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TECHNICAL DATA RECORD		PAGE 1 OF 33		<i>W.A. Rhoades</i>	
AUTHOR R. L. Johnson		DEPT & GROUP NO. 721-13		DATE 4-29-63	
				GO NO 7569	
TITLE Core Transfer Function Models for SNAP System Kinetics		S A NO 2005		TWR	
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PROGRAM Supporting R&D		SUBACCOUNT TITLE Basic Control and Stability Analysis		UNCL. ☒ AEC ☐ DOD ☐ CONF. ☐ SECRET ☐	
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		DATE 5/1/63			
STATEMENT OF PROBLEM To formulate a suitable mathematical model of the reactor core kinetics and thermodynamics for use in a transfer function study of SNAP system kinetics					
ABSTRACT <p>A distributed parameter transfer function is developed for SNAP type reactor cores. The coolant may be either boiling, non-boiling, or both. In the boiling case the Bankoff model is used to find the spatial distribution of voids.</p> <p>The transfer functions obtained are not simple polynomials. However, they are readily analyzed for stability using the 7090 program TRAFIC, which uses a Nyquist diagram approach.</p> <p>Only a few cases have been examined up to the present. Some general conclusions resulting from these calculations are included. It is planned to use the transfer functions obtained in the analysis of more complex system in the near future.</p>					

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I. REACTOR POWER

If the reactor power is assumed to be space uniform it may be calculated from (Reference 1)

$$\frac{dP(t)}{dt} = \frac{\rho(t)}{\bar{\tau}} P(t) + \frac{1}{\bar{\tau}} \sum_{i=1}^N \lambda_i \beta_i \int_{-\infty}^t P(t') e^{-\lambda_i(t-t')} dt' \quad (1.1)$$

In this equation P is the total reactor power, $\bar{\tau}$ the prompt mean lifetime, λ_i the decay constant for the i th group of delayed neutrons and β_i the fraction of the fissions which give rise to precursors in the i th group. $\rho(t)$ is the prompt reactivity which can be conveniently divided into external and feedback reactivity as follows:

$$\rho(t) = \delta(t) + \rho(t) - \beta \quad \beta = \sum_{i=1}^N \beta_i \quad (1.2)$$

In this equation, $\delta(t)$ is the externally applied reactivity from the control rods and $\rho(t)$ is the feedback reactivity.

To find the linearized power equation put

$$P(t) = P_0 + p(t) \quad p(t) \ll P_0$$

and substitute into (1.1). The resulting equation for $p(t)$ is

$$\frac{dp(t)}{dt} = \frac{P_0}{\bar{\tau}} [\delta(t) + \rho(t)] - (\beta/\bar{\tau}) p(t) + \frac{1}{\bar{\tau}} \sum_{i=1}^N \lambda_i \beta_i \int_0^t p(t') e^{-\lambda_i(t-t')} dt' \quad (1.3)$$

which contains only terms of first order in $p(t)$, all higher order terms having been neglected. The Laplace transform of (1.3) is

$$s \bar{p}(s) = \frac{P_0}{\bar{\tau}} [\bar{\delta}(s) + \bar{\rho}(s)] - (\beta/\bar{\tau}) \bar{p}(s) + \frac{1}{\bar{\tau}} \left(\sum_{i=1}^N \frac{\lambda_i \beta_i}{s + \lambda_i} \right) \bar{p}(s) \quad (1.4)$$

From (1.4) the solution for $\bar{\rho}(s)$ is

$$\bar{\rho}(s) = \frac{\frac{P_0}{\tau} \bar{\beta}(s)}{s + \beta/\tau - \frac{1}{\tau} \sum_{i=1}^N \frac{\lambda_i \beta_i}{s + \lambda_i}} + \bar{H}(s) \quad (1.5)$$

where

$$\frac{P_0}{\tau} \bar{\rho}(s) = -\bar{H}(s) \bar{\rho}(s) \quad (1.6)$$

defines the feedback reactivity transfer function. The problem is to find an expression for $\bar{H}(s)$ which will serve as a useful physical approximation.

II. ROD TEMPERATURE CHANGE AND HEAT RELEASE

The change in power level causes a change in rod temperature and as a result a change in the heat rate to the coolant. If conduction in the longitudinal direction is neglected the rod temperature is a solution to the equation.

$$\frac{Q(t)}{k} + \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} = \frac{1}{K_t} \frac{\partial T}{\partial t} \quad (2.1)$$

where $0 \leq r \leq a$. The rod radius is a , and K_t is the thermal diffusivity of the rod material. The rods are assumed to be homogeneous for simplicity. The source term $Q(t)$ is the power density which has been assumed to be independent of any spatial coordinate. In this case

$$Q(t) = \frac{P(t)}{V} = \frac{\text{total reactor power}}{\text{fuel volume}}$$

To find $T(t)$ in terms of $Q(t)$ it will be assumed that the rod surface temperature is unaffected by small power changes. This is a particularly valid assumption when the coolant is boiling. When the coolant is not boiling it is a very good approximation as long as k ,

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the thermal conductivity of the rod, is much smaller than the heat transfer coefficient. For systems containing NaK as coolant the ratio k/h is quite small.

As a consequence of these assumptions, Equation (2.1) can be solved for the change in temperature. Let $Q(t) = Q_0 + Q'(t)$
 $T(r,t) = T_0(r) + T'(r,t)$. Substituting into (2.1) it is found that

$$\frac{Q'(t)}{k} + \frac{\partial T'}{\partial \pi^2} + \frac{1}{\kappa} \frac{\partial T'}{\partial \kappa} = \frac{1}{K_t} \frac{\partial T'}{\partial t} \quad (2.3)$$

The boundary conditions for the primed quantities are

$$T'(r,0) = 0 \quad T'(a,t) = 0$$

Equation (2.3) can now be solved by standard techniques for the two quantities of interest, namely the space averaged change in rod temperature and the change in heat flux to the coolant.

