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A REMARK ON THE SECOND FUNDAMENTAL
THEOREM IN ONE-VELOCITY
TRANSPORT THEORY

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operated by

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for the

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by

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Abstract

It is shown that in one-velocity transport theory with linearly anisotropic scattering the second fundamental theorem always gives an underestimate of the average nonescape probability of neutrons from critical slabs, spheres, and cylinders if the buckling is calculated with zero extrapolation length.

1. Introduction

In the approximate calculation of the criticality of bare, homogeneous reactors one sometimes replaces the finite reactor by an infinite one of the same composition and adds the requirements that the angular flux satisfy the wave equation and the scalar flux vanish on the extrapolated boundary of the finite reactor. The resulting criticality problem can then be solved and the criticality equation obtained from it can be interpreted to mean that the nonescape probability from the associated finite reactor is the Fourier transform of the infinite-medium fission-to-absorption kernel evaluated at the buckling of the infinite medium flux. This statement is the so-called second fundamental theorem of reactor physics.¹

Strictly speaking, the criticality equation obtained in this procedure is correct only in an infinite medium. The connection with the original finite reactor is made by noticing that the replacement of the finite reactor by an infinite reactor ought not to affect the flux deep in the interior significantly if the finite reactor is large enough and the infinite medium solution is suitably chosen.

The accuracy of this method has been carefully analyzed in the past. Using bucklings deduced from the extrapolation distance obtained by solving the Milne problem, Inönü² and Dresner³ were able to show that in one-velocity transport theory and the second fundamental theorem yields results of surprising accuracy even for reactors with

dimensions comparable to a mean free path.

In spite of this high accuracy, however, the use of the second fundamental theorem to estimate the criticality of finite reactors is equivocal because of the ambiguity present in the choice of buckling. Searching for a rigorous connection between the finite and the infinite medium problems in one-velocity transport theory with isotropic scattering, Dresner⁴ was able to prove that if the buckling corresponded to zero extrapolation distance, then the second fundamental theorem always gave a lower limit to the nonescape probability.

Dresner's original proof of this fact was based on the integral equation formulation of transport theory. An attempt to generalize this proof to the case of linearly anisotropic scattering⁵ met with only partial success, the desired result being proven essentially only for positive $\bar{\mu}$, the mean cosine of the scattering angle. The present paper supplies a complete proof of this inequality in the case of linearly anisotropic scattering for slabs, spheres, and infinite cylinders.

In the first part of the paper only slab reactors will be considered. In the latter part the proof will be extended to spheres and infinite cylinders.

2. Solution of the Infinite Medium Equation

To solve the infinite medium equation we use a method originally suggested by Garabedian⁶ and later by Hurwitz and Zweifel⁷ and by Davison.⁸ Let us choose the total mean free path in the reactor medium as the unit of length. In these units the one-velocity Boltzmann equation can be written:

$$\begin{aligned} \text{div } [\omega \psi(r, \omega)] + \psi(r, \omega) \\ = k \int \psi(r, \omega') f(\omega' \cdot \omega) d\omega' \end{aligned} \quad (1)$$

where $\psi(r, \omega)$ is the collision density at r of neutrons traveling in a direction specified by the unit vector ω ; $f(\omega \cdot \omega')$ is the normalized angular distribution of secondary neutrons resulting from collision of primaries of direction ω' ; and k is the number of secondary neutrons produced per collision. In slab geometry Eq. (1) reduces to

$$\begin{aligned} \mu \frac{d\psi(z, \mu)}{dz} + \psi(z, \mu) \\ = k \int_{-1}^{+1} f(\mu, \mu') \psi(z, \mu') d\mu' \end{aligned} \quad (2)$$

where μ is the cosine of the angle between ω and the z axis,

$\psi(z, \mu) = 2\pi \psi(z, \omega)$, and

$$f(\mu, \mu') = \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' f(\omega \cdot \omega') \quad (3)$$

If Eq. (2) is Fourier transformed the result is

$$\begin{aligned} (-iB\mu + 1)\psi(B, \mu) \\ = k \int_{-1}^{+1} f(\mu, \mu') \psi(B, \mu') d\mu' \end{aligned} \quad (4)$$

Here B is the transform variable, and we have set it equal to the buckling in view of the fact that the infinite medium solution ψ is assumed to satisfy the wave equation.

