pi\}_j (L'j | \mathcal{X} | Li)^\pi m_{i\gamma}^{L\pi} \\
 &= \sum_{\beta} e_{\beta}^{L'\pi} (M^{L'\pi})_{\beta j}^{-1} (L'j | \mathcal{X} | Li)^\pi m_{i\gamma}^{L\pi} ,
 \end{aligned}$$

so that

$$(L'\beta | \mathcal{X}^{(2)} | L\gamma)^\pi = \sum_j (M^{L'\pi})_{\beta j}^{-1} (L'j | \mathcal{X} | Li)^\pi m_{i\gamma}^{L\pi} , \quad (107)$$

where $(L'j | \mathcal{X} | Li)$ is defined by

$$\begin{aligned}
 (-)^L (2L' + 1/5)^{1/2} \{ \mathcal{X}^{(2)}(1,2), \{L\pi\}_i \}^{L'} &= \sum_j \{L'\pi\}_j (L'j | \mathcal{X} | Li)^\pi \\
 &= (-)^L (2L' + 1/5)^{1/2} \rho^{-2} \{ \{ \rho \rho \}^2, \{L\pi\}_i \}^{L'} .
 \end{aligned} \quad (108)$$

In order to evaluate Equation (108), it is simplest to use the recoupling coefficients

$$U_L(l_1, l_2, l_3, l_4) \equiv U \begin{pmatrix} 1 & 1 & l_1 \\ 1 & 1 & l_2 \\ l_3 & l_4 & L \end{pmatrix} \quad (109)$$

$$= \left[(2l_1+1)(2l_2+1)(2l_3+1)(2l_4+1) \right]^{1/2} \begin{Bmatrix} 1 & 1 & l_1 \\ 1 & 1 & l_2 \\ l_3 & l_4 & L \end{Bmatrix} .$$

Then it follows that

$$\begin{aligned} \left\{ \{pp\}^2, \{pp\}^2 \right\}^{L'} &= \sum_{ll'} U_{L', (22, ll')} \left\{ \{pp\}^l \{pp\}^{l'} \right\}^{L'} \\ &= \delta_{L', 0} U_0(22, 00) \{pp\}^0 \{pp\}^0 + 2\delta_{L', 2} U_2(22, 02) \{pp\}^0 \{pp\}^2 \\ &\quad + U_{L', (22, 22)} \left\{ \{pp\}^2 \{pp\}^2 \right\}^{L'} \end{aligned} \quad (110)$$

$$= \delta_{L', 0} \frac{U_0(22, 00)}{1 - U_0(22, 22)} \{pp\}^0 \{pp\}^0 + \delta_{L', 2} \frac{2U_2(22, 02)}{1 - U_2(22, 22)} \{pp\}^0 \{pp\}^2,$$

where the procedure in (110) and the following is only useful for $L' < 3$; these are the only values of interest here. Table I and

$$\{A, B\}^0 = 3^{-1/2} \underline{A \cdot B} \quad (111)$$

give

$$\begin{aligned} \left\{ \{pp\}^2, \{pp\}^2 \right\}^0 &= \frac{2p^4}{3 \cdot 5^{1/2}} \\ \left\{ \{pp\}^2, \{pp\}^2 \right\}^2 &= \frac{2p^2}{21^{1/2}} \{pp\}^2 \\ \left\{ \{pp\}^2, \{pp\}^2 \right\}^1 &= 0. \end{aligned} \quad (112)$$

