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Convergent Plasma Test Particle Problem

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A charged test particle passing through an electron plasma is subject to a drag force. Previous calculations of this drag force, when the test particle's velocity is maintained at a constant value, have led to logarithmically divergent results. In this paper a convergent and consistent expression is derived for the drag using a modified linearization of the Vlasov equation.

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I. INTRODUCTION

A charged particle moving through a fully ionized plasma will experience a drag force due to collisions with the plasma electrons and ions. The calculation of this drag, on a "test particle" constrained to move with constant velocity, is referred to as the plasma test particle problem.

The simplest formulation of the test particle problem neglects terms of the order of the electron mass divided by the ion mass; in this case the ions are assumed to maintain a uniform distribution. The electron plasma and test particle are described by the Vlasov (self consistent field) equation,

$$\begin{aligned} \frac{\partial f(\underline{r}, \underline{v}, t)}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{r}} - \frac{ee_T}{m} \frac{\partial \phi(|\underline{r} - \underline{r}_T|)}{\partial \underline{r}} \cdot \frac{\partial f}{\partial \underline{r}} \\ - \frac{e^2 n}{m} \int d\underline{v}' \int d\underline{r}' \frac{\partial \phi(|\underline{r} - \underline{r}'|)}{\partial \underline{r}} f(\underline{r}', \underline{v}', t) \cdot \frac{\partial f}{\partial \underline{v}} = 0 \end{aligned} \quad (1)$$

where $\phi(r) = 1/r$, and $f(\underline{r}, \underline{v}, t)$ is the distribution function for the electrons, normalized to unity; the electrons have charge $-e$, mass m , and (in the absence of the test charge) average density n . The test particle with charge $-e_T$ has the position vector $\underline{r}_T = \underline{v}_T t$ where $\underline{v}_T$ is a constant velocity.

The drag force on the test particle is equal to the test particle charge times the electric field induced in the plasma evaluated at $\underline{r}_T$,

$$\ddot{x} = e_T e n \int dv \int dr \frac{\partial \phi(|\underline{r} - \underline{r}_T|)}{\partial r} f(\underline{r}, \underline{v}, t) \quad (2)$$

In Section II two approximate solutions of Eq. (1) are discussed. Both solutions lead to logarithmically divergent drag expressions. In Section III a method of solving Eq. (1) is described that leads to a drag expression which does not diverge. This expression for the drag is exact in the limit $v_T \rightarrow 0$. When v_T exceeds the mean electron speed, $(\theta/m)^{1/2}$, the result is still qualitatively correct.

II. REVIEW

Previous treatments of the test particle problem have used one of two limiting cases of Eq. (1). In one case the test particle is assumed to be influenced primarily by close binary collisions with electrons, and in the other case the test particle is assumed to be influenced primarily by distant many body collisions with plasma electrons.

In the binary collision approach the self consistent field term (last term) in Eq. (1) is neglected. The remaining terms describe the motion of electrons interacting with the Coulomb field of the test particle. For a boundary condition we assume that electrons are in a Maxwell-Boltzmann velocity distribution before interacting with the test particle,

$$f = f^{(0)}(\underline{v}) = (m/2\pi\theta)^{3/2} \exp(-m\underline{v}^2/2\theta), \quad (3)$$

where $\theta = k_B T$ is the kinetic temperature. The drag, Eq. (2), obtained with

this approximation is equivalent to the drag obtained from a simple binary collision calculation.¹ When $v_T < (\theta/m)^{1/2}$ the drag expression is

$$\mathcal{F}_{B.C.} = -e_T^2 \omega_p^2 \ln(3B/R_L) v_T^{-3} \int_0^{v_T} dv f^{(0)}(v) \quad (4)$$

where $\omega_p = (4\pi e^2 n/m)^{1/2}$ is the plasma frequency, $R_L = ee_T/m$ is the Landau length, and B is a (unspecified) maximum impact parameter. The drag expression diverges logarithmically as $B \rightarrow \infty$. This divergence is the result of our failure to include collective plasma interactions (last term in Eq. (1)). It is known that collective interactions screen the test particle,² and should limit the maximum impact parameter to approximately the Debye length,

$$B \approx R_D = (4\pi e^2 n/\theta)^{-1/2}$$

If we think of the test particle as influenced primarily by the simultaneous resultant of many weak interactions then the effect of the test particle on the plasma can be treated as a perturbation. In this case a linearized solution of Eq. (1) can be obtained. The distribution function is written

$$f(\underline{r}, \underline{v}, t) = f^{(0)}(\underline{v}) + h(\underline{r}, \underline{v}, t) \quad (5)$$

Using Eq. (5) in Eq. (1) and neglecting products of h and $e_T \partial\phi/\partial \underline{r}$ a linear integral equation is obtained for h . The approximate linear solution

for the electron distribution can be used in Eq. (2) to obtain an expression for the drag,^{3,4}

$$\mathcal{F}_L = -e_T^2 u_p^2 \ln(R_D/b) v_T^{-3} \int_0^{v_T} dv f^{(0)}(v) \quad (6)$$

where b is some (unspecified) minimum distance at which the linearized solution breaks down. It is easy to see how this breakdown occurs. In the neighborhood of the test particle the integral term in Eq. (1) is unimportant compared to the direct interaction term. In that region Eq. (1) is approximated by