In order to solve Eq. (4) let us set

$$\psi(B, \mu) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} \psi_{\ell}(B) P_{\ell}(\mu) \quad (5a)$$

$$f(\underline{\omega} \cdot \underline{\omega}') = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} f_{\ell} P_{\ell}(\underline{\omega} \cdot \underline{\omega}') \quad (5b)$$

If we divide Eq. (4) by the factor $1 - iB\mu$, multiply by $P_{\ell}(\mu)$, and integrate over μ we obtain

$$\psi_{\ell}(B) = k \sum_{m=0}^{\infty} (2m+1) A_{\ell m} f_m \psi_m(B) \quad (6a)$$

where

$$A_{\ell m} = \frac{1}{2} \int_{-1}^{+1} \frac{P_{\ell}(\mu) P_m(\mu)}{1 - iB\mu} d\mu \quad (6b)$$

In deriving Eq. (6) we have employed the addition theorem for the Legendre polynomials (5) from which it follows that

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi' \int_0^{2\pi} d\phi P_{\ell}(\underline{\omega} \cdot \underline{\omega}') = 2\pi P_{\ell}(\mu) P_{\ell}(\mu') \quad (7)$$

Finally, Hurwitz and Zweifel have shown⁷ that the $A_{\ell m}$ satisfy the recurrence relation

$$\frac{2\ell+1}{iB} A_{\ell m} - (\ell+1) A_{\ell+1, m} - \ell A_{\ell-1, m} = \frac{\delta_{\ell m}}{iB} \quad (8)$$

This formula together with their symmetry and the special value

$$A_{00} = \frac{\arctan B}{B} \quad (9)$$

serves to determine all the $A_{\ell m}$.

If the angular moments, f_ℓ , vanish for all ℓ larger than a certain one, L , it is clear that of the set of coupled equations (6a) the first $L + 1$ are a self-contained subset. We shall only be interested in the case of linear anisotropy, i.e., the case when only f_0 and f_1 do not vanish. In such a case it is easy to show that $f_0 = 1$ and $f_1 = \bar{\mu}$, the mean cosine of the angle of scattering. Equation (6a) then reduces to the coupled pair^{*}

$$\psi_0(B) = kA_{00}\psi_0(B) + 3kA_{01}\bar{\mu}\psi_1(B) \quad (10a)$$

$$\psi_1(B) = kA_{10}\psi_0(B) + 3kA_{11}\bar{\mu}\psi_1(B) \quad (10b)$$

If we now set $P = k^{-1}$ it follows from Eqs. (10) that

$$\begin{vmatrix} A_{00} - P & 3\bar{\mu}A_{01} \\ A_{10} & 3\bar{\mu}A_{11} - P \end{vmatrix} = 0 \quad (11)$$

From Eqs. (8) and (9) we can write

$$A_{00} = K(B) \quad (12a)$$

$$A_{01} = A_{10} = i \frac{1 - K(B)}{B} \quad (12b)$$

$$A_{11} = \frac{1 - K(B)}{B^2} \quad (12c)$$

*That the coefficients in Eq. (10) really represent the Fourier transform of the infinite medium single-collision kernel is shown by direct calculation in Ref. 5. Thus the procedure outlined here is really identical with that of the second fundamental theorem.

where $K(B)$ is an abbreviation for the function $\arctan B/B$. P is then the larger root of the quadratic equation

$$P^2 - P(K + 3\bar{\mu} A_{11}) + 3\bar{\mu} A_{11} = 0 \quad (13)$$

For the finite reactor $k^{-1} = P$ is the nonescape probability.

