Dual $\pi\pi$ Partial-Wave Amplitudes.

I. Threshold Behavior*

B. K. Chung and David Feldman

Department of Physics

Brown University

Providence, Rhode Island 02912

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ABSTRACT

A formulation of the integral representations of the dual $\pi\pi$ partial-wave amplitudes is presented. One of the many advantages of these representations is that one can extract the threshold coefficients without difficulty. This is demonstrated in a calculation of the scattering lengths and effective ranges for the s, p, d, and f waves.

I. INTRODUCTION

Subsequent to Veneziano's original paper,¹ a dual $\pi\pi$ scattering amplitude which meets the requirements of Regge asymptotic behavior and crossing symmetry and which exhibits zero-width resonance poles was proposed by Lovelace² and Shapiro.³ The partial-wave structure of this amplitude and, in particular, its asymptotic behavior have been studied in the complex angular-momentum plane and in the complex energy-squared plane by many authors,⁴⁻¹¹ partly in connection with the K-matrix unitarization scheme, and partly in connection with dispersion-theoretic studies, among other considerations.

In contrast to the partial-fraction expansion of partial waves which has been used by most authors, we present a unified treatment of the corresponding integral representations, which turn out to be more useful in many respects. In this paper, we study, in some detail, the projection of the partial waves in the complex energy-squared plane and their threshold behavior; in a sequel, we will examine their asymptotic behavior and cut structure.

In Sec. II, we perform a partial-wave projection in the s-channel physical region, and, in Sec. III, carry out the analytic continuation into the entire complex energy-squared plane. We exhibit, in Sec. IV, the normal threshold behavior, with coefficients which can be easily calculated. Here, we see, for the first time, the advantage of our partial-wave formulation. By way of application, we calculate the scattering lengths and effective ranges for the s, p, d, and f waves in Sec. V. An Appendix is devoted to questions of kinematics and notation.

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II. PARTIAL-WAVE PROJECTION IN THE s -CHANNEL PHYSICAL REGION

The dual model for the elastic $\pi\pi$ scattering amplitude was introduced^{2,3} by identifying the isospin-two amplitude $A^2(s,t,u)$ of Eq. (A.3) with $F(t,u)$, where

$$F(x,y) = \frac{\Gamma(1-\alpha(x))\Gamma(1-\alpha(y))}{\Gamma(1-\alpha(x)-\alpha(y))} + \text{secondary terms}, \quad (2.1)$$

and $\alpha(x) = ax+b$ represents a linear Regge trajectory with a and b real.

The other amplitudes are then determined by Eqs. (A.2-4), being given by

$$\begin{aligned} A^0(s,t,u) &= \frac{1}{2}gF(t,u) - \frac{3}{2}g[F(s,t) + F(s,u)], \\ A^1(s,t,u) &= g[F(s,u) - F(s,t)], \\ A^2(s,t,u) &= -gF(t,u), \end{aligned} \quad (2.2)$$

where g is the overall coupling constant. In our discussion, we take the simplest form of $F(x,y)$, neglecting the secondary terms, and assume that $a>0$ and $0<b<1$ (inasmuch as $b>1$ violates Froissart's bound¹² and $b=1$ corresponds to the Pomeranchuk trajectory).

The amplitudes given by Eq. (2.2) are analytic functions of complex s , t , and u , having only simple poles on the real axis. In the s -channel physical region, s , t , and u are real; also, $s = 4v+4m_\pi^2 \geq 4m_\pi^2$, $t = -2v(1-\cos\theta) \leq 0$, and $u = -2v(1+\cos\theta) \leq 0$, where $v \equiv q^2$ and q is the c.m. momentum. We therefore find, in view of the identity $\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z$,

$$F(s,t) = \frac{\alpha(s)}{\sin\pi\alpha(s)} \sin\pi(\alpha(s)+\alpha(t)) \int_0^1 du u^{\alpha(s)+\alpha(t)-1} (1-u)^{-\alpha(t)}. \quad (2.3)$$