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \frac{ee_T}{m} \frac{\partial \phi(|\mathbf{r} - \mathbf{r}_T|)}{\partial \mathbf{r}} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \quad (7)$$

This is the equation used to obtain the binary collision drag expression, Eq. (4). The equation used to obtain Eq. (6), in the neighborhood of the test particle, is the linearized version of this equation

$$\frac{\partial h}{\partial t} + \mathbf{v} \cdot \frac{\partial h}{\partial \mathbf{r}} = \frac{ee_T}{m} \frac{\partial \phi(|\mathbf{r} - \mathbf{r}_T|)}{\partial \mathbf{r}} \cdot \frac{\partial f^{(0)}}{\partial \mathbf{v}} \quad (8)$$

The time asymptotic electron distribution obtained from the solution of Eq. (8) is a function of $\xi = \mathbf{r} - \mathbf{r}_T$,

$$f(\xi, \mathbf{v}) = f^{(0)}(\mathbf{v}) + \frac{ee_T}{m} \int_0^\infty d\tau \frac{\partial}{\partial \xi} \phi(|\xi - (\mathbf{v} - \mathbf{v}_T)\tau|) \cdot \frac{\partial f^{(0)}(\mathbf{v})}{\partial \mathbf{v}} \quad (9)$$

The interpretation of this result is clear. The linear theory develops the distribution function, through the first term of a Taylor series, about the Maxwell-Boltzmann distribution at $\xi = +\infty$. The momentum given to an electron coming from $\xi = +\infty$ to ξ , according to the linear theory, is

$$\Delta P = ee_T \int_0^\infty d\tau \frac{\partial}{\partial \xi} \phi(|\xi - (v - v_T)\tau|).$$

This is a "straight path" approximation, as $\xi \rightarrow 0$ the momentum change is infinite and this accounts for the divergence in the drag expression, Eq. (6).

Several authors^{5,6,7} have combined these two approaches to the test particle problem. In terms of Eq. (4) and Eq. (6) one sets $B = b$ and adds,

$$\begin{aligned} \bar{F} &= \bar{F}_{B.C.} + \bar{F}_L \\ &= -e_T^2 u_P^2 \ln(R_D/R_L) v_T^{-3} \int_0^{v_T} dv f^{(0)}(v) \end{aligned} \quad (10)$$

Since this result relies on two expressions which involve arbitrary cutoffs, it is clear that there is still an ambiguity in the logarithm argument.

III. METHOD OF SOLUTION

In this section we describe a modified linearization of Eq. (1).

Instead of using a spatially uniform Maxwell-Boltzmann distribution for the unperturbed state of the plasma we use the stationary solution of Eq. (1) when the test particle is at rest, $v_T = 0$. In this case the distribution function can be separated into a product of the Maxwell-Boltzmann velocity distribution, Eq. (3), times a Boltzmann-Gibbs type factor, which is a function of $|\underline{r} - \underline{r}_T|$,

$$f(\underline{r}, \underline{v}) = f^{(0)}(\underline{v}) \exp\{-R_L \phi(|\underline{r} - \underline{r}_T|)\} \quad (11)$$

where $R_L = ee_T/\theta$. The "generalized" potential $\phi(r)$ is determined from Eq. (1) with $v_T = 0$. We will see in Section IV that a good approximation for $\phi(r)$ is the Debye-Huckel potential

$$\phi(r) = \frac{1}{r} \exp\{-r/R_D\}$$

where $R_D = (4\pi e^2 n/\theta)^{-1/2}$ is the Debye length.

When the test particle is moving with a constant velocity the equilibrium distribution is distorted and we write

$$f(\underline{r}, \underline{v}, t) = [f^{(0)}(\underline{v}) + h(\underline{r}, \underline{v}, t)] \exp\{-R_L \phi(|\underline{r} - \underline{r}_T|)\} \quad (12)$$

It is possible to obtain a linearized solution for h . We do this in Section V. The linearization neglects products of the quantities h , $(ee_T/m) \partial \phi(|\underline{r} - \underline{r}_T|)/\partial \underline{r}$, and $R_L \phi(|\underline{r} - \underline{r}_T|)$. Of course these terms are not small when $\underline{r}$ is sufficiently close to $\underline{r}_T$, but we will see that this is

unimportant for the drag problem when $v_T < (\theta/m)^{\frac{1}{2}}$. When $v_T \geq (\theta/m)^{\frac{1}{2}}$ the situation is marginal, but the drag expression must still exhibit the qualitative features of an exact solution.