The eigenvalue we have just calculated is doubly degenerate, corresponding at once to solutions proportional to e^{-iBz} and e^{iBz} . Since we are interested only in those infinite medium solutions which can approximate the true reactor flux we impose the condition of reality on $\psi(z, \mu)$. This requires us to consider a linear combination of the above exponentials, viz.,

$$\psi(z, \mu) = \psi(B, \mu) e^{-iBz} + \text{complex conjugate} \quad (14a)$$

$$= 2 \operatorname{Re} \psi(B, \mu) \cos Bz + 2 \operatorname{Im} \psi(B, \mu) \sin Bz \quad (14b)$$

3. Comparison of the Infinite Medium and Finite Reactor Solutions

The angular flux $\phi(z, \mu)$ in a finite slab reactor of thickness T also satisfies Eq. (2) with an eigenvalue k_f as well as the boundary conditions:

$$\phi\left(\frac{T}{2}, \mu\right) = 0 \quad \mu < 0 \quad (15a)$$

$$\phi\left(-\frac{T}{2}, \mu\right) = 0 \quad \mu > 0 \quad (15b)$$

Thus

$$-\mu \frac{d\phi(z, -\mu)}{dz} + \phi(z, -\mu) = k_f \int_{-1}^{+1} f(\mu, \mu') \phi(z, -\mu') d\mu' \quad (16)$$

If we multiply Eq. (2) by $\phi(z, -\mu)$, Eq. (16) by $\psi(z, \mu)$, integrate over μ and z , and subtract we obtain

$$\begin{aligned}
 (k-k_F) & \int_{-\frac{T}{2}}^{+\frac{T}{2}} dz \int_{-1}^{+1} d\mu \int_{-1}^{+1} d\mu' \psi(z, \mu) f(\mu, \mu') \phi(z, -\mu') \\
 & = \int_{-\frac{T}{2}}^{+\frac{T}{2}} dz \int_{-1}^{+1} d\mu \mu \left\{ \phi(z, -\mu) \frac{d\psi(z, \mu)}{dz} + \psi(z, \mu) \frac{d\phi(z, -\mu)}{dz} \right\}
 \end{aligned} \tag{17}$$

The RHS of Eq. (17) can be written as

$$\begin{aligned}
 & \int_{-1}^{+1} \mu \left\{ \phi(z, -\mu) \psi(z, \mu) \right\} \begin{matrix} z = T/2 \\ z = -T/2 \end{matrix} d\mu \\
 & = \int_{-1}^0 \mu \phi\left(\frac{T}{2}, -\mu\right) \psi\left(\frac{T}{2}, \mu\right) d\mu - \int_0^1 \mu \phi\left(-\frac{T}{2}, -\mu\right) \psi\left(-\frac{T}{2}, \mu\right) d\mu
 \end{aligned} \tag{18}$$

by using Eq. (15). As we shall show in the next section

$$\psi\left(\frac{T}{2}, \mu\right) < 0 \quad \mu < 0 \tag{19a}$$

$$\psi\left(-\frac{T}{2}, \mu\right) < 0 \quad \mu > 0 \tag{19b}$$

Thus it follows from Eq. (18) that the RHS of Eq. (17) is > 0 ; for $\phi(z, \mu)$, being a physically realizable flux, can never be negative.

Using analogous expansions to that of Eq. (5a) for $\psi(z, \mu)$ and $\phi(z, \mu)$, the integral on the LHS of Eq. (17) can be written as

$$\int_{-\frac{T}{2}}^{+\frac{T}{2}} \frac{1}{2} \left\{ \phi_0(z) \psi_0(z) - 3\bar{\mu} \phi_1(z) \psi_1(z) \right\} dz \quad (20)$$

Since Eq. (2) is homogeneous we can multiply ψ by any constant without any effect. In particular we do this so that $\psi_0(R) = 1$. It then follows from Eqs. (10), (12), and (14) that

$$\psi_0(z) = 2 \cos Bz \quad (21a)$$

$$\psi_1(z) = 2 \left(\frac{B}{3\bar{\mu}} \right) \frac{K-P}{1-K} \sin Bz \quad (21b)$$