Similarly,

$$\left\{ \{pp\}^2 \{p\lambda\}^1 \right\}^{L'} = U_{L', (21, 0L')} \{pp\}^0 \{p\lambda\}^{L'} + \delta_{L', 2} U_2(21, 20) \{p\lambda\}^0 \{pp\}^2$$

$$\begin{aligned}
 & + U_{L, (21, 21)} \left\{ \{ \rho \rho \}^2 \{ \rho \lambda \}^1 \right\}^{L'} + U_{L, (21, 22)} \left\{ \{ \rho \rho \}^2 \{ \rho \lambda \}^2 \right\}^{L'} \\
 & \left\{ \{ \rho \rho \}^2 \{ \rho \lambda \}^2 \right\}^{L'} = U_{L, (22, 0L')} \{ \rho \rho \}^0 \{ \rho \lambda \}^{L'} + \delta_{L', 2} U_2(22, 20) \{ \rho \lambda \}^0 \{ \rho \rho \}^2 \\
 & + U_{L, (22, 21)} \left\{ \{ \rho \rho \}^2 \{ \rho \lambda \}^1 \right\}^{L'} + U_{L, (22, 22)} \left\{ \{ \rho \rho \}^2, \{ \rho \lambda \}^2 \right\}^{L'}.
 \end{aligned} \tag{113}$$

These equations have the solution

$$\begin{aligned}
 \left\{ \{ \rho \rho \}^2 \{ \rho \lambda \}^1 \right\}^{L'} &= \Delta_{L'}^{-1} \begin{vmatrix} U_{L, (21, 0L')} & -U_{L, (21, 22)} \\ U_{L, (22, 0L')} & 1 - U_{L, (22, 22)} \end{vmatrix} \{ \rho \rho \}^0 \{ \rho \lambda \}^{L'} \\
 &+ \Delta_2^{-1} \begin{vmatrix} U_2(21, 20) & -U_2(21, 22) \\ U_2(22, 20) & 1 - U_2(22, 22) \end{vmatrix} \delta_{L', 2} \{ \rho \lambda \}^0 \{ \rho \rho \}^2 \\
 &\left\{ \{ \rho \rho \}^2 \{ \rho \lambda \}^2 \right\}^{L'} = \Delta_{L'}^{-1} \begin{vmatrix} 1 - U_{L, (21, 21)} & U_{L, (21, 0L')} \\ -U_{L, (22, 21)} & U_{L, (22, 0L')} \end{vmatrix} \{ \rho \rho \}^0 \{ \rho \lambda \}^{L'} \\
 &+ \Delta_2^{-1} \begin{vmatrix} 1 - U_2(21, 21) & U_2(21, 20) \\ -U_2(22, 21) & U_2(22, 20) \end{vmatrix} \delta_{L', 2} \{ \rho \lambda \}^0 \{ \rho \rho \}^2
 \end{aligned} \tag{114}$$

$$\Delta_{L'} = \begin{vmatrix} 1 - U_{L, (21, 21)} & -U_{L, (21, 22)} \\ -U_{L, (22, 21)} & 1 - U_{L, (22, 22)} \end{vmatrix}$$

$$\left\{ \{ \rho \rho \}^2 \{ \rho \lambda \}^2 \right\}^0 = (1 - U_0(22, 22))^{-1} U_0(22, 00) \{ \rho \rho \}^0 \{ \rho \lambda \}^0 .$$

Substitution of values from Table I gives

$$\begin{aligned} \left\{ \{ \rho \rho \}^2 \{ \rho \lambda \}^1 \right\}^1 &= - (\rho^2/15)^{1/2} \{ \rho \lambda \}^1 \\ \left\{ \{ \rho \rho \}^2 \{ \rho \lambda \}^2 \right\}^1 &= (\rho^2/5)^{1/2} \{ \rho \lambda \}^1 \\ \left\{ \{ \rho \rho \}^2 \{ \rho \lambda \}^1 \right\}^2 &= - (\rho^2/3)^{1/2} \{ \rho \lambda \}^2 + (\rho \cdot \lambda / 3)^{1/2} \{ \rho \rho \}^2 \\ \left\{ \{ \rho \rho \}^2 \{ \rho \lambda \}^2 \right\}^2 &= (\rho^2/21)^{1/2} \{ \rho \lambda \}^2 + (\rho \cdot \lambda / 21)^{1/2} \{ \rho \rho \}^2 \\ \left\{ \{ \rho \rho \}^2 \{ \rho \lambda \}^2 \right\}^0 &= 2\rho^2 \rho \cdot \lambda / 3 \cdot 5^{1/2} \end{aligned} \quad (115)$$