IV. THE GENERALIZED POTENTIAL

To obtain the function $\phi(|\underline{r}-\underline{r}_T|)$ we look for the time independent solution of Eq. (1) in which the distribution function has the form of Eq. (11). We find that $\phi(r)$ must be a solution of

$$\underline{v} \cdot \left\{ \frac{\partial}{\partial \underline{r}} \phi(r) - \frac{\partial \phi(r)}{\partial \underline{r}} - \frac{en}{e_T} \int d\underline{r}' \frac{\partial \phi(|\underline{r}-\underline{r}'|)}{\partial \underline{r}} [\exp[-R_L \phi(r')] - 1] \right\} = 0 \quad (13)$$

where the -1 in the integral accounts for the uniform density of positive charge. Since this must be satisfied for all $\underline{v}$, the vector in brackets is identically zero. Taking the divergence of this vector, and using the identity

$$\nabla^2 \phi(r) = -4\pi \delta(\underline{r}) \quad (14)$$

we find that $\phi(r)$ must be a solution of

$$\nabla^2 \phi(r) = 4\pi e e_T^{-1} n [\exp[-R_L \phi(r)] - 1] - 4\pi \delta(\underline{r}) \quad (15)$$

An approximate solution of Eq. (15) is obtained by expanding the exponential term through the first order in ϕ . The solution of the resulting equation, that vanishes at infinity, is the Debye-Huckel potential

$$\phi(r) = \frac{1}{r} \exp\{-r/R_D\} \quad (16)$$

where $R_D = (4\pi e^2 n / \theta)^{-1/2}$ is the Debye length.

An exact solution for ϕ (which can only be obtained numerically) would replace the simple shielding factor $\exp(-r/R_D)$ by a more complicated function of r , but reducing to this form at large r ($r > R_D$). For our purposes Eq. (16) is sufficiently accurate.

V. THE MODIFIED LINEARIZATION

Inserting Eq. (12) in Eq. (1) and neglecting products of the quantities h , $R_L \phi$, and $e_T \partial \phi / \partial r$, which are assumed small for purposes of linearization, we obtain a linear equation for $h(r, v, t)$,

$$\begin{aligned} \frac{\partial h(r, v, t)}{\partial t} + v \cdot \frac{\partial h}{\partial \mathbf{r}} - \frac{e e_T}{m} \frac{\partial}{\partial \mathbf{r}} \phi(|\mathbf{r} - \mathbf{r}_T|) \cdot \frac{\partial f^{(0)}}{\partial \mathbf{v}} \\ - \frac{e^2 n}{m} \int d\mathbf{v}' \int d\mathbf{r}' \frac{\partial \phi(|\mathbf{r} - \mathbf{r}'|)}{\partial \mathbf{r}} h(\mathbf{r}', \mathbf{v}', t) \cdot \frac{\partial f^{(0)}}{\partial \mathbf{v}} - R_L f^{(0)}(\mathbf{v})(\mathbf{x} - \mathbf{x}_T) \\ \cdot \frac{\partial}{\partial \mathbf{r}} \phi(|\mathbf{r} - \mathbf{r}_T|) = 0 \end{aligned} \quad (17)$$

where $\underline{r}_T = \underline{v}_T t$. Note that Eq. (13) has been used to obtain Eq. (17).

In obtaining Eq. (17) we have neglected terms that are not small in the neighborhood of the test particle. To get an estimate of the validity of the linearization we consider an approximate form of Eq. (17). The third and last terms of Eq. (17) combine to give

$$R_L f^{(0)}(\underline{v}) \underline{v}_T \cdot \frac{\partial}{\partial \underline{r}} \phi(|\underline{r} - \underline{r}_T|).$$

Since the time dependence of $h(\underline{r}, \underline{v}, t)$ is of the form $\underline{r} - \underline{v}_T t$ or $\underline{r} - \underline{r}_T$ the time derivative of h can be replaced by $-\underline{v}_T \cdot \partial / \partial \underline{r} h$. In addition, the integral term in Eq. (17) is small compared to $\phi(|\underline{r} - \underline{r}_T|)$ when $\underline{r}$ is close to $\underline{r}_T$. With these reductions the approximate form of Eq. (17), in the neighborhood of the test particle, is

$$\frac{\partial}{\partial \underline{r}} \cdot \left\{ (\underline{v} - \underline{v}_T) h(\underline{r} - \underline{r}_T, \underline{v}) + R_L \underline{v}_T f^{(0)}(\underline{v}) \phi(|\underline{r} - \underline{r}_T|) \right\} = 0. \quad (18)$$

The linear solution for h will fail when h is on the order of $f^{(0)}(\underline{v})$. There are two cases depending on the magnitude of $\underline{v}_T$. If $\underline{v}_T$ is less than the mean electron speed, $\underline{v}_T \ll (\theta/m)^{1/2}$, Eq. (18) indicates that the linearization breaks down when

$$(\theta/m)^{-1/2} \underline{v}_T R_L \phi(|\underline{r} - \underline{r}_T|) \geq 1.$$

Since the distribution function, Eq. (12), decreases exponentially to zero as $R_L \phi(|\underline{r} - \underline{r}_T|)$ increases the failure of the linearized solution

for h is negligible for small v_T , and is exact as $v_T \rightarrow 0$. If v_T is on the order of the mean electron speed, $v_T \geq (g/m)^{1/2}$, the linearization fails when

$$R_L \phi(|\underline{r}-\underline{r}_T|) \geq 1.$$

This case is marginal, since the exponential factor in the distribution function is reduced to e^{-1} when h has attained a value comparable to $f^{(0)}(\underline{v})$. The distribution function should still retain the qualitative features of an exact solution.