To solve (2.3) take the Laplace transform with respect to t (dropping the primes for convenience). The transformed equation is

$$\frac{\bar{Q}(s)}{k} + \frac{d^2 \bar{T}(n,s)}{dn^2} + \frac{1}{\kappa} \frac{d\bar{T}(n,s)}{dn} = \frac{s}{K_t} \bar{T}(n,s) \quad (2.4)$$

In this equation s is the transformed variable. The solution to this equation which is finite at $r = 0$ is

$$\bar{T}(n,s) = A(s) I_0\left(\sqrt{\frac{s}{K_t}} n\right) + \frac{K_t}{k} \frac{\bar{Q}(s)}{s} \quad (2.5)$$

as is easily verified. The function I_0 is the Bessel function of zero order of the second kind. $A(s)$ is determined from the condition that $\bar{T}$ vanish at $r = a$. Then

$$A(s) = - \frac{K_t}{k} \frac{\bar{Q}(s)}{s I_0\left(\sqrt{\frac{s}{K_t}} a\right)}$$

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The complete solution is

$$\bar{T}(r,s) = \frac{K_t}{k} \frac{\bar{Q}(s)}{s} - \frac{K_t}{k} \frac{\bar{Q}(s)}{s} \frac{I_0(\sqrt{\frac{s}{K_t}} r)}{I_0(\sqrt{\frac{s}{K_t}} a)} \quad (2.6)$$

The change in heat flux at the surface due to this change in temperature is given by

$$-k \frac{\partial \bar{T}}{\partial r} \Big|_{r=a} = \bar{q}(t)$$

or

$$-k \frac{d\bar{T}(a,s)}{dr} = \bar{q}(s)$$

From (2.6)

$$\bar{q}(s) = \frac{K_t \sqrt{\frac{s}{K_t}}}{s} \frac{I_1(\sqrt{\frac{s}{K_t}} a)}{I_0(\sqrt{\frac{s}{K_t}} a)} \bar{Q}(s) \quad (2.7)$$

Equation (2.7) can be inverted in the usual way. The only poles of $\bar{q}(s)/\bar{Q}(s)$ are at the zeros of $I_0(\sqrt{\frac{s}{K_t}} a)$. These lie on the negative real axis at $s_n = -\frac{K_t}{a^2} \xi_n^2$ in which ξ_n is the nth zero of $J_0(z)$. (Zero order Bessel function.) The residue at the nth pole of $(\bar{q}(s)/\bar{Q}(s)) e^{st}$ is $\frac{2K_t}{a} e^{-\frac{K_t}{a^2} \xi_n^2 t}$

Then by the convolution theorem, and since the inverse of $\bar{Q}(s)$ is $Q(t)$, the perturbed heat flux is given by

$$\bar{q}(t) = \frac{2K_t}{a} \int_0^t Q(\tau) \sum_{n=0}^{\infty} e^{-\frac{K_t}{a^2} \xi_n^2 (t-\tau)} d\tau \quad (2.8)$$

The average temperature change in the rod is needed as a function of power in order to form the corresponding reactivity feedback term

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The average is defined as

$$\langle T(t) \rangle = \frac{2}{a^2} \int_0^a T(r,t) r dr \quad (2.9)$$

The temperature as a function of r can be found from (2.6). Inverting $\bar{T}(r,s)/\bar{Q}(s)$, by residues and then using the convolution theorem as before it is found that

$$T(r,t) = -\frac{2K_t}{ka} \int_0^t Q(\tau) \sum_{n=0}^{\infty} e^{-\frac{K_t}{a^2} s_n^2 (t-\tau)} \frac{I_0\left(\frac{\sqrt{s_n}}{K_t} r\right)}{\frac{\sqrt{s_n}}{K_t} I_1\left(\frac{\sqrt{s_n}}{K_t} a\right)} d\tau \quad (2.10)$$

This is multiplied by rdr and integrated from 0 to a . In the integral,

$$\int_0^a I_0\left(\frac{\sqrt{s_n}}{K_t} r\right) r dr$$

put $u = \frac{\sqrt{s_n}}{K_t} r$; $dr = \frac{du}{\frac{\sqrt{s_n}}{K_t}}$. The integral then becomes

$$\frac{1}{\frac{\sqrt{s_n}}{K_t}} \int_0^{\frac{\sqrt{s_n}}{K_t} a} I_0(u) u du = \frac{1}{\frac{\sqrt{s_n}}{K_t}} \left(\frac{\sqrt{s_n}}{K_t} a\right) I_1\left(\frac{\sqrt{s_n}}{K_t} a\right)$$

then

$$\langle T(t) \rangle = -\left(\frac{2K_t}{k}\right) \frac{2}{a^2} \int_0^t Q(\tau) \sum_{n=0}^{\infty} \frac{e^{-\frac{K_t}{a^2} s_n^2 (t-\tau)}}{\frac{\sqrt{s_n}}{K_t}} d\tau$$

and since

$$\frac{s_n}{K_t} = -\frac{s_n^2}{a^2}$$

$$\langle T(t) \rangle = \frac{4K_t}{k} \int_0^t Q(\tau) \sum_{n=0}^{\infty} \frac{e^{-\frac{K_t}{a^2} s_n^2 (t-\tau)}}{s_n^2} d\tau \quad (2.11)$$

From (2.11) and (2.8) the values

$$\bar{q}(s) = \frac{2K_t}{a} \left(\sum_{n=0}^{\infty} \frac{1}{s + \frac{K_t}{a^2} s_n^2} \right) \bar{Q}(s) \quad (2.12)$$

and

$$\langle \bar{T}(s) \rangle = \frac{4K_t}{k} \left(\sum_{n=0}^{\infty} \frac{1}{s_n^2 (s + \frac{K_t}{a^2} s_n^2)} \right) \bar{Q}(s) \quad (2.13)$$

are found by interchanging the order of summation and integration and taking the Laplace transform term by term. Alternately they can be obtained from (2.7) and (2.6) by expansion in partial fractions.