Changing the variable u to x such that $u = 1/(1+e^{-x})$, and substituting the expansion¹³

$$e^{iz \cos\theta} = \sum_{\ell=0}^{\infty} (2\ell+1) i^\ell j_\ell(z) P_\ell(\cos\theta),$$

where $j_\ell(z)$ is the spherical Bessel function [$j_0(z) = (\sin z)/z$, $j_1(z) = (\sin z)/z^2 - (\cos z)/z$, etc.], we obtain, from (2.3),

$$F(s, t) = \frac{1}{2i} \frac{\alpha(s)}{\sin \pi \alpha(s)} \sum_{\ell=0}^{\infty} (2\ell+1) i^\ell P_\ell(\cos \theta) \times \left\{ e^{i\pi(\alpha(s)-2av+b)} \int_{-\infty}^{\infty} dx \frac{e^{-x(2av-b+1)}}{[1+e^{-x}]^{\alpha(s)+1}} j_\ell(z) - e^{-i\pi(\alpha(s)-2av+b)} \int_{-\infty}^{\infty} dx \frac{e^{-x(2av-b+1)}}{[1+e^{-x}]^{\alpha(s)+1}} j_\ell(w) \right\}, \quad (2.4)$$

where $z = -i2av(i\pi+x)$ and $w = -i2av(-i\pi+x)$. In a similar fashion, we can proceed to find a corresponding expansion for $F(s, u)$; the result is that one has simply to replace $-z$ and $-w$ for z and w in Eq. (2.4).

We define the partial waves $V_\ell^{(\pm)}(s)$ by

$$F(s, t) \pm F(s, u) = \sum_{\ell=0}^{\infty} (2\ell+1) V_\ell^{(\pm)}(s) P_\ell(\cos \theta), \quad (2.5)$$

and, using $j_\ell(-z) = e^{i\pi\ell} j_\ell(+z)$, we see easily that

$$V_\ell^{(\pm)}(s) = [1 \pm e^{i\pi\ell}] \frac{\alpha(s)}{\sin \pi \alpha(s)} \frac{1}{2} i^{\ell-1} h_\ell(v), \quad (2.6)$$

where

$$h_\ell(v) = e^{i\pi(\alpha(s)-2av+b)} \int_{-\infty}^{\infty} dx \frac{e^{-x(2av-b+1)}}{[1+e^{-x}]^{\alpha(s)+1}} j_\ell(z) - e^{-i\pi(\alpha(s)-2av+b)} \int_{-\infty}^{\infty} dx \frac{e^{-x(2av-b+1)}}{[1+e^{-x}]^{\alpha(s)+1}} j_\ell(w). \quad (2.7)$$

Similarly, we define the partial waves $V_\ell(s)$ by

$$F(t,u) = \sum_{\ell=0}^{\infty} (2\ell+1) V_{\ell}(s) P_{\ell}(\cos\theta) \quad (2.8)$$

and obtain

$$V_{\ell}(s) = [1+e^{i\pi\ell}] \frac{1}{2} i^{\ell} (4av-2b+1) g_{\ell}(v) , \quad (2.9)$$

where

$$g_{\ell}(v) = \int_{-\infty}^{\infty} dx \frac{e^{-x(2av-b+1)}}{[1+e^{-x}]^{4av-2b+2}} j_{\ell}(i2avx) . \quad (2.10)$$

Equations (2.6) and (2.9) show the signature factors explicitly.

III. ANALYTIC CONTINUATION

Up to now, we have restricted ourselves to the physical region. Even in this region, we observe that the integrands of $h_{\ell}(v)$ and $g_{\ell}(v)$, as functions of complex x , have essential singularities only along the imaginary axis and are analytic elsewhere. Therefore, so long as we can neglect the contribution from the far-distant arc, we can shift the path of integration to any ray in the first and third or the fourth and second quadrants in the complex x plane.

When v is real and positive (i.e., it assumes a physical value), the integrands of $h_{\ell}(v)$ are bounded by

$$M(\ell, v) |\pm i\pi + x|^{\ell} \exp[-(1-b)\text{Re } x] \text{ as } \text{Re } x \rightarrow +\infty ,$$

and by

$$N(\ell, v) |\pm i\pi + x|^{\ell} \exp[+(4a\pi^2 + 2b)\text{Re } x] \text{ as } \text{Re } x \rightarrow -\infty ,$$

where $M(\ell, v)$ and $N(\ell, v)$ are functions of ℓ and v only. Here, we have used the inequality,

$$\Gamma(\nu+1) |J_\nu(z)| \leq \left| \frac{1}{2}z \right|^\nu e^{|\operatorname{Im} z|} \quad \text{for } \nu \geq -\frac{1}{2}.$$

This shows that the contribution from the far-distant arc indeed vanishes. We see also that the integrand of $g_\ell(\nu)$ is bounded by $|x|^\ell \exp[-(1-b)|\operatorname{Re} x|]$ as $|\operatorname{Re} x| \rightarrow \infty$, which shows again that the contribution from the far-distant arc vanishes.