Finally,

$$\begin{aligned} \left\{ \{ \rho \rho \}^2, \{ \lambda \lambda \}^2 \right\}^{L'} &= \sum_{\ell \ell'} U_{L', (22, \ell \ell')} \left\{ \{ \rho \lambda \}^{\ell}, \{ \rho \lambda \}^{\ell'} \right\}^{L'} \\ \left\{ \{ \rho \lambda \}^{\ell}, \{ \rho \lambda \}^{\ell'} \right\}^{L'} &= \sum_{k k'} U_{L', (\ell \ell', k k')} \left\{ \{ \rho \rho \}^k, \{ \lambda \lambda \}^{k'} \right\}^{L'} \\ \left\{ \{ \rho \rho \}^2, \{ \lambda \lambda \}^2 \right\}^{L'} &= \delta_{L', 0} U_0(22, 00) \{ \rho \lambda \}^0 \{ \rho \lambda \}^0 + 2\delta_{L', 1} U_1(22, 01) \{ \rho \lambda \}^0 \{ \rho \lambda \}^1 \\ &\quad + 2\delta_{L', 2} U_2(22, 02) \{ \rho \lambda \}^0 \{ \rho \lambda \}^2 + \sum_{\substack{k, k' \\ \ell, \ell' \neq 0}} U_{L', (22, \ell \ell')} U_{L', (k k', \ell \ell')} \left\{ \{ \rho \rho \}^k, \{ \lambda \lambda \}^{k'} \right\}^{L'} \end{aligned} \quad (116)$$

$$= \left(1 - \sum_{\ell\ell' \neq 0} U_{\ell'}^2(22, \ell\ell')\right)^{-1} \left[\begin{aligned} & \delta_{L',0} \left(U_0(22,00) \{ \rho\lambda \}^0 \{ \rho\lambda \}^0 + \sum_{\ell\ell' \neq 0} U_{\ell'}(22, \ell\ell') \right. \\ & \quad \left. U_0(00, \ell\ell') \{ \rho\rho \}^0 \{ \lambda\lambda \}^0 \right) \\ & + \delta_{L',1} \left(2U_1(22,01) \{ \rho\lambda \}^0 \{ \rho\lambda \}^1 \right) \\ & + \delta_{L',2} \left(2U_2(22,02) \{ \rho\lambda \}^0 \{ \rho\lambda \}^2 \right. \\ & \quad \left. + \sum_{\ell\ell' \neq 0} U_{\ell'}^2(22, \ell\ell') U_2(02, \ell\ell') \left[\{ \rho\rho \}^0 \{ \lambda\lambda \}^2 + \{ \lambda\lambda \}^0 \{ \rho\rho \}^2 \right] \right) \end{aligned} \right].$$

Substitution from Table I gives

$$\{ \rho\rho \}^2 \{ \lambda\lambda \}^2 \}^0 = (\rho\lambda)^2 / 5^{1/2} - \rho^2 \lambda^2 / 3 \cdot 5^{1/2}.$$

$$\{ \rho\rho \}^2 \{ \lambda\lambda \}^2 \}^1 = (2\rho\lambda / 5^{1/2}) \{ \rho\lambda \}^1 \quad (117)$$

$$\{ \rho\rho \}^2 \{ \lambda\lambda \}^2 \}^2 = (6\rho\lambda / 21^{1/2}) \{ \rho\lambda \}^2 - (2\rho^2 / 21^{1/2}) \{ \lambda\lambda \}^2 - (2\lambda^2 / 21^{1/2}) \{ \rho\rho \}^2.$$

Therefore

$$\{ \rho\rho \}^2, \{ 0 \} \}^2 = \{ 2 \}_2 \quad (2j | \mathcal{X} | 0)^+ = \delta_{j,2} / \rho^2 = \frac{2}{r(1 + s \cos \varphi)} \delta_{j,2}$$

$$-(3/5)^{1/2} \{ \rho\rho \}^2, \{ 1 \} \}^1 = (\rho^2 / 5) \{ 1 \} \quad (1 | \mathcal{X} | 1) = \frac{1}{5}$$

