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MASTER

Generalized Goldstone Theorems in an Effective Lagrangian Framework

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Abstract

A set of sum rules among coupling tensors of different rank is derived for spontaneously broken continuous symmetries in an effective Lagrangian framework. The sum rules, among which is the Goldstone theorem, constitute in this framework the physical content of that part of the spontaneously broken symmetry not contained in the conventionally realized subgroup which leaves the vacuum invariant, the latter sub-symmetry manifesting itself in Wigner-Eckart relations among coupling tensors of the same rank.

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In a spontaneously broken¹ symmetry (SBS) group G of a Hamiltonian H , the vacuum state is invariant only against a lower symmetry G' , a subgroup of G , and hence zero mass excitations or Goldstone-Nambu-particles^{1,2} (GNP's) are required to support the noninvariance of the vacuum against group operations in G but not in G' which leave H invariant. The single particle states form irreducible G' multiplets, the mass matrix is invariant against G' but not G , and group operations in G but not in G' take single particle states into multiparticle states which have zero mass particles in them and take the vacuum into zero mass particle states.^{1,2} Multiplets irreducible under G exist³ but do not correspond to particle states with fixed particle number.^{1,2} Wigner-Eckart selection rules arising from the G' symmetry result in the usual relations among scattering amplitudes for fixed n -particle processes.

It is interesting to inquire into the nature and physical content of the residual symmetry of H under operations not in G' , that is, Wigner-Eckart selection rules involving these operations. If experience with broken chiral symmetry is a guide, and if the soft meson limits endemic in chiral symmetry calculations⁴ take one to an underlying spontaneously broken theory with a multiplet of mesons playing the role of GNP's,⁵ then the invariance of the theory in this limit against group operations not in G' results not in the usual Wigner-Eckart relations among n -particle amplitudes but in theorems for the emission or absorption of GNP's,⁴ that is, relations among amplitudes having different numbers of particles in external states. This is not surprising since the fixed particle number states⁶ can not diagonalize the conserved charges which are generators of group operations in G but not in G' . That the results of a SBS can be extracted by Wigner-Eckart relations among states not having fixed particle number, but diagonal in the spontaneously broken generators, has been shown

for the particular case of theories invariant against c-number translations of scalar boson fields.⁷ This method, while complete and straightforward, is also awkward and tedious, involving first the diagonalization process and then the restatement of the resultant Wigner-Eckart relation in terms of fixed particle number states. More importantly, the method does not easily generalize to the physical case in which the vacuum degeneracy is lifted by the addition of explicit symmetry breaking terms which remove the vacuum degeneracy and give the GNP's a non-zero mass.

Apart from the practical purpose of establishing all physical consequences of SBS over and above what is well known¹ [namely (i) conventional invariance under G' , the symmetry of the vacuum; and (ii) the appearance of GNP's as required by the Goldstone theorem] the question of full consequence of the SBS is interesting from a formal point of view because it is related to the converse problem: given a theory invariant under G' with a vacuum also invariant under G' , with a spectrum which includes a multiplet of zero mass mesons, under what conditions is it possible to extend the symmetry G' to a higher symmetry G which is spontaneously broken?

A simple SBS framework in which these questions can be precisely posed and one of some practical importance and current interest is the effective Lagrangian⁸ or tree approximation approach⁹ in which the Lagrangian $\mathcal{L}(\varphi)$, a polynomial or entire function on a real set of scalar or pseudoscalar fields φ_i , with $\mathcal{L}$ symmetric against a group of continuous orthogonal¹⁰ transformations G on the fields,² has a minimal constant field strength $\tilde{\varphi}_i \neq 0$, with $\tilde{\varphi}_i$ a solution of

$$\frac{\partial \mathcal{L}}{\partial \varphi_i} = 0 \quad (1)$$

where

$$\begin{aligned}\mathcal{L} &\equiv \frac{1}{2} \partial^\mu \varphi_i \partial_\mu \varphi_i + \sum_n \frac{1}{n!} \frac{\partial^n \mathcal{L}}{\partial \varphi_i \partial \varphi_i \partial \varphi_k \dots} \Big|_0 \varphi_i \varphi_j \varphi_k \dots \\ &\equiv \frac{1}{2} \partial^\mu \varphi_i \partial_\mu \varphi_i + \sum_n \frac{1}{n!} M_{ijk\dots}^{(n)}(0) \varphi_i \varphi_j \varphi_k \dots\end{aligned}\quad (2)$$