We therefore shift the path of integration so that we have, for ν real and positive,

$$\begin{aligned} h_\ell(\nu) &= e^{i\pi(\alpha(s)-2a\nu+b)} \int_{-\infty}^{\infty} dx e^{i\delta} \frac{e^{-xe^{i\delta}(2a\nu-b+1)}}{[1+e^{-xe^{i\delta}}]^{\alpha(s)+1}} j_\ell(z) \\ &\quad - e^{-i\pi(\alpha(s)-2a\nu+b)} \int_{-\infty}^{\infty} dx e^{i\delta} \frac{e^{-xe^{i\delta}(2a\nu-b+1)}}{[1+e^{-xe^{i\delta}}]^{\alpha(s)+1}} j_\ell(w), \end{aligned} \quad (3.1)$$

where $z = -i2a\nu(i\pi+xe^{i\delta})$ and $w = -i2a\nu(-i\pi+xe^{i\delta})$; similarly,

$$g_\ell(\nu) = \int_{-\infty}^{\infty} dx e^{i\delta} \frac{e^{-xe^{i\delta}(2a\nu-b+1)}}{[1+e^{-xe^{i\delta}}]^{4a\nu-2b+2}} j_\ell(i2a\nu xe^{i\delta}). \quad (3.2)$$

In both cases we have $|\delta| < \frac{\pi}{2}$.

Now we remove the restriction that ν is real and positive, and extend ν into its complex plane. We see that as long as the phase θ of $\nu = |\nu|e^{i\theta}$ is such that the integrands of $h_\ell(\nu)$ and $g_\ell(\nu)$ are bounded at $|x| = \infty$, so as to give the integrals validity, $h_\ell(\nu)$ and $g_\ell(\nu)$ are analytic functions in this domain. Since the integrands of $h_\ell(\nu)$ of (3.1) are bounded by

$$M(\nu, \ell) |\pm i\pi+xe^{i\delta}|^\ell \exp\{-x[2a|\nu| \cos(\theta+\delta) + (1-b)\cos\delta] + 2a|\nu||x| \cos(\theta+\delta)\}$$

as $x \rightarrow +\infty$, and by

$$N(v, \ell) | \pm i\pi + x e^{i\delta} |^{\ell} \exp \{ +x [2a|v| \cos(\theta+\delta) + (4a^2 m_{\pi}^2 + 2b) \cos \delta] - 2a|v|x| \cos(\theta+\delta) | \}$$

as $x \rightarrow -\infty$, we find that the integrals defining $h_{\ell}(v)$ exist if and only if

$$2a|v| \cos(\theta+\delta) + (1-b) \cos \delta - 2a|v| | \cos(\theta+\delta) | > 0$$

and

$$2a|v| \cos(\theta+\delta) + (4a^2 m_{\pi}^2 + 2b) \cos \delta - 2a|v| | \cos(\theta+\delta) | > 0 .$$

From these conditions, we deduce that so long as

$$| \theta + \delta | \leq \frac{\pi}{2} \text{ and } | \delta | < \frac{\pi}{2} , \quad (3.3)$$

$h_{\ell}(v)$ [Eq. (3.1)] is analytic in the complex v plane. We note also that even if $\theta = \pi$, $h_{\ell}(v)$ is well defined provided $|v|$ is less than the smaller of $(1-b)/4a$ and $(4a^2 m_{\pi}^2 + 2b)/4a$. Similarly, the integral defining $g_{\ell}(v)$ [Eq. (3.2)] exists for complex v if (3.3) holds, because the integrand of $g_{\ell}(v)$ is bounded by $|x|^{\ell} \exp \{ -2a|vx| \cos(\theta+\delta) - |x|[(1-b) \cos \delta + 2a|vx| \cos(\theta+\delta)] \}$ as $|x| \rightarrow \infty$. We see also from this bound that the integral still exists for $\theta = \pi$ provided $|v| < (1-b)/4a$.

We concluded therefore that $h_{\ell}(v)$, given by (3.1), and $g_{\ell}(v)$, given by (3.2), are analytic functions in the entire complex v plane except along the negative real axis, starting at $-v_L$ [where v_L is the lesser of $(1-b)/4a$ and $(4a^2 m_{\pi}^2 + 2b)/4a$ for $h_{\ell}(v)$ and $(1-b)/4a$ for $g_{\ell}(v)$]. They are free of singularities except possibly at infinity.