III. TEMPERATURE CHANGE IN COOLANT

For a forced convection, non-boiling system the energy equation for the coolant can be written as

$$\frac{\varphi(t)}{Wc_p} = \frac{\partial T_c}{\partial z} + \frac{1}{U} \frac{\partial T_c}{\partial t} \quad (3.1)$$

Equation (3.1) neglects friction losses and applies to an incompressible fluid. The conduction term in the direction of flow has been neglected and the transverse temperature gradients have been averaged. $\varphi(t)$ is the total heat flux at the boundaries in Btu/ft sec, U is the coolant velocity which may depend on time, W is the total mass flow rate and C_p is the coolant specific heat.

Equation (3.1) can only be handled easily when the flow rate is constant. As before assume that $T = T_0 + T'$, $\varphi(t) = \varphi_0 + \varphi'(t)$ to get

$$\frac{\varphi'(t)}{Wc_p} = \frac{\partial T'_c}{\partial z} + \frac{1}{U} \frac{\partial T'_c}{\partial t} \quad (3.2)$$

At $t = 0$ $T'_c(z, 0) = 0$ everywhere. In addition the inlet temperature must be specified, $T'_c(0, t) = f(t)$. Taking the transform of (3.2) with respect to t , (again dropping the primes)

$$\frac{\bar{\varphi}(s)}{WC_p} = \frac{d\bar{T}_c(z,s)}{dz} + \frac{S}{U} \bar{T}_c(z,s) \quad (3.3)$$

and solving for $\bar{T}_c(z,s)$

$$\bar{T}_c(z,s) = \frac{\bar{\varphi}(s)}{A_c \rho_c C_p} \frac{1}{s} \left[1 - e^{-\frac{S}{U} z} \right] + \bar{f}(s) e^{-\frac{S}{U} z} \quad (3.4)$$

where $\bar{f}(s)$ is the transform of the coolant inlet temperature, A_c the coolant channel flow area. Note that $\bar{\varphi}(s)$ is the total heat transfer area times $\bar{q}(s)$ so that all the temperature feedback terms are now available in terms of $Q(t)$, which is proportional to the power.

IV. REACTIVITY FEEDBACK TRANSFER FUNCTION

For small changes in the temperature the feedback reactivity is supposed to be proportional to the change in temperature. Then $\rho(t)$ from (1.2) can be written as

$$\rho(t) = -\alpha_T \langle T(t) \rangle - \alpha_c \langle T_c(t) \rangle \quad (4.1)$$

Where α_T is the fuel temperature coefficient and α_c is the coolant temperature coefficient. Strictly speaking the feedback terms should be importance weighted in each region and integrated over all regions of the reactor. Such a step is beyond the reach of this analysis. However, diffusion calculations have shown that for the reactors under discussion the neutron importance tends to be space uniform if the power is. In this case Equation (4.1) is justified. The Laplace transform of (4.1) gives

$$\bar{\rho}(s) = -\alpha_T \langle \bar{T}(s) \rangle - \alpha_c \langle \bar{T}_c(s) \rangle \quad (4.2)$$

where $\langle \bar{T}_c(s) \rangle$ is the space averaged change in coolant temperature given by

$$\langle \bar{T}_c(s) \rangle = \frac{1}{L} \int_{z=0}^L \bar{T}_c(z,s) dz$$

$$\begin{aligned} \langle \bar{T}_c(s) \rangle = & \frac{\varphi(s)}{A_c \ell \rho_c c_p} \left(\frac{1}{s} \left[\ell - \frac{u}{s} (1 - e^{-\frac{s\ell}{u}}) \right] \right) \\ & + \frac{u}{\ell s} \bar{f}(s) \left[1 - e^{-\frac{s\ell}{u}} \right] \end{aligned} \quad (4.3)$$

From (4.2), (4.3), (2.12), and (2.13), the core transfer function can now be obtained. The source term $\bar{Q}(s)$ is proportional to the change in power $\bar{p}(s)$. That is,

$$\bar{Q}(s) = \frac{1}{\pi a^2 N \ell} \bar{p}(s)$$

$$\bar{p}(s) = (2\pi N a) \bar{q}(s)$$

where N is the total number of fuel rods, a is the rod radius, and ℓ is the total core length. Then

$$\langle \bar{T}(s) \rangle = \frac{4K_t}{N\pi a^2 \ell A} \left(\sum_{n=0}^{\infty} \frac{1}{s_n^2 (s + \frac{K_t}{a^2} s_n^2)} \right) \bar{p}(s) \quad (4.4)$$

$$\bar{q}(s) = \frac{4K_t}{a^2 \ell} \left(\sum_{n=0}^{\infty} \frac{1}{s + \frac{K_t}{a^2} s_n^2} \right) \bar{p}(s) \quad (4.5)$$