$\mathcal{L}$ may also be expanded about $\tilde{\varphi}_i$,

$$\begin{aligned}\mathcal{L}(\varphi) &= \frac{1}{2} \partial^\mu \varphi_i \partial_\mu \varphi_i + \sum_n \frac{1}{n!} M_{ijk\dots}^{(n)}(\tilde{\varphi})(\varphi_i - \tilde{\varphi}_i)(\varphi_j - \tilde{\varphi}_j)(\varphi_k - \tilde{\varphi}_k)\dots \\ &= \frac{1}{2} \partial^\mu \varphi'_i \partial_\mu \varphi'_i + \sum_n \frac{1}{n!} M_{ijk\dots}^{(n)}(\tilde{\varphi}) \varphi'_i \varphi'_j \varphi'_k \dots \\ &\equiv \mathcal{L}'(\varphi')\end{aligned}\quad (3)$$

where $\varphi'_i \equiv \varphi_i - \tilde{\varphi}_i$ are displaced fields with $\tilde{\varphi}'_i \equiv 0$ a solution of $\frac{\partial \mathcal{L}'(\varphi')}{\partial \varphi'_i} = 0$. If $\tilde{\varphi}_i$ is identified^{2,8,9,11} with $\langle 0 | \varphi_i | 0 \rangle$ and φ'_i with $\langle 0 | \varphi'_i | 0 \rangle$ then φ'_i are physical fields with zero vacuum expectation values. $\mathcal{L}'(\varphi')$, the Lagrangian in terms of physical fields, is not G-invariant but is invariant against the set of transformations $\{g'\}$ which leave $\tilde{\varphi}$ invariant

$$g' \tilde{\varphi} = \tilde{\varphi} \quad (4)$$

where $\{g'\}$ form the little group $G' \equiv G_{\tilde{\varphi}}$, a subgroup of G, with $G_{\tilde{\varphi}}$ the vacuum symmetry.

It is clear that the $G_{\tilde{\varphi}}$ invariance is realized in the usual way, that is, by the $G_{\tilde{\varphi}}$ irreducible multiplets, and a $G_{\tilde{\varphi}}$ invariant mass matrix and coupling tensors $M^{(n)}$.² Apart from this, de Mottoni and Fabri² have shown that to operations in G but not in $G_{\tilde{\varphi}}$, that is, those for which $g\tilde{\varphi} \neq \tilde{\varphi}$, there corresponds a field $\chi \neq 0$,

$$g\tilde{\varphi} \equiv e^{\lambda Q} \tilde{\varphi} \equiv \tilde{\varphi} + \lambda \chi + O(\lambda^2) \quad (5)$$

which is an eigenvector of the mass matrix $M^{(2)}(\tilde{\varphi})$ with eigenvalue zero, and that the number of such GNP fields is the dimension of the orbit of $\tilde{\varphi}$.