From Eq. (3.1), we observe that

$$h_{\ell}^*(v) = e^{-i\pi(\alpha(s^*) - 2av^* + b)} \int_{-\infty}^{\infty} dx e^{-i\delta} \frac{e^{-xe^{-i\delta}(2av^*-b+1)}}{[1+e^{-xe^{-i\delta}}]^{\alpha(s^*)+1}} j_{\ell}(z^*)$$

$$- e^{+i\pi(\alpha(s^*) - 2av^* + b)} \int_{-\infty}^{\infty} dx e^{-i\delta} \frac{e^{-xe^{-i\delta}(2av^*-b+1)}}{[1+e^{-xe^{-i\delta}}]^{\alpha(s^*)+1}} j_{\ell}(w^*) ,$$

where $z^* = +i2av^*(-i\pi+xe^{-i\delta})$ and $w^* = +i2av^*(+i\pi+xe^{-i\delta})$. We know that the factor $e^{i\delta}$ in $h_{\ell}(v)$ must appear as $e^{-i\delta}$ in $h_{\ell}(v^*)$ in virtue of the constraints (3.3). By making use of $j_{\ell}(-z) = e^{i\pi\ell} j_{\ell}(+z)$, we find

$$[i^{\ell-1} h_{\ell}(v)]^* = i^{\ell-1} h_{\ell}(v^*) ,$$

which in turn gives

$$[V_{\ell}^{(\pm)}(s)]^* = V_{\ell}^{(\pm)}(s^*) . \quad (3.4)$$

Similarly, we see from Eq. (3.2) that

$$[i^{\ell} g_{\ell}(v)]^* = i^{\ell} g_{\ell}(v^*) ,$$

which yields

$$V_{\ell}^*(s) = V_{\ell}(s^*) . \quad (3.5)$$

IV. THRESHOLD BEHAVIOR

Even though the integrals defining $h_{\ell}(v)$ and $g_{\ell}(v)$ in Eqs. (3.1) and (3.2) look very complicated, they are very useful and powerful representations of the partial waves. In this section, we show how simply they yield the threshold coefficients.

We substitute the power-series expansion of $j_\ell(z)$,

$$j_\ell(z) = \frac{\sqrt{\pi}}{2} \sum_{m=0}^{\infty} (-)^m \frac{\left(\frac{z}{2}\right)^{\ell+2m}}{m! \Gamma(m+\ell+\frac{3}{2})},$$

into (3.1) and (3.2), obtaining the normal threshold behavior of the partial waves:

$$\begin{aligned} h_\ell(v) &= v^\ell H_\ell(v), \\ g_\ell(v) &= v^\ell G_\ell(v), \end{aligned} \quad (4.1)$$

where

$$\begin{aligned} H_\ell(v) &= \frac{\sqrt{\pi}}{2} (-ia)^\ell \sum_{m=0}^{\infty} \frac{(av)^{2m}}{m! \Gamma(m+\ell+\frac{3}{2})} \\ &\times \left\{ e^{i\pi(\alpha(s)-2av+b)} \int_{-\infty}^{\infty} dx e^{i\delta} \frac{e^{-xe^{i\delta}(2av-b+1)}}{[1+e^{-xe^{i\delta}}]^{\alpha(s)+1}} [i\pi+xe^{i\delta}]^{\ell+2m} \right. \\ &\quad \left. - e^{-i\pi(\alpha(s)-2av+b)} \int_{-\infty}^{\infty} dx e^{i\delta} \frac{e^{-xe^{i\delta}(2av-b+1)}}{[1+e^{-xe^{i\delta}}]^{\alpha(s)+1}} [-i\pi+xe^{i\delta}]^{\ell+2m} \right\}, \end{aligned}$$

i.e.,

$$\begin{aligned} H_\ell(v) &= 2i \frac{\sqrt{\pi}}{2} (-ia)^\ell \sum_{m=0}^{\infty} \frac{(av)^{2m}}{m! \Gamma(m+\ell+\frac{3}{2})} \\ &\times \left[\left(\frac{d}{dy} \right)^{\ell+2m} \sin \pi(\alpha(s)-2av+b+y) \int_{-\infty}^{\infty} dx e^{i\delta} \frac{e^{-xe^{i\delta}(2av-b+1-y)}}{[1+e^{-xe^{i\delta}}]^{\alpha(s)+1}} \right]_{y=0}, \end{aligned} \quad (4.2)$$

$$G_{\ell}(v) = \frac{\sqrt{\pi}}{2}(-ia)^{\ell} \sum_{m=0}^{\infty} \frac{(av)^{2m}}{m! \Gamma(m+\ell+\frac{3}{2})} \left[\left(\frac{d}{dy} \right)^{\ell+2m} \int_{-\infty}^{\infty} dx e^{i\delta} \frac{e^{-xe^{i\delta}(2av-b+1+y)}}{[1+e^{-xe^{i\delta}}]^{4av-2b+2}} \right]_{y=0}.$$