Thus

$$\psi_1(z) = - \frac{1}{3\bar{\mu}} \left(\frac{K-P}{1-K} \right) \frac{d\psi_0(z)}{dz} \quad (22)$$

If we substitute Eq. (22) into the expression (20) and integrate by parts we obtain for (20)

$$\int_{-\frac{T}{2}}^{+\frac{T}{2}} \frac{1}{2} \left\{ \phi_0(z) \psi_0(z) - \left(\frac{K-P}{1-K} \right) \psi_0(z) \frac{d\phi_1(z)}{dz} \right\} dz \quad (23)$$

The integrated terms in expression (23) vanish since the buckling B has been chosen equal to $\frac{\pi}{T}$, i.e., so that it corresponds to zero extrapolation distance.

Since ϕ satisfies an equation of continuity, viz.,

$$\frac{d\phi_1(z)}{dz} = (k_f - 1) \phi_0(z), \quad (24)$$

the expression in (23) can finally be written

$$\frac{1}{2} \left\{ 1 - (k_f - 1) \frac{K-P}{1-K} \right\} \int_{-\frac{T}{2}}^{+\frac{T}{2}} \phi_0(z) \psi_0(z) dz \quad (25)$$

Since both $\phi_0(z)$ and $\psi_0(z)$ are positive for $|z| < T/2$, Eq. (17) is equivalent to the inequality

$$(k - k_f) \left(1 - (k_f - 1) \frac{K-P}{1-K} \right) > 0 \quad (26)$$

Eq. (26) is equivalent to the two possible alternatives, viz.:

$$k > k_f \quad \text{and} \quad 1 > (k_f - 1) \frac{K-P}{1-K} \quad (27a)$$

or

$$k < k_f \quad \text{and} \quad 1 < (k_f - 1) \frac{K-P}{1-K} \quad (27b)$$

When $\bar{\mu} \leq 0$, $P \geq K$, since for $\bar{\mu} = 0$ $P = K$, and P is a monotone decreasing function of $\bar{\mu}$.^{4,5} Since $k_f \geq 1 \geq K$, when $\bar{\mu} \leq 0$, the second and therefore both parts of Eq. (27a) must hold.

Now let us consider if there is ever any positive value of $\bar{\mu}$ for which Eq. (27b) is valid. If we plot $k - k_f$ as abscissa and

$1 - (k_f - 1) \frac{K-P}{1-K}$ as ordinate for all values of $\bar{\mu}$, Eqs. (27) say that the resulting curve must lie only in the first and third quadrants. If it lies in both it must pass through the origin, since it is a continuous curve. Thus if there are values of $\bar{\mu}$ for which Eq. (27b) is true, there must be at least one value of $\bar{\mu}$ for which

$$k = k_f \quad (28a)$$

and

$$1 = (k_f - 1) \frac{K-P}{1-K} \quad (28b)$$

Let $\bar{\mu}_0$ be the smallest such value. If we eliminate k_f from Eqs. (28) they can be rewritten as

$$P^2 - 2P + K = 0 \quad (29)$$

The larger root of Eq. (29) is

$$P = 1 + \sqrt{1-K} \quad (30)$$

while the larger root of Eq. (13) is

$$P = \frac{K + 3\bar{\mu}_0 A_{11}}{2} + \sqrt{\left(\frac{K + 3\bar{\mu}_0 A_{11}}{2}\right)^2 - 3\bar{\mu}_0 A_{11}} \quad (31)$$

Setting these two values equal gives the condition

$$(1-K) \left[\left(\frac{K + 3\bar{\mu}_0 A_{11}}{2} \right)^2 - 3\bar{\mu}_0 A_{11} \right] = 0 \quad (32)$$

Since $K < 1$ for all finite bucklings, the square bracket Δ in Eq. (32) must vanish.

In general, Δ will either have no real roots or two real roots $\bar{\mu}_{01} \leq \bar{\mu}_{02}$. From Eq. (32) it follows that the product of the two roots is always positive while half their sum equals $\frac{1}{3}(2-K)/A_{11}$. It can be shown that $2-K > A_{11}$; therefore $\bar{\mu}_{01} + \bar{\mu}_{02} > \frac{2}{3}$. Both roots must thus be positive, and one at most can be less than $\frac{1}{3}$.