VI. DRAG ON A TEST PARTICLE

The drag on a test particle is given by Eq. (2) and Eq. (12),

$$\underline{\mathcal{F}} = ee_T n \int d\underline{r} \frac{\partial \phi(|\underline{r}-\underline{r}_T|)}{\partial \underline{r}} \cdot \exp\{-R_L \phi(|\underline{r}-\underline{r}_T|)\} \int d\underline{v} h(\underline{r}, \underline{v}, t) \quad (19)$$

where the first term of the distribution function does not contribute to the drag, since it is spherically symmetric about $\underline{r}_T$. The integral in Eq. (19) can also be expressed in terms of Fourier space transforms

$$\underline{\mathcal{F}} = ee_T n (2\pi)^{-3} \int d\underline{k} Q(-\underline{k}) \cdot \exp\{i\underline{k} \cdot \underline{v}_T t\} \int d\underline{v} \tilde{h}(\underline{k}, \underline{v}, t) \quad (20)$$

where $\tilde{h}$ is the Fourier space transform of h , and Q denotes the Fourier

transform of the remaining factor in the integrand,

$$Q(k) = \int d\mathbf{r} \frac{\partial \phi(\mathbf{r})}{\partial \mathbf{r}} \exp\{-R_L \phi(\mathbf{r}) - i\mathbf{k} \cdot \mathbf{r}\} \quad (21)$$

Eq. (20) is a convenient form for the drag expression, since the function $\int d\mathbf{v} \tilde{h}(\mathbf{k}, \mathbf{v}, t)$ is easily obtained from a Fourier transformation of Eq. (17). To be specific, consider the initial value problem $h = 0$ for $t \leq 0$. The Fourier space and time transform of $h(\mathbf{r}, \mathbf{v}, t)$ is defined as

$$\tilde{h}(\mathbf{k}, \mathbf{v}, \omega) = (2\pi)^{-1} \int_0^\infty dt \int d\mathbf{r} h(\mathbf{r}, \mathbf{v}, t) \exp\{-i\mathbf{k} \cdot \mathbf{r} + i\omega t\} \quad (22)$$

where t is integrated only over positive values for the initial value problem. From the Fourier transformation of Eq. (17) we find

$$\int d\mathbf{v} \tilde{h}(\mathbf{k}, \mathbf{v}, \omega) = - \frac{R_L (\mathbf{k} \cdot \mathbf{v}_T) (kR_D)^2 \tilde{\phi}(\mathbf{k})}{(2\pi i) \omega (\omega - \mathbf{k} \cdot \mathbf{v}_T)} \frac{1 + W\left[\frac{\omega}{k} \left(\frac{m}{2\theta}\right)^{\frac{1}{2}}\right]}{(kR_D)^2 + W\left[\frac{\omega}{k} \left(\frac{m}{2\theta}\right)^{\frac{1}{2}}\right]} \quad (23)$$

where

$$W\left[\frac{\omega}{k} \left(\frac{m}{2\theta}\right)^{\frac{1}{2}}\right] = R_D^2 \omega_P^2 \int d\mathbf{v} \frac{\mathbf{k} \cdot \frac{\partial f(\mathbf{v})}{\partial \mathbf{v}}}{\omega - \mathbf{k} \cdot \mathbf{v}} \quad (24)$$

To invert the Fourier time transform we need the analytic continuation of W from $\text{Im} \omega > 0$ to the remaining portion of the ω -plane. This is accomplished in Appendix A where it is shown that for real t ,

$$W(t) = 1 - 2t \exp(-t^2) \int_0^t dx \exp(x^2) + i(\pi)^{\frac{1}{2}} t \exp(-t^2) \quad (25)$$

This function is equal to $-\frac{1}{2} d/dt Z(t)$ where $Z(t)$ is the plasma dispersion function.⁸ It can be shown that $|W(t)| \leq 1$.