It is instructive to repeat their proof of the Goldstone theorem in

this framework because a generalization produces the sum rules among the coupling tensors: Since $\frac{\partial \mathcal{L}}{\partial \varphi_i}$ transforms like φ_i under G , $g\tilde{\varphi}$ is also a solution of $\frac{\partial \mathcal{L}}{\partial \varphi_i} = 0$ if $\tilde{\varphi}$ is and therefore

$$0 = \left. \frac{\partial \mathcal{L}}{\partial \varphi_i} \right|_{g\tilde{\varphi}} - \left. \frac{\partial \mathcal{L}}{\partial \varphi_i} \right|_{\tilde{\varphi}} = \left. \frac{\partial^2 \mathcal{L}}{\partial \varphi_i \partial \varphi_j} \right|_{\tilde{\varphi}} (g\tilde{\varphi} - \tilde{\varphi})_j + O[(g\tilde{\varphi} - \tilde{\varphi})^2] \quad (6)$$

implies for an infinitesimal transformation $g\tilde{\varphi} = \tilde{\varphi} + \lambda\chi$,

$$0 = \lambda M_{ij}^{(2)}(\tilde{\varphi}) \chi_j + O(\lambda^2) \quad (7)$$

with χ therefore a GNP field, $M^{(2)}(\tilde{\varphi})\chi = 0$.

Thus the spontaneous breakdown imposes a strong physical constraint on the mass matrix $M^{(2)}(\tilde{\varphi})$, a residual effect of the higher symmetry of $\mathcal{L}(\varphi)$. Generalizing Eq.(6) to second rank couplings,

$$\begin{aligned} M_{ij}^{(2)}(g\tilde{\varphi}) - M_{ij}^{(2)}(\tilde{\varphi}) &= \left. \frac{\partial^2 \mathcal{L}}{\partial \varphi_i \partial \varphi_j} \right|_{g\tilde{\varphi}} - \left. \frac{\partial^2 \mathcal{L}}{\partial \varphi_i \partial \varphi_j} \right|_{\tilde{\varphi}} \\ &= \left. \frac{\partial^3 \mathcal{L}}{\partial \varphi_k \partial \varphi_i \partial \varphi_j} \right|_{\tilde{\varphi}} (g\tilde{\varphi} - \tilde{\varphi})_k + O(\lambda^2) = \lambda M_{ijk}^{(3)}(\tilde{\varphi}) \chi_k + O(\lambda^2) \end{aligned} \quad (8)$$

However, since $\frac{\partial^2 \mathcal{L}}{\partial \varphi_i \partial \varphi_j}$ transforms like a second rank tensor under G ,

$$\begin{aligned} M_{ij}^{(2)}(g\tilde{\varphi}) &= g_{i\ell} g_{jm} M_{\ell m}^{(2)}(\tilde{\varphi}) = (1 + \lambda Q)_{i\ell} (1 + \lambda Q)_{jm} M_{\ell m}^{(2)}(\tilde{\varphi}) + O(\lambda^2) \\ &= M_{ij}^{(2)}(\tilde{\varphi}) + \lambda \{ Q_{i\ell} M_{\ell j}^{(2)}(\tilde{\varphi}) + Q_{jm} M_{im}^{(2)}(\tilde{\varphi}) \} + O(\lambda^2) \end{aligned} \quad (9)$$

With φ real and g orthogonal, Q is antisymmetric¹⁰ and we have

$$M_{ij}^{(2)}(g\tilde{\varphi}) - M_{ij}^{(2)}(\tilde{\varphi}) = \lambda [Q, M]_{ij} + O(\lambda^2) \quad (10)$$

whence, combining with Eq.(8),

$$M_{ijk}^{(3)}(\tilde{\varphi}) \chi_k = [Q, M^{(2)}(\tilde{\varphi})]_{ij} \quad (11)$$

which is a relation between third and second rank tensors with non-trivial physical content. The coupling between χ and physical fields $\psi^{(1)}$ and $\psi^{(2)}$ (eigenvectors of $M^{(2)}$) for example is given by

$$M_{ijk}^{(3)} \psi_i^{(1)} \psi_j^{(2)} \chi_k = \psi_i^{(1)} [Q, M^{(2)}]_{ij} \psi_j^{(2)} = (m_2^2 - m_1^2) \psi_i^{(1)} Q_{ij} \psi_j^{(2)} \quad (12)$$