Δ is the discriminant of Eq. (13). If $\bar{\mu}_{01} \leq \frac{1}{3}$, $\Delta \geq 0$ for $\bar{\mu} \leq \bar{\mu}_{01}$ and $\Delta < 0$ for $\bar{\mu}_{01} < \bar{\mu} \leq \frac{1}{3}$. If $\bar{\mu}_{01} \geq \frac{1}{3}$, $\Delta \geq 0$ for $\bar{\mu} \leq \frac{1}{3}$. (For linear anisotropy $|\bar{\mu}|$ can never exceed $\frac{1}{3}$.) Thus, under those circumstances for which Eq. (13) has real solutions, Eq. (27a) always holds and thus

$$P_f \equiv k_f^{-1} > k^{-1} \equiv P \quad \text{q.e.d.} \quad (33)$$

Eq. (33) is still subject to the condition (19) to whose proof we now turn our attention.

4. Calculation of the Boundary Condition (19)

In order to calculate $\psi(\frac{T}{2}, \mu)$ for $\mu \leq 0$, we invert the differential operator in Eq. (2) and obtain

$$\psi(z, \mu) = k \int_{-\infty \operatorname{sgn} \mu}^z e^{(z'-z)/\mu} \frac{dz'}{\mu} \int_{-1}^{+1} f(\mu, \mu') \psi(z', \mu') d\mu' \quad (34)$$

In the case of linear anisotropy $f(\mu, \mu')$ is given by

$$f(\mu, \mu') = \frac{1}{2} \left\{ 1 + 3\bar{\mu}\mu\mu' \right\}, \quad (35)$$

which when substituted into Eq. (34) yields

$$\psi(z, \bar{\mu}) = k \int_{-\infty}^z \frac{e^{(z'-z)/\mu}}{\text{sgn } \mu} [\psi_0(z') + 3\bar{\mu}\mu \psi_1(z')] \frac{dz'}{2\mu} \quad (36)$$

Substitution of Eqs. (21) into Eq. (36) gives

$$\psi(z, \mu) = k \int_{-\infty}^z \frac{e^{(z'-z)/\mu}}{\text{sgn } \mu} (\cos Bz' + B\mu \frac{K-P}{1-K} \sin Bz') \frac{dz'}{\mu} \quad (37)$$

If we make the substitutions $z = \frac{T}{2}$, $y = z' - \frac{T}{2}$, $\mu = -|\mu| < 0$ in (37)

we find that

$$\psi(\frac{T}{2}, -|\mu|) = k \int_0^{\infty} e^{-y/|\mu|} \left\{ -\sin By - |\mu| B \frac{K-P}{1-K} \cos By \right\} \frac{dy}{|\mu|} \quad (38a)$$

$$= -k \frac{B|\mu|}{1 + B^2|\mu|^2} \frac{1-P}{1-K} < 0 \quad (38b)$$

Since Eq. (37) is invariant under the simultaneous inversion

$z \rightarrow -z$, $\mu \rightarrow -\mu$, it follows that

$$\psi(-\frac{T}{2}, |\mu|) < 0 \quad (38c)$$

This concludes the proof that $P_f > P$ for slabs.

5. Generalization to Spheres and Infinite Cylinders

We now generalize the foregoing procedure to spheres and infinite cylinders. As a solution of the infinite medium Boltzmann equation we take a linear superposition of the slab solutions, viz.,

$$\psi(\underline{r}, \omega) = \int_{|\underline{k}|=1} 2 \operatorname{Re} [\psi(\underline{B}, \underline{k} \cdot \omega) e^{-i \underline{B} \underline{k} \cdot \underline{r}}] g(\underline{k}) d^2 k \quad (39)$$

where $\psi(\underline{B}, \underline{k} \cdot \omega)$ is identical with $\psi(\underline{B}, \mu)$, $g(\underline{k})$ is as yet arbitrary, and the $\underline{k}$ -integral is over the surface of the unit sphere. From (39) it follows that