The Fourier time inversion of Eq. (23) is

$$\int dv \tilde{h}(k, v, t) = \int_{-\omega+1\gamma}^{\omega+1\gamma} d\omega \exp(-i\omega t) \int dv \tilde{h}(k, v, \omega) \quad (26)$$

where γ is chosen so the ω path of integration lies above any singularities of the integrand. The integral is evaluated by completing the contour in the lower half ω -plane. The function $[(kR_D)^2 + W]$ is $(kR_D)^2$ times the electron plasma dielectric constant,⁹ and has no zeros for $\text{Im}\omega \geq 0$. There is no pole at $\omega = 0$ since the numerator of Eq. (23) vanishes at $\omega = 0$. Therefore, the only pole in Eq. (23) with $\text{Im}\omega \geq 0$ is $\omega = k \cdot v_{Te}$. For then, in time asymptotic drag the poles with $\text{Im}\omega < 0$ are of no concern, since they contribute exponentially damped terms. In view of these remarks the time asymptotic value of Eq. (26) is

$$\int dv \tilde{h}(k, v, t \rightarrow \infty) = R_L (kR_D)^2 \tilde{h}(k) \exp(-ik \cdot v_{Te} t) \frac{1 - W\left[\frac{k \cdot v_{Te}}{k} \left(\frac{m}{2\theta}\right)^{1/2}\right]}{(kR_D)^2 + W\left[\frac{k \cdot v_{Te}}{k} \left(\frac{m}{2\theta}\right)^{1/2}\right]} \quad (27)$$

The drag force is determined from Eq. (20) and Eq. (27)

$$\vec{F} = e e_T n R_L (2\pi)^{-3} \int d\vec{k} Q(-\vec{k}) (kR_D)^2 \tilde{h}(k) \frac{1 - W\left[\frac{k \cdot v_{Te}}{k} \left(\frac{m}{2\theta}\right)^{1/2}\right]}{(kR_D)^2 + W\left[\frac{k \cdot v_{Te}}{k} \left(\frac{m}{2\theta}\right)^{1/2}\right]} \quad (28)$$

Although the integral in Eq. (28) is extremely complicated it is possible to evaluate it to lowest order in R_L/R_D . This is a weak restriction, since

$R_L/R_D < 10^{-5}$ for many interesting plasmas. To lowest order in R_L/R_D the generalized potential is approximated by the Debye-Huckel potential Eq. (16) which has the Fourier transform

$$\tilde{\Phi}(k) = 4\pi R_D^2 / [1 + (kR_D)^2] \quad (29)$$

In Appendix B it is shown that, to lowest order in R_L/R_D , the function $\tilde{Q}$ is given in terms of a Kelvin function

$$\tilde{Q}(k) = -(2\pi)^2 i \frac{k}{k^2} \text{hei}_2[(4R_L k)^{\frac{1}{2}}] \quad (30)$$

Substituting Eq. (29) and Eq. (30) in Eq. (28), a slight rearrangement gives

$$\tilde{\mathcal{F}} = \frac{e_T^2}{2\pi} \int d\mathbf{k} \frac{k}{k^2} \text{hei}_2[(4R_L k)^{\frac{1}{2}}] \left\{ \frac{1}{1 + (kR_D)^2} - \frac{W\left[\frac{k \cdot \mathbf{v}_T}{k} \left(\frac{m}{2\theta}\right)^{\frac{1}{2}}\right]}{(kR_D)^2 + W\left[\frac{k \cdot \mathbf{v}_T}{k} \left(\frac{m}{2\theta}\right)^{\frac{1}{2}}\right]} \right\} \quad (31)$$

The first term, an odd function of $\mathbf{k}$, gives no contribution. For the remaining term polar coordinates are used for $\mathbf{k}$, $(k, \eta = \cos \theta, \phi)$, with $\mathbf{v}_T$ as polar axis. After the ϕ integration is performed only the component of $\mathbf{k}$ parallel to $\mathbf{v}_T$ remains,

$$\tilde{\mathcal{F}} = -\tilde{\mathcal{F}} v_T / v_T$$

where

$$\mathcal{F} = e_T^2 \int_{-1}^1 \eta d\eta \int_0^\infty k dk \operatorname{hei}_2[(4R_L k)^{\frac{1}{2}}] \frac{W[\eta v_T (\frac{m}{2\theta})^{\frac{1}{2}}]}{(kR_D)^2 + W[\eta v_T (\frac{m}{2\theta})^{\frac{1}{2}}]} \quad (32)$$

Changing to dimensionless variables $t = \eta v_T (\frac{m}{2\theta})^{\frac{1}{2}}$, $x = kR_D$ we find

$$\mathcal{F} = 2ie_T^2 \omega_p^2 v_T^{-2} \int_{-\alpha}^{\alpha} t dt \int_0^\infty x dx \operatorname{hei}_2 \left[\left(4 \frac{R_L}{R_D} x \right)^{\frac{1}{2}} \right] \left\{ \frac{W(t)}{x^2 + W(t)} \right\} \quad (33)$$

where $\alpha = v_T (\frac{m}{2\theta})^{\frac{1}{2}}$. Referring to Eq. (25) we see that only the imaginary part of the x integration is nonvanishing. In Appendix C the x integral is evaluated to lowest order in R_L/R_D ,

$$i \int_0^\infty x dx \operatorname{hei}_2 \left[\left(4 \frac{R_L}{R_D} x \right)^{\frac{1}{2}} \right] \left\{ \frac{W(t)}{x^2 + W(t)} \right\} = \frac{1}{\pi} W_I \ln \left[\frac{\epsilon}{\lambda^2} \frac{R_D}{R_L} \right] - \frac{1}{2\pi} \operatorname{Im}(W \ln W) + O\left(\frac{R_L}{R_D}\right) \quad (34)$$