The integrals in (4.2) yield¹⁴ simply beta functions, thus giving

$$H_{\ell}(v) = 2\pi i \frac{\sqrt{\pi}(-ia)^{\ell}}{2\Gamma(\ell+\frac{3}{2})} \sum_{m=0}^{\infty} \frac{(av)^{2m}}{m! \Gamma(m+\ell+\frac{3}{2})} \left[\left(\frac{d}{dy} \right)^{\ell+2m} \frac{\Gamma(2av-b+1-y)}{\Gamma(1-4am_{\pi}^2-2b-2av-y)} \right]_{y=0}, \quad (4.3)$$

$$G_{\ell}(v) = \frac{\sqrt{\pi}}{2}(-ia)^{\ell} \sum_{m=0}^{\infty} \frac{(av)^{2m}}{m! \Gamma(m+\ell+\frac{3}{2})} \left[\left(\frac{d}{dy} \right)^{\ell+2m} B(2av-b+1+y, 2av-b+1-y) \right]_{y=0}.$$

If we expand $H_{\ell}(v)$ and $G_{\ell}(v)$ around $v = 0$, i.e.,

$$\begin{aligned} H_{\ell}(v) &= C_{\ell} + D_{\ell}v + O(v^2), \\ G_{\ell}(v) &= E_{\ell} + F_{\ell}v + O(v^2), \end{aligned} \quad (4.4)$$

we find that

$$\begin{aligned} C_{\ell} &= \frac{\sqrt{\pi}(-ia)^{\ell}}{2\Gamma(\ell+\frac{3}{2})} \frac{2\pi i}{\Gamma(1+\ell+b)} \left(\frac{d}{dy} \right)^{\ell} \frac{\Gamma(1-b-y)}{\Gamma(1-\lambda-2b-y)}, \\ D_{\ell} &= \frac{\sqrt{\pi}(-ia)^{\ell}}{2\Gamma(\ell+\frac{3}{2})} \frac{4\pi ai}{\Gamma(1+\ell+b)} \left(\frac{d}{dy} \right)^{\ell} \frac{\Gamma(1-b-y)}{\Gamma(1-\lambda-2b-y)} \left[\psi(1-b-y) + \psi(1-\lambda-2b-y) - 2\psi(1+\lambda+b) \right], \\ E_{\ell} &= \frac{\sqrt{\pi}(-ia)^{\ell}}{2\Gamma(\ell+\frac{3}{2})} \left(\frac{d}{dy} \right)^{\ell} B(1-b+y, 1-b-y), \\ F_{\ell} &= \frac{\sqrt{\pi}(-ia)^{\ell}}{2\Gamma(\ell+\frac{3}{2})} 2a \left(\frac{d}{dy} \right)^{\ell} B(1-b+y, 1-b-y) \left[\psi(1-b+y) + \psi(1-b-y) - 2\psi(2-2b) \right], \end{aligned} \quad (4.5)$$

where $y = 0$, $\lambda = 4am_{\pi}^2$, and $\psi(z) = \frac{d}{dz} \ln \Gamma(z)$.

V. SCATTERING LENGTHS AND EFFECTIVE RANGES

As a simple application of our partial-wave formulation, we compute the scattering lengths and effective ranges for the s, p, d, and f waves.

Since we have only the real part of the amplitude in the dual model, we identify the real part of the right-hand side of Eq. (A.10) with its left-hand side, where $A^I(s, \theta)$ is given by (2.2). That is to say, we have

$$\frac{v^\ell [(a_\ell^I)^{-1} + \frac{1}{2}r_\ell^I v + 0(v^2)]}{[(a_\ell^I)^{-1} + \frac{1}{2}r_\ell^I v + 0(v^2)]^2 + v^{2\ell+1}} = \frac{1}{2\sqrt{s}} \int_{-1}^1 d(\cos\theta) A^I(s, \theta) P_\ell(\cos\theta) . \quad (5.1)$$

If we define x_ℓ^I and y_ℓ^I such that

$$\int_{-1}^1 d(\cos\theta) A^I(s, \theta) P_\ell(\cos\theta) = v^\ell [x_\ell^I + v y_\ell^I + 0(v^2)] , \quad (5.2)$$

then we can express a_ℓ^I and r_ℓ^I in terms of x_ℓ^I and y_ℓ^I as follows:

$$a_\ell^I = \frac{x_\ell^I}{4m_\pi} , \quad (5.3)$$

$$r_\ell^I = -2a_{0\ell0}^I + \frac{1}{m_\pi^2 a_\ell^I} - \frac{y_\ell^I}{2m_\pi (a_\ell^I)^2} .$$