$$\omega \cdot \nabla \psi(\underline{r}, \omega) + \psi(\underline{r}, \omega) = \int_{|\underline{k}|=1} 2 \operatorname{Re} [(1 - i \underline{B} \cdot \underline{k} \cdot \omega) \psi(\underline{B}, \underline{k} \cdot \omega) e^{-i \underline{B} \underline{k} \cdot \underline{r}}] g(\underline{k}) d^2 k \quad (40a)$$

$$= \int_{|\underline{k}|=1} 2 \operatorname{Re} [e^{-i \underline{B} \underline{k} \cdot \underline{r}} k \int_{-1}^{+1} f(\underline{k} \cdot \omega, \underline{k} \cdot \omega') \psi(\underline{B}, \underline{k} \cdot \omega') d(\underline{k} \cdot \omega')] g(\underline{k}) d^2 k \quad (40b)$$

$$= \int_{|\underline{k}|=1} 2 \operatorname{Re} [e^{-i \underline{B} \underline{k} \cdot \underline{r}} k \int f(\omega \cdot \omega') \psi(\underline{B}, \underline{k} \cdot \omega') d\omega'] g(\underline{k}) d^2 k$$

$$= k \int f(\omega \cdot \omega') \psi(\underline{r}, \omega') d\omega' \quad (40d)$$

Thus $\psi(\underline{r}, \omega)$ given by Eq. (39) is a solution of Eq. (1) belonging to the eigenvalue $k = P^{-1}$ given by Eq. (13).

6. Comparison of the Infinite Medium and Finite Reactor Solution

By the same procedure which led us to Eq. (17) we can show that

$$\begin{aligned}
& (k-k_f) \int_V d^3r \int d\omega \int d\omega' \psi(\underline{r}, \omega) f(\omega \cdot \omega') \phi(\underline{r}, -\omega') \\
& = \int_V d^3r \int d\omega \omega \cdot \left\{ \phi(\underline{r}, -\omega) \nabla \psi(\underline{r}, \omega) + \psi(\underline{r}, \omega) \nabla \phi(\underline{r}, -\omega) \right\}
\end{aligned} \tag{41}$$

where V denotes the volume occupied by the finite reactor and $\phi(\underline{r}, \omega)$ is the true angular flux inside it. The RHS may be written as before as

$$\int_S d^2r \int d\omega \omega \cdot \underline{n} \phi(\underline{r}, -\omega) \psi(\underline{r}, \omega) \tag{42}$$

where S is the surface of V and $\underline{n}$ its outward normal. Since $\phi(\underline{r}, -\omega)$ is zero when $\underline{n} \cdot \omega > 0$, the integration in Eq. (42) extends only the angular range where $\underline{n} \cdot \omega < 0$. If as before $\psi(\underline{r}, \omega)$ is < 0 when $\underline{n} \cdot \omega$ is < 0 , expression (42) and the RHS of Eq. (41) will be positive.

The LHS of (41) can be written similarly to before as

$$(k-k_f) \int_V \frac{1}{4\pi} \left\{ \psi_0(\underline{r}) \phi_0(\underline{r}) - 3\bar{\mu} \psi_1(\underline{r}) \cdot \phi_1(\underline{r}) \right\} d^3r \tag{43}$$

where

$$\phi_0(\underline{r}) = \int \phi(\underline{r}, \omega) d\omega \tag{44a}$$

and

$$\phi_1(\underline{r}) = \int \omega \phi(\underline{r}, \omega) d\omega \quad (44b)$$

and similarly for $\psi_0(\underline{r})$ and $\psi_1(\underline{r})$.