where $W(t) = W_R + iW_I$, $e = 2.718\cdots$ is the base of the natural logarithms, and $\ln \lambda = 0.577\cdots$ is Euler's constant. With the identity

$$2\pi^{-1} \int_{-\alpha}^{\alpha} t dt W_I(t) = \int_0^{v_T} dv_{\sim} f^{(0)}(v_{\sim})$$

where the 3-fold integral on the right hand side is over the sphere of radius v_T , expressions (33) and (34) give the drag force to lowest order in R_L/R_D ,

$$\mathcal{F}_{\sim} = -e_T^2 \omega_p^2 \ln \left[\frac{\epsilon}{\lambda^2} \frac{R_D}{R_L} U(\alpha) \right] v_T^{-3} \int_0^{v_T} dv_{\sim} f^{(0)}(v_{\sim}) \quad (35)$$

with $\alpha = v_T (m/2\theta)^{\frac{1}{2}}$ and

$$\ln U(\alpha) = -\pi^{-1} \int_{-\alpha}^{\alpha} dt \, t \, \text{Im}(W \ln W) / \int_0^{v_T} dv f^{(0)}(v) \quad (36)$$

A numerical integration of the numerator in Eq. (36) was used to obtain the graph of $U(\alpha)$ shown in Figure 1. The limiting values of this function are

$$U(0) = e^{-\frac{1}{2}} = 0.6065 \quad (37)$$

$$U(\alpha) \approx \alpha = v_T \left(\frac{m}{2\theta} \right)^{\frac{1}{2}} \quad \alpha \gg 3.$$

The function $U(\alpha)$ is a monotonically increasing function of v_T . This was expected since the short range cutoff, the average distance of closest approach in a collision, should be less than R_L , while a decrease in the shielding efficiency, with increasing v_T , means the long range cutoff should be greater than R_D .

It has not been necessary to introduce an arbitrary cutoff to obtain the drag expression Eq. (35). The discussion of Section V indicated that Eq. (35) is exact as $v_T \rightarrow 0$. When $v_T > (\theta/m)^{\frac{1}{2}}$ this result should still retain the qualitative features of an exact solution. One would expect a more nearly exact solution to alter the velocity dependence of the logarithm argument at large v_T .

VII. COMPARISON WITH OTHER RESULTS

It is interesting to compare the convergent drag expression, Eq. (35), with Eq. (4) and Eq. (6). Expression (6) was obtained from Eq. (1) linearized about a spatially homogeneous distribution $f^{(0)}(\underline{v})$. This leads to a force integral that is identical to Eq. (33), except for the factor $-\pi e i_2 \left[\left(4 R_L x / R_D \right)^{\frac{1}{2}} \right]$. It is this additional factor, however, which guarantees the convergence of the drag integral at large x . In the simple linearization of Eq. (1) the neglect of terms that are not small when $\left| \underline{r} - \underline{r}_T \right|$ is small is equivalent to the expansion

$$-\pi e i_2 \left[\left(4 \frac{R_L}{R_D} x \right)^{\frac{1}{2}} \right] \approx 1 - \frac{\pi}{4} \frac{R_L}{R_D} x + \dots \quad (38)$$

That is, using only the first term of Eq. (38) in Eq. (33) results in Eq. (6), but it is clear that the expansion (38) is not valid for arbitrarily large values of x .

To compare Eq. (35) with the binary collision calculation it is necessary to reconsider the solution of Eq. (7). We look for a solution of the form

$$f(\underline{r}, \underline{v}, t) = \exp \left\{ -R_L \phi_{B.C.}(|\underline{r} - \underline{r}_T|) \right\} \cdot [f^{(0)}(\underline{v}) + h_{B.C.}(\underline{r}, \underline{v}, t)] \quad (39)$$

Using the modified linearization method described in Section III we find

$$\phi_{B.C.}(r) = r^{-1} \quad (40)$$

and

$$\int d\mathbf{v} \tilde{h}_{B.C.}(\mathbf{k}, \mathbf{v}, t=\infty) = R_L \tilde{\phi}(\mathbf{k}) \left\{ 1 - W \left[\frac{\mathbf{k} \cdot \mathbf{v}}{k} \frac{T}{\left(\frac{m}{2\theta} \right)^{1/2}} \right] \right\}. \quad (41)$$

The drag expression is then

$$\mathcal{F}_{B.C.} = ee_T n R_L (2\pi)^{-3} \int d\mathbf{k} Q_{B.C.}(-\mathbf{k}) \tilde{\phi}(\mathbf{k}) \left\{ 1 - W \left[\frac{\mathbf{k} \cdot \mathbf{v}}{k} \frac{T}{\left(\frac{m}{2\theta} \right)^{1/2}} \right] \right\} \quad (42)$$

where $Q_{B.C.}$ is given by Eq. (21) with $\phi = \phi_{B.C.}$. Comparing Eq. (42) with Eq. (28) we see that the inclusion of collective plasma interactions has two consequences; (a) the pure Coulomb potential $\phi(r)$ is replaced with a shielded potential $\phi(r)$, and (b) the plasma dielectric constant appears in the denominator of the force integral. Both of these changes contribute to the small k convergence of the force integral.