Thus trilinear couplings involving at least one GNP are given by two particle matrix elements of the spontaneously broken charge Q associated with the GNP. Equation (12) states, among other things, that GNP's do not couple trilinearly to particles with same mass, that the couplings of two GNP's to a third particle is proportional to the third particle's mass, and that the mutual coupling of three GNP's always vanishes.¹²

This procedure easily generalizes to coupling tensors of arbitrary rank. With the notation

$$M^{(n)}(\psi^{(1)}, \psi^{(2)}, \psi^{(3)}, \dots, \psi^{(n)}) \equiv M_{ijk\dots p}^{(n)} \psi_i^{(1)} \psi_j^{(2)} \psi_k^{(3)} \dots \psi_p^{(n)} \quad (13)$$

we have

$$M^{(n+1)}(\psi^{(1)}, \psi^{(2)}, \psi^{(3)}, \dots, \psi^{(n)}, \chi) = \sum_{\text{perm}} M^{(n)}(Q\psi^{(1)}, \psi^{(2)}, \psi^{(3)}, \dots, \psi^{(n)}) \quad (14)$$

where $\sum_{\text{perm}}$ means the sum of all possible distributions of Q among the fields $\psi^{(1)}, \dots, \psi^{(n)}$. The sum rules relate the $(n+1)$ rank coupling of n arbitrary fields to the n -rank coupling of the n fields $Q\psi^{(1)}, \psi^{(2)}, \dots, \psi^{(n)}$ (plus permutations). For $n = 1$, Eq.(14) reduces to the Goldstone theorem and with $n > 1$ we have its generalization. If one of the fields $\psi^{(1)}, \dots, \psi^{(n)}$ is itself a GNP [not necessarily the same as the χ in Eq.(14)] then at least all but one of the $M^{(n)}$ terms may be further reduced to $M^{(n-1)}$ couplings, and so on. If all the $\psi^{(1)}, \dots, \psi^{(n)}$ are multiples of $\tilde{\phi}$, the vacuum direction, then

$$M^{(n+1)}(\tilde{\phi}, \tilde{\phi}, \dots, \tilde{\phi}, \chi) = n M^{(n)}(\tilde{\phi}, \tilde{\phi}, \dots, \tilde{\phi}, \chi) = n! M^{(2)}(\tilde{\phi}, \chi) = 0 \quad (15)$$

That is, fields with $\tilde{\varphi}$ quantum numbers do not couple to a single GNP.

Had we applied the process leading to Eq.(14) to a transformation in $G_{\tilde{\varphi}}$, with generator Q , $Q\tilde{\varphi} = 0$, we would have obtained

$$M^{(n)}(\psi^{(1)}, \psi^{(2)}, \dots, Q\tilde{\varphi}) = 0 = \sum_{\text{perm}} M^{(n)}(Q\psi^{(1)}, \psi^{(2)}, \dots, \psi^{(n)}) \quad (16)$$

which simply states the $G_{\tilde{\varphi}}$ invariance of the $M^{(n)}$ couplings, implying the usual Wigner-Eckart relations for the conventionally realized symmetry $G_{\tilde{\varphi}}$; for example, with $n = 2$, Eq.(16) implies

$$[Q, M^{(2)}] = 0, \quad Q \text{ a generator of } G_{\tilde{\varphi}}. \quad (17)$$

If $\mathcal{L}(\varphi)$ is a polynomial of degree N , then $M^{(N+1)} = 0 \Rightarrow \sum_{\text{perm}} M^{(n)}(Q\psi^{(1)}, \psi^{(2)}, \dots, \psi^{(n)}) = 0$, implying the full G invariance of the highest rank coupling.