By direct calculation it follows from Eqs. (39) that

$$\psi_0(\underline{r}) = 4\pi \int_{|\underline{K}|=1} \cos(\underline{B} \cdot \underline{K} \cdot \underline{r}) g(\underline{K}) d^2 K \quad (45a)$$

and

$$\psi_1(\underline{r}) = 4\pi \left(\frac{\underline{B}}{3\bar{\mu}} \right) \frac{\underline{K}-\underline{P}}{1-\underline{K}} \int_{|\underline{K}|=1} \sin(\underline{B} \cdot \underline{K} \cdot \underline{r}) \underline{K} g(\underline{K}) d^2 K \quad (45b)$$

so that

$$\psi_1(\underline{r}) = - \frac{1}{3\bar{\mu}} \frac{\underline{K}-\underline{P}}{1-\underline{K}} \nabla \psi_0(\underline{r}) \quad (45c)$$

The equation of continuity for $\phi(\underline{r}, \omega)$ connects ϕ_0 and ϕ_1 is as follows

$$\nabla \cdot \phi_1(\underline{r}) = (\underline{k}_F - 1) \phi_0(\underline{r}) \quad (46)$$

Next we employ Eqs. (45c) and (46) to rewrite the last term in Eq. (43):

$$\int_V \psi_1(\underline{r}) \cdot \phi_1(\underline{r}) d^3 r = - \frac{1}{3\bar{\mu}} \frac{\underline{K}-\underline{P}}{1-\underline{K}} \int_V \phi_1(\underline{r}) \cdot \nabla \psi_0(\underline{r}) d^3 r \quad (47a)$$

$$= \frac{1}{3\bar{\mu}} \frac{\underline{K}-\underline{P}}{1-\underline{K}} \left[(\underline{k}_F - 1) \int_V \phi_0(\underline{r}) \psi_0(\underline{r}) d^3 r - \int_S \underline{n} \cdot \phi_1(\underline{r}) \psi_0(\underline{r}) d^2 r \right] \quad (47b)$$

By reference to Eq. (45a) we can see that $\psi_0(\underline{r})$ satisfies the wave equation. We now choose $g(\underline{K})$ so that $\psi_0(\underline{r})$ is > 0 in V and vanishes on S , which can always be done. Then the surface integral in Eq. (47b) vanishes. The substitution of Eq. (47b) in Eq. (43) again yields Eq. (26). From this point on the argument proceeds as in section 3.

7. The Boundary Condition on $\psi(\underline{r}, \underline{\omega})$ for Spheres and Infinite Cylinders

The last thing we must do is prove that on S $\psi(\underline{r}, \underline{\omega})$ is < 0 for $\underline{n} \cdot \underline{\omega} < 0$. This we do in a manner analogous to that of section 4. If $\underline{r}_S$ is a point on S and $\underline{\omega}$ a direction such that $\underline{n} \cdot \underline{\omega} < 0$, then by inverting the differential operator in Eq. (1) we can write

$$\psi(\underline{r}_S, \underline{\omega}) = k \int_0^\infty e^{-v} dv \int f(\underline{\omega} \cdot \underline{\omega}') \psi(\underline{r}_S - v\underline{\omega}, \underline{\omega}') d\omega' \quad (48a)$$

$$= \frac{k}{4\pi} \int_0^\infty e^{-v} \left\{ \psi_0(\underline{r}_S - v\underline{\omega}) + 3\bar{\mu}\underline{\omega} \cdot \underline{\psi}_1(\underline{r}_S - v\underline{\omega}) \right\} dv \quad (48b)$$

$$- \frac{k}{4\pi} \int_0^\infty e^{-v} \left\{ \psi_0(\underline{r}_S - v\underline{\omega}) - \frac{K-F}{1-K} \underline{\omega} \cdot \nabla \psi_0(\underline{r}_S - v\underline{\omega}) \right\} dv \quad (48c)$$

Now

$$\underline{\omega} \cdot \nabla \psi_0(\underline{r}_S - v\underline{\omega}) = - \frac{d}{dv} \psi_0(\underline{r}_S - v\underline{\omega}); \quad (49)$$

hence, by partial integration

$$- \int_0^{\infty} e^{-v} \underline{\omega} \cdot \nabla \psi_0(\underline{r}_S - v \underline{\omega}) dv = \left[e^{-v} \psi_0(\underline{r}_S - v \underline{\omega}) \right]_0^{\infty} + \int_0^{\infty} e^{-v} \psi_0(\underline{r}_S - v \underline{\omega}) dv \quad (50)$$