Then using (4.4) and (4.5) in (4.2) the reactivity change, or rather its transform, is given by

$$\begin{aligned} \bar{\rho}(s) = & -\alpha_T \left(\frac{4K_t}{h} \right) \left(\frac{1}{\pi a^2 N \ell} \right) \left(\sum_{n=0}^{\infty} \frac{1}{s_n^2 (s + \frac{K_t}{a^2} s_n^2)} \right) \bar{p}(s) \\ & - \alpha_c \left(\frac{4K_t}{a^2 \ell} \right) \left(\frac{1}{s} \left[\ell - \frac{u}{s} (1 - e^{-\frac{s\ell}{u}}) \right] \right) \left(\frac{1}{A_c \ell \rho_c c_p} \right) \left(\sum_{n=0}^{\infty} \frac{1}{s + \frac{K_t}{a^2} s_n^2} \right) \bar{p}(s) \\ & - \alpha_c \left(\frac{u}{\ell s} \right) \left[1 - e^{-\frac{s\ell}{u}} \right] \bar{D}(s) \bar{p}(s) \end{aligned} \quad (4.6)$$

where the transform of the inlet temperature $\bar{T}(s)$ has been written in terms of the external system transfer function, that is

$$\bar{T}(s) = \bar{D}(s) \bar{P}(s) \quad (4.7)$$

On comparing (4.6) with (1.6) the expression for the feedback transfer function follows immediately.

$$\begin{aligned} \bar{H}(s) = \frac{P_0}{T} \left\{ \alpha_T \left(\frac{4K_t}{R} \right) \left(\frac{1}{N\pi a^2 L} \right) \sum_{n=0}^{\infty} \frac{1}{s_n^2 (s + \frac{K_t}{a^2} s_n^2)} + \right. \\ \left. \alpha_c \left[\left(\frac{4K_t}{a^2 L} \right) \left(\frac{1}{A_c \ell \rho_c c_p} \right) \left(\frac{\ell}{s} - \frac{U}{s^2} (1 - e^{-\frac{s\ell}{U}}) \right) \sum_{n=0}^{\infty} \frac{1}{s + \frac{K_t}{a^2} s_n^2} \right. \right. \\ \left. \left. + \frac{U}{\ell s} (1 - e^{-\frac{s\ell}{U}}) \bar{D}(s) \right] \right\} \quad (4.8) \end{aligned}$$

Although (4.8) looks formidable, it can be easily examined graphically for stability using TRAFIC (3) provided the series converge rapidly enough as they generally do. This program takes $\bar{H}(j\omega)$ as input and constructs Bode and Nyquist plots. The external system transfer function will generally be a simple polynomial involving the transfer and heat exchanger delay times of the external circuit.

V. BOILING COOLANT

If the coolant boils the formulation of the transfer function becomes considerably more involved. This is due to two causes:

(1) there are now two distinct regions in the coolant and they vary in dimension as the boiling interface changes position. (2) the difficulty involved in predicting the spatial distribution of voids.

When considered as a homogeneous mixture of steam and liquid the flow is clearly compressible since the mixture density varies in both time and space. On the other hand when considered as separated flow of steam and liquid each component may be considered as incompressible, to

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a good approximation, when flowing at the velocity expected in a reactor core. In the appendix a slight modification of the Bankoff, homogeneous void model is developed; the resulting equation applies only to the vapor phase, which may be then treated as compressible. The equation found there is

$$\frac{\varphi(z,t)}{A_c} = \frac{\partial}{\partial z}(\chi w h_{fg}) + \frac{\partial}{\partial t}(\frac{\chi w h_{fg}}{u_g}) - \frac{\partial p}{\partial t} + \frac{F_f}{A_c} \quad (5.1)$$

In this equation, χ is the quality, w the mass flow rate in $\text{lbs/ft}^2 \text{ sec}$ $\varphi(z,t)$ is again the heat added at the boundary, and u_g is the steam velocity. A_c is the coolant channel cross section, p the pressure, and the term F_f represents friction losses.

As in the preceeding assume that a perturbation about steady quantities occurs,

$$\begin{aligned} \varphi(z,t) &= \varphi_0(z) + \varphi'(z,t) \\ \chi(z,t) &= \chi_0(z) + \chi'(z,t) \\ u_g(z,t) &= u_g(z) + u_g'(z,t) \end{aligned} \quad (5.2)$$

and substitute into (5.1). Assuming that the additional input of energy goes primarily to increasing the quality and not changing the pressure or doing friction work and also that w and h_{fg} are constant, the substitution of (5.2) into (5.1) gives

$$\frac{\varphi'(z,t)}{w h_{fg}} = \frac{\partial \chi'}{\partial z} + \frac{\chi_0(z)}{u_g(z)} \frac{\partial}{\partial t} \left(\frac{\chi'}{\chi_0(z)} \right) - \frac{\chi_0(z)}{u_g^2(z)} \frac{\partial u_g'}{\partial t} + \frac{\chi_0}{u_g^3} \frac{\partial u_g'}{\partial z} \quad (5.3)$$

where the steady relation

$$\frac{\varphi_0(z)}{w h_{fg}} = \frac{d\chi_0(z)}{dz} \quad \text{and} \quad W = A_c w$$

have been used. Equation (5.3) contains all the first order terms resulting from the perturbation. Neglecting the change in steam velocity with time Equation (5.3) becomes

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$$\frac{\varphi'(z,t)}{Wh_{fg}} = \frac{\partial \chi'}{\partial z} + \frac{1}{u_g(z)} \frac{\partial \chi'}{\partial t} \quad (5.4)$$

The neglect of the terms containing $\frac{\partial u'}{\partial t}$ is a good approximation in the region near where boiling begins, becoming less accurate as the distance down the boiling channel increases. At most these terms are about 1/4 of the term containing $\frac{\partial \chi'}{\partial t}$. Thus (5.4) is a good approximation if the boiling channel is not too long and a fair approximation if the whole core is boiling.