Consider now the case in which explicit symmetry breaking terms appear in $\mathcal{L}(\varphi)$: if to $\mathcal{L}(\varphi)$ is added a G -symmetry breaking but $G_{\tilde{\varphi}}$ invariant term, $-\delta\tilde{\mathcal{L}}(\varphi)$, $\mathcal{L}'(\varphi) = \mathcal{L}(\varphi) - \delta\tilde{\mathcal{L}}(\varphi)$, with new vacuum $\tilde{\varphi}$ a solution of

$$0 = \left. \frac{\partial \mathcal{L}'}{\partial \varphi_i} \right|_{\tilde{\varphi}} = \left. \frac{\partial \mathcal{L}}{\partial \varphi_i} \right|_{\tilde{\varphi}} - \delta \left. \frac{\partial \tilde{\mathcal{L}}}{\partial \varphi_i} \right|_{\tilde{\varphi}} \quad (18)$$

and if $\tilde{\mathcal{L}}(\varphi)$ is a polynomial of degree N' , then the sum rules

$$M^{(n+1)}(\psi^{(1)}, \psi^{(2)}, \dots, \psi^{(n)}, \chi) = \sum_{\text{perm}} M^{(n)}(Q\psi^{(1)}, \psi^{(2)}, \dots, \psi^{(n)}) \quad (19)$$

persist, even in the presence of explicit symmetry breaking, $\tilde{\mathcal{L}}(\varphi)$, for $n > N'$. The sum rules with $n \leq N'$ are modified by terms arising from the $\tilde{M}^{(n)}$, $n \leq N'$, which do not transform like $M^{(n)}$. For example, if to $\mathcal{L}(\varphi)$ is added a linear breaking term, popular¹¹ in some spontaneously broken effective Lagrangians

$$\mathcal{L}'(\varphi) \equiv \mathcal{L}(\varphi) - \delta\tilde{\varphi}_i \varphi_i \quad (20)$$

with $\delta\tilde{\varphi}_i\varphi_i$ the $G_{\tilde{\varphi}}$ invariant term linear in the fields φ , then Eq.(14) persists for $n > 1$ whereas for $n = 1$ we have, instead of the Goldstone theorem,

$$\left. \frac{\partial \mathcal{L}'}{\partial \varphi_i} \right|_{g\tilde{\varphi}} = \lambda M'_{ij}(2) \chi_j + O(\lambda^2) \quad (21)$$

with

$$\begin{aligned} \left. \frac{\partial \mathcal{L}'}{\partial \varphi_i} \right|_{g\tilde{\varphi}} &= \left. \frac{\partial \mathcal{L}}{\partial \varphi_i} \right|_{g\tilde{\varphi}} - \delta\tilde{\varphi} = g_{ij} \left. \frac{\partial \mathcal{L}}{\partial \varphi_j} \right|_{\tilde{\varphi}} - \delta\tilde{\varphi}_i \\ &= \delta g_{ij} \tilde{\varphi}_j - \delta\tilde{\varphi}_i = \delta\tilde{\varphi} + \delta\lambda(Q\tilde{\varphi})_i - \delta\tilde{\varphi}_i + O(\lambda^2) \\ &= \delta\lambda\chi_i + O(\lambda^2) \end{aligned} \quad (22)$$

and thus

$$M'_{ij}(2)(\tilde{\varphi})\chi_j = \delta\chi_j \quad (23)$$

with $Q\tilde{\varphi} = \chi$ still an eigenvector of the mass matrix but now with eigenvalue $\delta = \frac{m^2}{\chi}$.

A host of other sum rules may be derived by considering G transformations to $O(\lambda^m)$ of $M^{(n)}$, resulting in sum rules among tensors of rank $(n+m)$, $(n+m-1), \dots, n$, involving the couplings of m GNP's to n other particles; these are however not independent of Eq.(14) but merely the result of successive reduction of these sum rules.

It is noteworthy that all the sum rules are independent of the specific form of $\mathcal{L}(\varphi)$ (which is an arbitrary entire function of the G invariant forms) and therefore do not suffer from the arbitrariness of many explicit effective Lagrangian models. Since the rules are entirely, in this framework, the result of the G -non-variant vacuum and the transformation properties dictated by the higher symmetry, it is possible that they can in some form be extended, without the embarrassment of a Lagrangian, to a more sophisticated axiomatic or S-matrix approach to SBS.¹

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