The integrated term vanishes since for $v = 0$, $\psi_0 = 0$. Thus

$$\psi(\underline{r}_S, \underline{\omega}) = \frac{k}{4\pi} \frac{1-P}{1-K} \int_0^{\infty} e^{-v} \psi_0(\underline{r}_S - v \underline{\omega}) dv \quad (51)$$

Fig. 1

For spheres and infinite cylinders we can show by direct calculation that the last integral is negative. In the case of spheres we use the coordinates shown in Fig. 1. Since for spheres $\psi_0(\underline{r}) = \sin Br/r$,

$$\int_0^{\infty} \psi_0(\underline{r}_S - v \underline{\omega}) e^{-v} dv = e^{\frac{\pi}{B} \cos \theta} \int_{\pi/B}^{\infty} \sin Br \cdot \frac{\exp \left\{ -\sqrt{r^2 - \pi^2 \sin^2 \theta / B^2} \right\}}{\sqrt{r^2 - \pi^2 \sin^2 \theta / B^2}} dr \quad (52)$$

The factor multiplying $\sin Br$ in Eq. (52) is monotone decreasing; since the area between successive nodes of the sine remains constant in magnitude but alternates in sign and the integral begins at $\frac{\pi}{B}$, the RHS of Eq. (52) is clearly < 0 .

For infinite cylinders we use the coordinates shown in Fig. 2 and

set $\psi_0(\underline{r}) = J_0(Br)$. Then

$$\int_0^\infty \psi_0(\underline{r}_s - v\omega) e^{-v} dv = e^{\left(\frac{\alpha \cos\theta/B}{\sin^2\theta \cos^2\phi + \cos^2\theta} \right)} \int_{\frac{\alpha}{B}}^\infty J_0(Br) \exp \left\{ - \frac{[(r^2 - \alpha^2/B^2)(\sin^2\theta \cos^2\phi + \cos^2\theta) + \alpha^2 \cos^2\theta/B^2]^{\frac{1}{2}}}{\sin^2\theta \cos^2\phi + \cos^2\theta} \right\} \frac{r dr}{[(r^2 - \alpha^2/B^2)(\sin^2\theta \cos^2\phi + \cos^2\theta) + \alpha^2 \cos^2\theta/B^2]^{\frac{1}{2}}} \quad (53)$$

where α is the first root of the zero order Bessel function ($= 2.405$).

Fig. 2

The factor multiplying $J_0(Br)$ in Eq. (53) is also monotone decreasing; since the area between adjacent nodes of J_0 decreases in magnitude and alternates in sign and the integral begins at $\frac{\alpha}{B}$, the RHS of Eq. (53) is also < 0 .

This completes the proof that $P_f > P$ for spheres and cylinders.

8. Discussion

The method of proof used in the present work cannot be used to prove that $P_f > P$ for any arbitrary solid, for it is not always the case that $\psi(\underline{r}_s, \omega) < 0$ for every surface point $\underline{r}_s$ and every inward direction ω of every solid. A simple counterexample is the flux directed into the edge of a square prism along a diagonal of the cross section. However, we have only shown that the condition $\psi(\underline{r}_s, \omega) < 0$ is sufficient to prove $P_f > P$;

its failure at some point does not necessarily imply that the integral (42) is positive or for that matter that P_f is not always $> P$.

By a combination of techniques the author has shown that $P_f > P$ in the following cases

- (i) for all convex shapes for isotropic scattering,⁴
- (ii) for all convex shapes for linearly anisotropic scattering,⁵
but with a restriction of $\bar{\mu}$ to certain positive values, and
- (iii) for slabs, spheres, and infinite cylinders for linearly anisotropic scattering for all $\bar{\mu}$.

It seems very reasonable on this basis to conjecture that $P_f > P$ for all convex solids and all orders of anisotropy, where in general P means the largest root the secular determinant of the set (6a).

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