From (5.4), the perturbation in the quality can be found and then by means of the Bankoff relation the change in void can be found.

The reactor power is assumed space independent. Then in the steady case

$$\varphi_0 = \frac{P_0}{\ell} = \frac{\text{total reactor steady power}}{\text{core length}}$$

From

$$\frac{\varphi_0}{Wh_{fg}} = \frac{d\chi_0}{dz}$$

the initial quality is

$$\chi_0(z) = \frac{1}{Wh_{fg}} \frac{P_0}{\ell} z$$

From the Bankoff Relation (see references)

$$u_g(z) = \frac{w\beta}{\rho_g K_B} \left[1 + \left(\frac{1}{\beta} - 1 \right) \chi_0 \right] \quad (5.5)$$

where $\beta = \rho_g/\rho_L$, the density ratio, and K_B is Bankoff's constant which in this case is considered to be close to 1.0. If now the parameters

$$g = \left(\frac{1}{\beta} - 1 \right) \frac{P_0}{Wh_{fg}\ell} \quad \gamma_1 = \frac{w\beta}{A_c \rho_g K_B}$$

are introduced the initial steam velocity is given by

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$$U_g(z) = \gamma_1 (1 + g z) \quad (5.6)$$

then (5.4) may be written as

$$\frac{\varphi'(t)}{W h_{fg}} = \frac{\partial \chi'}{\partial z} + \frac{1}{\gamma_1 (1 + g z)} \frac{\partial \chi'}{\partial t} \quad (5.7)$$

This equation is now solved for χ' in terms of the heat input subject to the conditions

$$\chi'(z, 0) = 0 \quad \chi'(0, t) = 0$$

If the Laplace transform of (5.7) is taken the result is if φ' is independent of z ,

$$\frac{\bar{\varphi}'(s)}{W h_{fg}} = \frac{d \bar{\chi}'(z, s)}{dz} + \frac{s}{\gamma_1 (1 + g z)} \bar{\chi}'(z, s) \quad (5.8)$$

Equation (5.8) is Bernoulli's equation and can be solved for $\bar{\chi}'(z, s)$ in the usual way. The solution which fits the given conditions is

$$\bar{\chi}'(z, s) = \frac{\gamma_1}{W h_{fg}} \frac{\bar{\varphi}'(s)}{s + \gamma_1 g} \left[(1 + g z) - (1 + g z)^{-\frac{s}{\gamma_1 g}} \right] \quad (5.9)$$

From this equation the transformed void is found as follows:

Using the Bankoff formula

$$\frac{1}{\chi} = 1 - \frac{1}{\beta} \left(1 - \frac{K_B}{\alpha} \right) \quad (5.10)$$

where α is the void, one finds by taking differentials that a small change in χ is related to a small change in α through

$$\frac{d\chi}{\chi^2} = \frac{K_B}{\beta} \frac{d\alpha}{\alpha^2}$$

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or upon replacing dX by X' and $d\alpha$ by α'

$$\alpha'(z,t) = \beta/k_0 \left(\frac{\alpha}{\chi}\right)_0^2 \chi'(z,t) \quad (5.11)$$

where $(\alpha/\chi)_0$ means this ratio evaluated at the steady condition. From (5.11) it follows that

$$\bar{\alpha}'(z,s) = \beta/k_0 \left(\frac{\alpha}{\chi}\right)_0^2 \bar{\chi}'(z,s) \quad (5.12)$$

Substituting $\bar{\chi}'(z,s)$ from (5.9) the transform for $\bar{\alpha}'(z,s)$ is found to be

$$\bar{\alpha}'(z,s) = \beta/k_0 \left(\frac{\alpha}{\chi}\right)_0^2 \frac{\gamma_1}{Whfg} \frac{\bar{\psi}'(s)}{(s+\gamma_1g)} \left[(1+gz) - (1+gz)^{-\frac{s}{\gamma_1g}} \right] \quad (5.13)$$

The quantity $(\alpha/\chi)_0$ can be found from (5.10) and the preceding expression for $X_0(z)$ by solving (5.10) for α and forming α/χ . The result is

$$\beta/k_0 \left(\frac{\alpha}{\chi}\right)_0^2 = \frac{k_0}{\beta(1+gz)^2} \quad (5.14)$$

Upon substituting (5.14) into (5.13) the desired expression for $\bar{\alpha}'(z,s)$ is finally obtained.

$$\bar{\alpha}'(z,s) = \frac{k_0}{\beta} \frac{\gamma_1}{Whfg} \frac{\bar{\psi}'(s)}{(s+\gamma_1g)} \left[\frac{1}{1+gz} - \frac{(1+gz)^{-\frac{s}{\gamma_1g}}}{(1+gz)^2} \right] \quad (5.15)$$

or

$$\bar{\alpha}'(z,s) = \frac{\bar{\psi}'(s)}{A_0 f g h f g (s+\gamma_1g)} \left[\frac{1}{1+gz} - \frac{(1+gz)^{-\frac{s}{\gamma_1g}}}{(1+gz)^2} \right]$$

after using the definition of γ_1 .

The quantity of interest is the total excess void in the system

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at any time t and it is obtained by integrating $\bar{\alpha}'(z,s)$ over the core open volume.

$$\bar{\Omega}'(s) = A_c \int_0^L \bar{\alpha}'(z,s) dz \quad (5.16)$$

Substituting (5.15) into (5.16) and carrying out the integration leads to

$$\bar{\Omega}'(s) = \left[\ln(1+g) - \frac{r_g}{s+r_g} \left(1 - \frac{(1+g)^{-\frac{s}{r_g}}}{(1+g)} \right) \right] \frac{\bar{\gamma}'(s)}{\beta_g h_g g(s+r_g)} \quad (5.17)$$

Equation (5.17) is the appropriate void feedback term for the reactor equation if the entire core were boiling. It must now be modified for a system in which the coolant enters subcooled.