On the other hand, we find from Eqs. (2.2, 5, 6, 8, 9), (4.1, 4) and (5.2)

that

$$x_\ell^0 = \frac{1}{2}g(1+e^{i\pi\ell})i^\ell [(1-2b)E_\ell + 3i\sigma C_\ell] ,$$

$$y_\ell^0 = \frac{1}{2}g(1+e^{i\pi\ell})i^\ell [4aE_\ell + (1-2b)F_\ell + 3i\xi C_\ell + 3i\sigma D_\ell] ,$$

$$x_\ell^1 = -g(1-e^{i\pi\ell})i^{\ell-1}\sigma C_\ell ,$$

(5.4)

$$y_{\ell}^1 = -g(1-e^{i\pi\ell})i^{\ell-1}(\xi C_{\ell} + \sigma D_{\ell}) ,$$

$$x_{\ell}^2 = -g(1+e^{i\pi\ell})i^{\ell}(1-2b)E_{\ell} ,$$

$$y_{\ell}^2 = -g(1+e^{i\pi\ell})i^{\ell}[4aE_{\ell} + (1-2b)F_{\ell}] ,$$

where

$$\sigma \sin \pi(\lambda+b) = \lambda+b \text{ and } \xi \sin \pi(\lambda+b) = 4a[1-\pi(\lambda+b)\cot \pi(\lambda+b)] .$$

Substituting (5.4) and (4.5) into (5.3), we can calculate the scattering lengths and effective ranges. For example, the scattering lengths for the s, p, d, and f waves are:

$$a_0^0 = \frac{g}{4m_{\pi}} \left\{ \frac{\Gamma(1-b)\Gamma(1-b)}{\Gamma(1-2b)} - 6 \frac{\Gamma(1-b)\Gamma(1-\lambda-b)}{\Gamma(1-\lambda-2b)} \right\} , \quad (5.5)$$

$$a_0^2 = - \frac{g}{2m_{\pi}} \frac{\Gamma(1-b)\Gamma(1-b)}{\Gamma(1-2b)} , \quad (5.6)$$

$$a_1^1 = \frac{g}{6m_{\pi}^3} \frac{\lambda\Gamma(1-b)\Gamma(1-\lambda-b)}{\Gamma(1-\lambda-2b)} [\psi(1-b) - \psi(1-\lambda-2b)] , \quad (5.7)$$

$$a_2^0 = \frac{g}{120m_{\pi}^5} \lambda^2 \Gamma(1-b) \left\{ \frac{\Gamma(1-b)}{\Gamma(1-2b)} \psi'(1-b) - \frac{3\Gamma(1-\lambda-b)}{\Gamma(1-\lambda-2b)} [(\psi(1-b)-\psi(1-\lambda-2b))^2 + \psi'(1-b) - \psi'(1-\lambda-2b)] \right\} , \quad (5.8)$$

$$a_2^2 = - \frac{g}{60m_{\pi}^5} \frac{\lambda^2 \Gamma(1-b)\Gamma(1-b)}{\Gamma(1-2b)} \psi'(1-b) , \quad (5.9)$$

$$\begin{aligned}
 a_3^1 = & \frac{g}{840m_\pi^7} \frac{\lambda^3 \Gamma(1-b) \Gamma(1-\lambda-b)}{\Gamma(1-\lambda-2b)} \{ [\psi(1-b) - \psi(1-\lambda-2b)]^3 \\
 & + 3[\psi(1-b) - \psi(1-\lambda-2b)] [\psi'(1-b) - \psi'(1-\lambda-2b)] \\
 & + \psi''(1-b) - \psi''(1-\lambda-2b) \} .
 \end{aligned} \tag{5.10}$$

Now, from Eqs. (5.5,6), we get, for $b \neq \frac{1}{2}$,

$$\frac{a_0^0}{a_0^2} = -\frac{1}{2} \left\{ 1 - 6 \frac{\Gamma(1-\lambda-b) \Gamma(1-2b)}{\Gamma(1-b) \Gamma(1-\lambda-2b)} \right\} , \tag{5.11}$$

from which we infer

(i) for $\lambda \equiv 4am_\pi^2 = 0$, $b \neq \frac{1}{2}$,

$$\frac{a_0^0}{a_0^2} = +\frac{5}{2} ; \tag{5.12}$$

(ii) for λ and b such that $0 < \lambda < 1$ and $b = \frac{1}{2} - \frac{\lambda}{4}$ (i.e., b satisfies