(118)

$$-\{ \rho\rho \}^2, \{ 1 \} \}^2 = -(\rho^2 / 3^{1/2}) \{ 2 \}_3 + (\rho\lambda / 3^{1/2}) \{ 2 \}_2$$

$$(2j | \mathcal{X} | 1) = \frac{s \sin \varphi}{1 + s \cos \varphi} 3^{-1/2} \delta_{j,2} - 3^{-1/2} \delta_{j,3}$$

$$5^{-1/2} \left\{ \{pp\}^2, \{2\}_j \right\}^0 = \delta_{j,1} (3(p \cdot \lambda)^2 - p^2 \lambda^2) / 15 + \delta_{j,2} 2p^4 / 15 + \delta_{j,3} 2p^2 p \cdot \lambda / 15$$

$$(0|\mathcal{L}|2_j) = \delta_{j,1} (r/30) \left(\frac{3s^2 \sin^2 \varphi}{1+s \cos \varphi} - 1 + s \cos \varphi \right) + \delta_{j,2} r(1+s \cos \varphi) / 15$$

$$+ \delta_{j,3} rs \sin \varphi / 15$$

$$(3/5)^{1/2} \left\{ \{pp\}^2, \{2\}_j \right\}^1 = \left[-\delta_{j,1} 2 \cdot 3^{1/2} p \cdot \lambda / 5 - \delta_{j,3} 3^{1/2} p^2 / 5 \right] \{1\}$$

$$(1|\mathcal{L}|2_j) = -\delta_{j,1} 2 \cdot 3^{1/2} s \sin \varphi / 5 (1+s \cos \varphi) - \delta_{j,3} 3^{1/2} / 5$$

$$\left\{ \{pp\}^2, \{2\} \right\}^2 = \{2\} \begin{pmatrix} -\frac{2p^2}{2!^{1/2}} & 0 & 0 \\ -\frac{2\lambda^2}{2!^{1/2}} & \frac{2p^2}{2!^{1/2}} & \frac{p \cdot \lambda}{2!^{1/2}} \\ \frac{6p \cdot \lambda}{2!^{1/2}} & 0 & \frac{p^2}{2!^{1/2}} \end{pmatrix}$$

$$(2|\mathcal{L}|2_j) = 2!^{-1/2} \begin{pmatrix} -2 & 0 & 0 \\ -2 \frac{1-s \cos \varphi}{1+s \cos \varphi} & 2 & \frac{s \sin \varphi}{1+s \cos \varphi} \\ 6 \frac{s \sin \varphi}{1+s \cos \varphi} & 0 & 1 \end{pmatrix}$$

Substitution of the above into (107) gives

$$(L'\beta | \mathcal{L}^2 | L\gamma)^+ = \begin{pmatrix} 0 & 0 & 1/30^{1/2} & -\alpha/10^{1/2} & -\beta/10^{1/2} \\ 0 & 1/5 & 0 & \beta/5^{1/2} & -\alpha/5^{1/2} \\ 1/30^{1/2} & 0 & -1/21^{1/2} & -\alpha/7^{1/2} & -\beta/7^{1/2} \\ -\alpha/10^{1/2} & \beta/5^{1/2} & -\alpha/7^{1/2} & 1/21^{1/2} & 0 \\ -\beta/10^{1/2} & -\alpha/5^{1/2} & -\beta/7^{1/2} & 0 & 1/21^{1/2} \end{pmatrix} \quad (119)$$

where α and β are given by (54). The matrix

$$(SL|Z|SL)^J = (-)^{J-\frac{1}{2}} 5^{1/2} (2S+1)(2S'+1) \begin{Bmatrix} 2 & 1 & 1 \\ \frac{1}{2} & S & S' \end{Bmatrix} \begin{Bmatrix} 2 & L' & L \\ J & S & S' \end{Bmatrix} \quad (120)$$