VI. MOTION OF THE BOILING INTERFACE

Let X_0 be the distance down channel at which boiling is initiated in the steady state corresponding to reactor power P . If $X(t)$ is the boiling interface location at time t measured from the channel inlet then the net total change in void is given by

$$\frac{\bar{\Omega}'(t)}{A_c} = \int_{z=0}^{L-X(t)} [\alpha'(z,t) + \alpha_0(z)] dz - \int_{z=0}^{L-X_0} \alpha_0(z) dz \quad (6.1)$$

This is equal to

$$\begin{aligned} \frac{\bar{\Omega}'(t)}{A_c} &= \int_{z=0}^{L-X(t)} \alpha'(z,t) dz + \int_{z=0}^{L-X(t)} \alpha_0(z) dz - \int_{z=0}^{L-X_0} \alpha_0(z) dz \\ \frac{\bar{\Omega}'(t)}{A_c} &\cong \int_{z=0}^{L-X_0} \alpha'(z,t) dz + \int_{z=0}^{L-X_0+X(t)} \alpha_0(z) dz - \int_{z=0}^{L-X_0} \alpha_0(z) dz \end{aligned} \quad (6.2)$$

The change in the limits in the first integral is justified by the

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assumption that the changes are small and in the second $X(t)$ has been replaced by $X(t) = X_0 - U(t)$ where $U(t)$ is a small displacement of the interface. Combining the last two integrals one gets

$$\frac{Q'(t)}{A_0} = \int_{z=0}^{L-X_0} \alpha'(z,t) dz + \int_{L-X_0}^{L-X_0+U(t)} \alpha_0(z) dz \quad (6.3)$$

The last integral may be approximated by $\alpha_0(\text{exit}) U(t)$ since $U(t)$ is small so that

$$Q'(t) = A_0 \int_{z=0}^{L-X_0} \alpha'(z,t) dz + A_0 U(t) \alpha_0(\text{exit}) \quad (6.4)$$

The first term is the same as that calculated in the preceding section except that the limits of integration are now 0 to $L-X_0$.

For the second term in (6.4) $\alpha_0(\text{exit})$ and $U(t)$ must be calculated. The distance traveled before the fluid reach saturation, $X(t)$ is given approximately by

$$X(t) = \frac{W c_p (T_s - T_i)}{\varphi(t)} \quad (6.5)$$

if φ does not change too much over the time interval of interest.

From (6.5)

$$X(t) = \frac{W c_p (T_s - T_i)}{\varphi_0 + \varphi'(t)} = \frac{W c_p (T_s - T_i)}{\varphi_0 (1 + \frac{\varphi'(t)}{\varphi_0})}$$

or
$$X(t) = X_0 \left(1 - \frac{\varphi'(t)}{\varphi_0} \right) \quad \varphi'(t) \ll \varphi_0 \quad (6.6)$$

Then

$$X_0 - X(t) = X_0 \frac{\varphi'(t)}{\varphi_0} \equiv U(t) \quad (6.7)$$

and so after transforming (6.7) gives

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$$\bar{U}(s) = \frac{X_0}{\varphi_0} \bar{\varphi}'(s) \quad (6.8)$$

where $\varphi_0 = \frac{P_0}{L} ; \quad X_0 = \frac{WC_p(T_s - T_i)}{q_0}$

The exit void can be calculated from the Bankoff relation and the initial quality distribution, $X_0(z)$.

From the Bankoff relation, (5.10),

$$\alpha_0(\text{exit}) = \frac{K_B}{\beta} \frac{X_0(\text{exit})}{1 + (\frac{1}{\beta} - 1)X_0(\text{exit})} \quad (6.9)$$

but $X_0(\text{exit}) = \frac{1}{Wh_{fg}} \frac{P_0(L - X_0)}{L}$

Which relation reduces (6.9) to

$$\alpha_0(\text{exit}) = \frac{K_B}{\beta} \frac{P_0(L - X_0)}{Wh_{fg}L + (\frac{1}{\beta} - 1)P_0(L - X_0)} \quad (6.10)$$

The Laplace transform of (6.4) is now taken giving

$$\bar{\Omega}'(s) = A_c \int_{z=0}^{L-X_0} \bar{\alpha}'(z,s) dz + A_c \bar{U}_s \alpha_0(\text{exit}) \quad (6.11)$$

The second term of (6.11) is, from (6.8), (6.10) and (4.5),

$$A_c \alpha_0(\text{exit}) \bar{U}(s) = \left[\frac{K_B(L - X_0)X_0 A_c}{\beta Wh_{fg}L + (1 - \beta)P_0(L - X_0)} \right] \left(\frac{4K_t}{a^2 L} \sum_{n=0}^{\infty} \frac{1}{s + \frac{K_t}{a^2 L} \beta_n^2} \right) \bar{\varphi}(s) \quad (6.12)$$

While the first term of (6.11) is obtained directly from (5.17) with (4.5) substituted for $\bar{\varphi}'(s)$.

$$A_c \int_{z=0}^{L-X_0} \bar{\alpha}'(z,s) dz = \quad (6.13)$$

$$\left\{ \ln[1 + q(L - X_0)] - \frac{\gamma_1 q}{s + \gamma_1 q} \left[1 - \frac{(1 + q(L - X_0))^{-\frac{s}{\gamma_1 q}}}{1 + q(L - X_0)} \right] \right\} \frac{1}{f_g h_{fg} q(s + \gamma_1 q)} \left(\frac{4K_t}{a^2 L} \sum_{n=0}^{\infty} \frac{1}{s + \frac{K_t}{a^2 L} \beta_n^2} \right) \bar{\varphi}(s)$$