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APPENDIX A

ANALYTIC CONTINUATION OF W

To analytically continue the function

$$W\left[\frac{\omega}{k} \left(\frac{m}{2\theta}\right)^{\frac{1}{2}}\right] = R_D^2 \omega_P^2 \int d\mathbf{v} (\omega - \mathbf{k} \cdot \mathbf{v})^{-1} \mathbf{k} \cdot \frac{\partial f^{(0)}}{\partial \mathbf{v}} \quad (\text{A.1})$$

from the $\text{Im}\omega > 0$ half plane into the remainder of the ω -plane we first reduce the three fold integration to a single integral,

$$W\left[\frac{\omega}{k} \left(\frac{m}{2\theta}\right)^{\frac{1}{2}}\right] = R_D^2 \omega_P^2 \int_{-\infty}^{\infty} du \left(\frac{\omega}{k} - u\right)^{-1} \frac{dg(u)}{du} \quad (\text{A.2})$$

where

$$g(u) = \int d\mathbf{v} f^{(0)}(\mathbf{v}) \delta\left(u - \frac{\mathbf{k} \cdot \mathbf{v}}{k}\right) = (m/2\pi\theta)^{\frac{1}{2}} \exp(-mu^2/2\theta) \quad (\text{A.3})$$

The analytic continuation of Eq. (A.2) is given by

$$W = R_D^2 \omega_P^2 \int_c du \left(\frac{\omega}{k} - u\right)^{-1} \frac{dg(u)}{du} \quad (\text{A.4})$$

where c , the path of integration for u , is the real axis when $\text{Im}\omega > 0$ and is shown in Figure 2 when $\text{Im}\omega \leq 0$. Introducing the dimensionless variables $t = (m/2\theta)^{\frac{1}{2}} \omega/k$, $x = u(m/2\theta)^{\frac{1}{2}}$, and using Eq. (A.3) we can write

$$W(t) = \frac{1}{2(\pi)^{\frac{1}{2}}} \int_c dx (t-x)^{-1} \frac{d}{dx} \exp(-x^2) \quad (\text{A.5})$$

For real t this is

$$W(t) = 1 - 2t \exp(-t^2) \int_0^t dx \exp(x^2) + i(\pi)^{\frac{1}{2}} t \exp(-t^2) \quad (A.6)$$

APPENDIX B

Evaluation of Q to Lowest Order in R_L/R_D

We want to evaluate the following integral to lowest order in R_L/R_D ,

$$Q(\underline{k}) = - \int d\underline{r} \frac{\underline{r}}{r^3} \exp\{-R_L \phi(\underline{r}) - i\underline{k} \cdot \underline{r}\} \quad (B.1)$$

For this purpose the generalized potential $\phi(\underline{r})$ can be replaced by the Coulomb potential $1/r$,

$$\begin{aligned} Q &= - \int d\underline{r} \frac{\underline{r}}{r^3} \exp\{-R_L/r - i\underline{k} \cdot \underline{r}\} \\ &= -i \frac{\partial}{\partial \underline{k}} \int d\underline{r} \frac{1}{r^3} \exp\{-R_L/r - i\underline{k} \cdot \underline{r}\} \end{aligned} \quad (B.2)$$

Using polar coordinates for $\underline{r}$ with $\underline{k}$ as polar axis

$$\begin{aligned} Q &= -i \frac{\partial}{\partial k} \int_0^\infty dr \frac{1}{r} \exp(-R_L/r) \int_{-1}^1 d\eta \int_0^{2\pi} d\phi \exp(-ikr\eta) \\ &= -4\pi i \frac{\partial}{\partial k} \int_0^\infty dr \frac{1}{kr^2} \exp(-R_L/r) \sin(kr) \end{aligned} \quad (B.3)$$

Making the change of variable $x = kr$ this can be written as

$$Q_m = 4\pi i R_L \frac{k}{k} \int_0^{\infty} dx \frac{1}{x^3} \exp(-R_L k/x) \cdot \sin x \quad (B.4)$$

The x integral is given in the Bateman tables (reference 10, I, 75),

$$Q_m(k) = -(2\pi)^2 i \operatorname{hei}_2 \left[(4R_L k)^{\frac{1}{2}} \right] k/k^2 \quad (B.5)$$

where the function in this formula is defined by¹¹

$$\operatorname{her}_v(x) \pm i \operatorname{hei}_v(x) = H_v^{(1)} [x \exp(\pm i3\pi/4)] \quad (B.6)$$

and $H_v^{(1)}$ is the Hankel function of the first kind of order v.