is

$$(SL|Z|S'L')^{1/2} = \begin{pmatrix} 0 & 0 & 0 & (2/3)^{1/2} \\ 0 & 0 & (5/9)^{1/2} & -(1/3)^{1/2} \\ 0 & (5/9)^{1/2} & (5/9)^{1/2} & (1/3)^{1/2} \\ (2/3)^{1/2} & -(1/3)^{1/2} & (1/3)^{1/2} & -(7/15)^{1/2} \end{pmatrix}$$

in the order $\frac{1}{2}0, \frac{1}{2}1, \frac{3}{2}1, \frac{3}{2}2$, so that (53), (119), and (120) give (55).

$$U_0(l l, k k) = \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{3} & \frac{1}{3^{1/2}} & \frac{5^{1/2}}{3} \\ \frac{1}{3^{1/2}} & \frac{1}{2} & -\frac{1}{2}(\frac{5}{3})^{1/2} \\ \frac{5^{1/2}}{3} & -\frac{1}{2}(\frac{5}{3})^{1/2} & \frac{1}{6} \end{pmatrix} \begin{matrix} 0 \\ 1 \\ 2 \end{matrix}$$

$$U_1(l l', k k') = \begin{pmatrix} 01 & 10 & 11 & 12 & 21 & 22 \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{2 \cdot 3^{1/2}} & \frac{5^{1/2}}{6} & -\frac{5^{1/2}}{6} & \frac{1}{2}(\frac{5}{3})^{1/2} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{2 \cdot 3^{1/2}} & -\frac{5^{1/2}}{6} & \frac{5^{1/2}}{6} & \frac{1}{2}(\frac{5}{3})^{1/2} \\ \frac{1}{2 \cdot 3^{1/2}} & \frac{1}{2 \cdot 3^{1/2}} & 0 & \frac{1}{2}(\frac{5}{3})^{1/2} & \frac{1}{2}(\frac{5}{3})^{1/2} & 0 \\ \frac{5^{1/2}}{6} & -\frac{5^{1/2}}{6} & \frac{1}{2}(\frac{5}{3})^{1/2} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{2 \cdot 3^{1/2}} \\ -\frac{5^{1/2}}{6} & \frac{5^{1/2}}{6} & \frac{1}{2}(\frac{5}{3})^{1/2} & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{2 \cdot 3^{1/2}} \\ \frac{1}{2}(\frac{5}{3})^{1/2} & \frac{1}{2}(\frac{5}{3})^{1/2} & 0 & -\frac{1}{2 \cdot 3^{1/2}} & -\frac{1}{2 \cdot 3^{1/2}} & 0 \end{pmatrix} \begin{matrix} 01 \\ 10 \\ 11 \\ 12 \\ 21 \\ 22 \end{matrix}$$

$$U_2(l l', k k') = \begin{pmatrix} 02 & 20 & 11 & 12 & 21 & 22 \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{2 \cdot 3^{1/2}} & \frac{1}{2} & -\frac{1}{2} & \frac{7^{1/2}}{6} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{2 \cdot 3^{1/2}} & -\frac{1}{2} & \frac{1}{2} & \frac{7^{1/2}}{6} \\ -\frac{1}{2 \cdot 3^{1/2}} & -\frac{1}{2 \cdot 3^{1/2}} & \frac{1}{2} & 0 & 0 & \frac{1}{2}(\frac{7}{3})^{1/2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{7^{1/2}}{6} & \frac{7^{1/2}}{6} & \frac{1}{2}(\frac{7}{3})^{1/2} & 0 & 0 & \frac{1}{6} \end{pmatrix} \begin{matrix} 02 \\ 20 \\ 11 \\ 12 \\ 21 \\ 22 \end{matrix}$$

TABLE I. RECOUPLING COEFFICIENTS

REFERENCES

1. M. Bolsterli and E. Jezak, Submitted to Phys. Rev.