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The reactivity feedback terms are now all available and if for simplicity the liquid temperature feedback is neglected in comparison with that of the voids and rod temperature changes the feedback reactivity is (the liquid phase feedback is actually of second order being the product of a small change in temperature and a small change in length).

$$\bar{\rho}(s) = -\alpha_T \langle \bar{T}(s) \rangle - \alpha_V \bar{\Omega}(s) \quad (6.14)$$

where $\bar{\Omega}'(s)$ is given by (6.11), (6.12) and (6.13) and $\langle \bar{T}(s) \rangle$ is given by (4.4). Let

$$F(q,s) = \frac{4K_t}{a^2 l} \frac{1}{\beta g h_f g} \left[\ln(1+g(l-x_0)) - \frac{\gamma_1 g}{s+\gamma_1 g} \left(1 - \frac{(1+g(l-x_0))^{-\frac{s}{\gamma_1 g}}}{1+g(l-x_0)} \right) \right]$$

$$G = \frac{4K_t}{a^2} \left[\frac{K_R(l-x_0)x_0 A_c}{\beta W h_f g l + (1-\beta) P_0(l-x_0)} \right]$$

then

$$\begin{aligned} \bar{\Omega}'(s) = & \frac{F(q,s)}{s+\gamma_1 g} \left(\sum_{n=0}^{\infty} \frac{1}{s + \frac{K_t}{a^2} s_n^2} \right) \bar{\rho}(s) \\ & + G \left(\sum_{n=0}^{\infty} \frac{1}{s + \frac{K_t}{a^2} s_n^2} \right) \bar{\rho}(s) \end{aligned} \quad (6.15)$$

In this case

$$\bar{\rho}(s) = -\alpha_T \langle \bar{T}(s) \rangle - \alpha_V \bar{\Omega}'(s) \quad (6.16)$$

Then from (4.4) and (6.15)

$$\bar{\rho}(s) = -\alpha_T \left(\frac{4K_t}{k} \right) \left(\frac{1}{\pi a^2 N l} \right) \left(\sum_{n=0}^{\infty} \frac{1}{s_n^2 (s + \frac{K_t}{a^2} s_n^2)} \right) \bar{\rho}(s) \quad (6.17)$$

$$- \alpha_V \frac{F(q,s)}{s+\gamma_1 g} \left(\sum_{n=0}^{\infty} \frac{1}{s + \frac{K_t}{a^2} s_n^2} \right) \bar{\rho}(s) - \alpha_V G \left(\sum_{n=0}^{\infty} \frac{1}{s + \frac{K_t}{a^2} s_n^2} \right) \bar{\rho}(s)$$

And then from (1.6), $\bar{H}(s)$, the reactivity feedback transfer function can be obtained as before.

CONCLUSIONS

Only Equation (6.17) has as yet been run on the Nyquist Plotter Program. Equation (4.8) will be investigated also but only after a control system and external loops have been added to the transfer function. The results of TRAFIC calculations using (6.17) are just as expected. The core is stable in the presence of boiling. In summary:

(1) If the fuel and void reactivity coefficients are both positive the core is unconditionally stable.

(2) If the fuel temperature coefficient alone were negative the core, in the presence of boiling, is unconditionally stable even for rather large (in absolute value) values of the fuel coefficient.

Instability was found only when the value of α_T used was about ten times its maximum expected value.

(3) If the fuel and void coefficients are both negative the core is unconditionally unstable.

Note that from (4.2) a negative coefficient means positive feedback due to the way in which the equation is written.

Stability investigations have also been carried out using PROOT, a digital polynomial root finder program. For this technique the characteristic equation must be a polynomial of relatively small degree. (For practical purposes not much greater than 10, otherwise the work of finding the coefficients becomes prohibitive.) To get the characteristic equation in the form of a polynomial of finite degree the spatial dependence of the voids must generally be suppressed in some manner. Assuming a linear spatial dependence, calculations were carried out which gave results generally in accordance with those listed above.

Some typical results are contained in Appendix II.

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APPENDIX I - Development of Equation (5.1)

Let w be the mass flow rate in $\text{lbs/ft}^2\text{sec}$ through the reactor. For compressible flow w may be a function of both z , the longitudinal distance up the channel, and t , the time. It is assumed that w is independent of transverse dimensions.

The steam quality X is defined as

$$X = \frac{w_g}{w} = \frac{\text{Mass flow rate gas}}{\text{Total mass flow rate}}$$

therefore

$$w_g = Xw = \text{mass flow rate of gas}$$

$$w_l = (1-X)w = \text{mass flow rate of liquid}$$

In a typical coolant channel of cross sectional area A , and transverse dimension y , take a mass balance as follows:

Let $f(z,t)$ be the lbs of gas being formed per ft^3 per sec due to heat addition. A steam balance on the element of volume $A dz$ gives

$$\text{Steam in} = A_c Xw + A_c f(z,t) dz = \frac{\text{lbs}}{\text{sec}}$$

$$\text{Steam out} = A_c Xw + A_c \frac{\partial}{\partial z} (Xw) dz = \frac{\text{lbs}}{\text{sec}}$$

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The net gain in mass is

$$dz A_c \frac{d}{dt} f(z,t) - A_c \frac{\partial}{\partial z} (\rho w) dz \quad (A.1)$$

This must equal the time rate of change of steam in the element of volume $A_c dz$. The weight of gas in a volume element $dx.dz.dy$ is

$$dm_g = \rho_g \alpha(y) dx dy dz = \rho_g \alpha(y) dA_c dz$$

since α , the void fraction is assumed to depend only on one transverse dimension (two dimensional channel).