Lovelace's condition² $am_\pi^2 + b = \frac{1}{2}$),

$$\frac{a_0^0}{a_0^2} = -\frac{7}{2} - 6\lambda \ln 2 + O(\lambda^2) . \tag{5.13}$$

The first term, $-7/2$, is Weinberg's ratio.¹⁵ To obtain (5.13), we have used

$$\psi(1) - \psi\left(\frac{1}{2}\right) = 2 \ln 2 .$$

Similarly, from (5.5-7), we deduce

$$2a_0^0 - 5a_0^2 = 18m_\pi^2 a_1^1 \Delta(\lambda, b) , \quad (5.14)$$

where

$$\lambda[\psi(1-b) - \psi(1-\lambda-2b)]\Delta(\lambda, b) = \frac{\Gamma(1-b)\Gamma(1-\lambda-2b)}{\Gamma(1-2b)\Gamma(1-\lambda-b)} - 1 . \quad (5.15)$$

If $\lambda \ll 1$, we can expand $\Delta(\lambda, b)$ in powers of λ , viz.,

$$\Delta(\lambda, b) = 1 + \lambda \Sigma + O(\lambda^2) , \quad (5.16)$$

where

$$2[\psi(1-b) - \psi(1-2b)]\Sigma = [\psi(1-b) - \psi(1-2b)]^2 - \psi'(1-b) - \psi'(1-2b) . \quad (5.17)$$

Lovelace's value² $b = 0.483$, for example, gives $\Sigma = -1.502$ and

$$2a_0^0 - 5a_0^2 = 18m_\pi^2 a_1^1 [1 - 1.502\lambda + O(\lambda^2)] . \quad (5.18)$$

To obtain numerical values for the scattering lengths and effective ranges, we take the experimental pion mass (0.139 GeV) and rho-meson mass (0.765 GeV), and adjust the parameters $\underline{a}$ and $\underline{b}$ such that

$$am_\rho^2 + b = 1 \text{ and } am_\pi^2 + b = \frac{1}{2} .$$

The second condition was taken by Lovelace² so that the dual model will satisfy Adler's self-consistency condition.¹⁶ We adjust the overall coupling constant g so that the p-wave resonance corresponds to the experimental rho-meson resonance width (0.125 GeV). We then find $g = 1.208$, $a = 0.883$, and $b = 0.483$. The resulting scattering lengths and effective ranges are listed below:

$$\begin{aligned} a_0^0 &= +0.230 \text{ m}_\pi^{-1} , & r_0^0 &= -6.43 \text{ m}_\pi^{-1} , \\ a_0^2 &= -0.062 \text{ m}_\pi^{-1} , & r_0^2 &= +44.7 \text{ m}_\pi^{-1} , \end{aligned}$$

$$\begin{aligned}
 a_1^1 &= +0.047 \, m_\pi^{-3} , & r_1^1 &= +17.1 \, m_\pi , \\
 a_2^0 &= +0.12 \times 10^{-2} \, m_\pi^{-5} , & r_2^0 &= +8.69 \times 10^2 \, m_\pi^3 , \\
 a_2^2 &= -0.43 \times 10^{-4} \, m_\pi^{-5} , & r_2^2 &= +5.99 \times 10^4 \, m_\pi^3 , \\
 a_3^1 &= +0.21 \times 10^{-4} \, m_\pi^{-7} , & r_3^1 &= +6.00 \times 10^4 \, m_\pi^5 .
 \end{aligned}$$

VI. CONCLUDING REMARKS

We would like to emphasize that the integral representations we have formulated for the partial waves of the dual $\pi\pi$ scattering amplitudes are very useful and powerful in many respects. As we have seen, one of the advantages of this formalism is that we can extract the threshold coefficients in a straightforward way.

To exploit this advantage, we have calculated the scattering lengths and effective ranges for the s, p, d, and f waves. We can of course extend this procedure to the higher angular-momentum waves as well, and may use them in the effective-range-approximation approach to unitarize the dual partial-wave amplitudes in the low-energy region.

The merits of these integral representations will probably be more manifest when we study the asymptotic behavior of the partial waves in the complex energy-squared plane. This subject will be investigated in a subsequent paper.