APPENDIX C

Evaluation of an Integral

We now evaluate, to lowest order in R_L/R_D , the integral

$$\operatorname{Im} \int_0^{\infty} x dx \operatorname{hei}_2 \left[\left(4 \frac{R_L}{R_D} x \right)^{\frac{1}{2}} \right] \frac{W(t)}{x^2 + W(t)} = \int_0^{\infty} dx \operatorname{hei}_2 [2bx^{\frac{1}{2}}] \frac{x^3 W_I}{(x^2 + W_R)^2 + W_I^2} \quad (C.1)$$

where $b = (R_L/R_D)^{\frac{1}{2}}$ and $W = W_R + iW_I$. The simplest procedure is to separate the range of integration into two parts $(0, 1/b)$ and $(1/b, \infty)$.

In the range $(0, 1/b)$ we can expand (reference 11, p. 320)

$$\operatorname{hei}_2 [2bx^{\frac{1}{2}}] \approx -\frac{1}{\pi} + \frac{b^2}{4} x + \dots \quad (C.2)$$

The contribution to Eq. (C.1) from the range $(0, 1/b)$ is

$$\begin{aligned}
 & -\frac{1}{\pi} \int_0^{1/b} dx \frac{x^3 W_I}{(x^2 + W_R)^2 + W_I^2} + O(R_L/R_D) \\
 & = -\pi^{-1} W_I \ln(1/b) + (2\pi)^{-1} \text{Im}(W \ln W) + O(R_L/R_D)
 \end{aligned} \tag{C.3}$$

In the second range $(1/b, \infty)$ we use the fact

$$|W| \leq 1 < 1/b$$

to expand the integrand in the right hand side of Eq. (C.1)

$$W_I \int_{1/b}^{\infty} dx \text{hei}_2[2bx^{\frac{1}{2}}] \left\{ \frac{1}{x} - \frac{2W_R}{x^3} + \dots \right\} \tag{C.4}$$

Making the change of variable $u = 2b(x)^{\frac{1}{2}}$ this becomes

$$W_I \int_{2b^{\frac{1}{2}}}^{\infty} du u^{-1} \text{hei}_2(u) + O(R_L/R_D) \tag{C.5}$$

This is a known integral (reference 11, p. 319)

$$(2)^{-\frac{1}{2}} u^{-1} W_I [\text{hei}_1(u) + \text{her}_1(u)] \Big|_{u=2b^{\frac{1}{2}}}^{u=\infty} \tag{C.6}$$

There is no contribution at $u = \infty$, and this becomes

$$-\frac{W_I}{(8b)^{\frac{1}{2}}} \left\{ \text{hei}_1[2(b)^{\frac{1}{2}}] + \text{her}_1[2(b)^{\frac{1}{2}}] \right\} \tag{C.7}$$

Since $2(b)^{\frac{1}{2}} \ll 1$ we expand expression (C.7) to obtain

$$\frac{1}{\pi} W_I \ln \left(\frac{\lambda^2}{\epsilon} \right) + O(R_L/R_D) \quad (C.8)$$

where $\ln \lambda = 0.577\dots$ is Euler's constant and $\epsilon = 2.718\dots$ is the base of the natural logarithms. Addition of expressions (C.3) and (C.8) gives the integral of Eq. (C.1) to lowest order in R_L/R_D ,

$$\text{Im} \int_0^\infty dx \, x \, \text{hei}_2 \left[\left(4 \frac{R_L}{R_D} x \right)^{\frac{1}{2}} \right] \frac{W(t)}{x^2 + W(t)} = - \frac{W_I}{2\pi} \ln \left[\frac{\epsilon}{\lambda^2} \frac{R_D}{R_L} \right] + \frac{1}{2\pi} \text{Im}(W \ln W) + O\left(\frac{R_L}{R_D}\right) \quad (C.9)$$

REFERENCES

1. M. N. Rosenbluth, W. M. MacDonald, and D. L. Judd, Phys. Rev. 107, 1 (1957).
2. D. Bohm, and D. Pines, Phys. Rev. 85, 338 (1952).
3. M. N. Rosenbluth and N. Rostoker, Phys. of Fluids 3, 1 (1960).
4. S. Gasiorowicz, M. Neuman, and R. J. Riddel, Jr., Phys. Rev. 101, 922 (1956).
5. A. A. Vlasov, Many-Particle Theory and Its Applications to Plasma, Gordon and Breach, N. Y. (1961).
6. J. Hubbard, Proc. Roy. Soc. A260, 114 (1961); and A262, 371 (1961).
7. O. Aono, Jour. of the Phys. Soc. Japan 17, 853 (1962).
8. B. D. Fried, S. D. Conte, The Plasma Dispersion Function, Academic Press, Inc., New York (1961).
9. L. Landau, Jour. of Phys. (USSR) 10, 25 (1946).
10. Bateman, Tables of Integral Transforms, I, McGraw-Hill (1954).
11. C. S. Whitehead, Quar. Jour. of Math. 42, 316 (1911).

Figure 1. Graph of $U(\alpha)$ as a function of $\alpha = v_T(m/2\theta)^{\frac{1}{2}}$

Figure 2. Contour for the Analytic Continuation of W .

Fig. 1

of fig. 2