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APPENDIX. KINEMATICS AND NOTATION

We consider the elastic $\pi\pi$ scattering amplitude for the process $a+b \rightarrow c+d$. We follow closely the notation used in Ref. 3, defining the S matrix by $S = 1 + i(2\pi)^4 \delta^4(\Sigma p) T(s, t, u)$, with $M^{dcba}(s, t, u) = T^{dcba}(s, t, u)/16\pi$. The quantities a, b, c , and d are Cartesian-basis vectors for the isospins of the particles, and s, t, u are the usual Mandelstam variables $(p_a + p_b)^2, (p_a + p_c)^2, (p_a + p_d)^2$, respectively, all p 's being physical energy-momentum vectors. The most general amplitude satisfying Bose statistics, isospin conservation, and crossing symmetry¹⁷ is

$$\begin{aligned} M^{dcba}(s, t, u) = & A(s, t, u) \delta_{ab} \delta_{cd} \\ & + B(s, t, u) \delta_{ac} \delta_{bd} \\ & + C(s, t, u) \delta_{ad} \delta_{bc} \quad , \end{aligned} \quad (A.1)$$

where

$$A(s, t, u) = A(s, u, t) = B(t, s, u) = C(u, t, s) \quad . \quad (A.2)$$

The total-isospin s-channel amplitudes are then given by

$$\begin{aligned} A^0(s, t, u) &= 3A + B + C \quad , \\ A^1(s, t, u) &= B - C \quad , \\ A^2(s, t, u) &= B + C \quad . \end{aligned} \quad (A.3)$$

We observe, in particular, that if $A^2(s, t, u)$ is known throughout the Mandelstam diagram, then the entire amplitude is determined by

$$A(s, t, u) = -\frac{1}{2}A^2(s, t, u) + \frac{1}{2}A^2(u, s, t) + \frac{1}{2}A^2(t, u, s) \quad . \quad (A.4)$$

We define the volume element of the one-particle contribution to Lorentz-invariant phase space to be $(2\pi)^{-3} d^3p/2E$. The Lorentz-invariant

normalization of states is then $\langle \vec{p}' | \vec{p} \rangle = (2\pi)^3 2E \delta^3(\vec{p}' - \vec{p})$. In terms of $M(s, t, u)$ and $A(s, \theta)$, we have $(d\sigma^I/d\Omega)_{el} = (4/s) |M^I|^2 = (4/s) |A^I(s, \theta)|^2$, and the unitarity condition $SS^\dagger = S^\dagger S = 1$ gives

$$\text{Im } A^I(s, \theta) = \frac{q}{4\pi\sqrt{s}} \int d\Omega' A^I(s, \theta'') [A^I(s, \theta')]^* \quad (\text{A.5})$$

for s below the inelastic threshold. Here, q is the c.m. momentum, $\cos\theta'' = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos\phi'$, and θ is the scattering angle in the c.m. system.

We define the partial waves $A_\ell^I(s)$ by

$$A^I(s, \theta) = \sum_{\ell=0}^{\infty} 2(2\ell+1) A_\ell^I(s) P_\ell(\cos\theta), \quad (\text{A.6})$$

where the extra factor 2 is due to Bose statistics. Elastic unitarity, Eq. (A.5), gives

$$\text{Im } A_\ell^I(s) = \frac{2q}{\sqrt{s}} |A_\ell^I(s)|^2. \quad (\text{A.7})$$

The phase shift $\delta_\ell^I(s)$ is defined by

$$A_\ell^I(s) = \frac{\sqrt{s}}{2q} e^{i\delta_\ell^I(s)} \sin\delta_\ell^I(s), \quad (\text{A.8})$$

so that it satisfies (A.7) in the elastic region.

The "scattering lengths" a_ℓ^I and the "effective ranges" r_ℓ^I are defined by

$$q^{2\ell+1} \cot\delta_\ell^I(s) = \frac{1}{a_\ell^I} + \frac{1}{2} r_\ell^I q^2 + O(q^4). \quad (\text{A.9})$$

It follows from Eqs. (A.6,8,9) that

$$\frac{1}{2\sqrt{s}} \int_{-1}^1 d(\cos\theta) A^I(s, \theta) P_\ell(\cos\theta) = q^{2\ell} / [(a_\ell^I)^{-1} + \frac{1}{2} r_\ell^I q^2 + O(q^4) - iq^{2\ell+1}]. \quad (\text{A.10})$$

Near a resonance, we obtain the Breit-Wigner relation

$$e^{i\delta_\ell^I(s)} \sin \delta_\ell^I(s) = M_R \Gamma_\ell^I / (M_R^2 - s - iM_R \Gamma_\ell^I) , \quad (\text{A.11})$$

where M_R and Γ_ℓ^I are the mass and the width of the resonance, respectively.

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