Then

$$m_g = \rho_g dz \int \alpha(y) dA_c = \rho_g dz A_c \bar{\alpha} \quad (A.2)$$

where

$$\bar{\alpha} = \frac{1}{A_c} \int \alpha(y) dA_c$$

is the mean void fraction

in the channel. Equation (A.2) gives the total mass in the channel of cross section A_c .

In a similar manner the mass flow of gas in a particular channel is given by

$$dw_g' = \rho_g \alpha(y) u_g(y) dA_c$$

and the total mass flow is

$$w_g' = \rho_g \bar{\alpha} u_g A_c$$

where

$$\bar{\alpha} u_g = \frac{1}{A_c} \int \alpha(y) u_g(y) dA_c \quad (A.3)$$

so that

$$\frac{w_g'}{A_c} = \rho_g \bar{\alpha} u_g = w_g \quad (A.4)$$

is the mass flow rate of gas per unit per second. From (A.4),

$$\rho_g = \frac{w_g}{\bar{\alpha} u_g} \quad . \quad \text{Then (A.2) becomes}$$

$$m_g = \frac{A_c \omega_g \bar{\alpha}}{\bar{\alpha} u_g} dz \quad (A.5)$$

If the velocity distribution is assumed to be very flat as it should be in turbulent flow then

$$\bar{\alpha} u_g = \bar{\alpha} u_{g0}$$

As a result Equation (A.5) can be written in terms of the quality and the total mass flow rate,

$$m_g = A_c \frac{\omega_g}{u_g} dz = A_c \frac{\chi \omega}{u_g} dz \quad (A.6)$$

Substituting (A.6) into (A.1) gives

$$f(z,t) dz = \frac{\partial}{\partial z} (\chi \omega) + \frac{\partial}{\partial t} \left(\frac{\chi \omega}{u_g} \right) \quad (A.7)$$

which is the conservation equation for the gas phase. A similar equation is obtained for the liquid phase

$$g(z,t) dz = \frac{\partial}{\partial z} [(1-\chi)\omega] + \frac{\partial}{\partial t} \left[\frac{(1-\chi)\omega}{u_l} \right] \quad (A.8)$$

where $g(z,t)$ is the source term for liquid. (since $g(z,t) = -f(z,t)$, addition of (A.7) and (A.8) leads to

$$\frac{\partial \omega}{\partial z} + \frac{\partial}{\partial t} \left[\frac{\chi \omega}{u_g} + \frac{(1-\chi)\omega}{u_l} \right] = 0 \quad (A.9)$$

which is the continuity equation for the two phase flow.

The energy equation can now be derived from the first law which states,

The time rate of change of energy in the element is equal to the net rate at which energy is added + time rate at which work is being done on the system.

Let E_g be the energy of the gas per pound (internal + kinetic),

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E_l that of the liquid. Then the energy in the differential volume $A_0 dz$ is

$$dz = [A_c \alpha \rho_g E_g + A_c (1-\alpha) \rho_l E_l]$$

where α , $1-\alpha$ are the average values $\bar{\alpha}$, $1-\bar{\alpha}$.

The energy flowing into the element is

$$A_c \bar{\alpha} W E_g + A_c (1-\bar{\alpha}) W E_l$$

The heat conducted into the element is

$$\varphi(z,t) dz$$

If $\varphi(z,t)$ is the wall heat flux in $\frac{\text{Btu}}{\text{ft}^2 \text{sec}}$.

Energy is also flowing out to the right in amount

$$(A_c \bar{\alpha} W E_g + A_c (1-\bar{\alpha}) W E_l) + \frac{\partial}{\partial z} (A_c \bar{\alpha} W E_g + A_c (1-\bar{\alpha}) W E_l) dz$$

In addition work is being done on the system in amount

$$\int_A \hat{\tau} \cdot \hat{u} dA$$

Where $\hat{u}$ is the velocity, $\hat{\tau}$ the stress, and the integration is over the surface of the volume element under consideration. Collecting all terms results in

$$\begin{aligned} & \frac{\partial}{\partial t} [\alpha \rho_g E_g + (1-\alpha) \rho_l E_l] + \frac{\partial}{\partial z} [\bar{\alpha} W E_g + (1-\bar{\alpha}) W E_l] \\ &= \frac{\varphi(z,t)}{A_c} + \frac{1}{A_c dz} \int_A \hat{\tau} \cdot \hat{u} dA \end{aligned} \quad (\text{A.10})$$

In calculating the work term the faces perpendicular to z where the largest force is the pressure will be taken separately. At z the integral is approximately

$$\frac{1}{A_c dz} [A_c \alpha \rho u_g + A_c (1-\alpha) \rho u_l]$$

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at $z + dz$,

$$- \frac{1}{A_c dz} \left\{ A_c \alpha p u_g + A_c (1-\alpha) p u_e + \frac{\partial}{\partial z} [A_c \alpha p u_g + A_c (1-\alpha) p u_e] dz \right\}$$

where the minus sign because on this face $\hat{c}$ and $\hat{u}$ are oppositely directed. The net result is

$$- \frac{\partial}{\partial z} [\alpha p u_g + (1-\alpha) p u_e]$$

The remainder of the integral is equal to the friction losses which will be denoted by F_f . Then Equation (A.10) becomes

$$\begin{aligned} & \frac{\partial}{\partial t} [\alpha p_g E_g + (1-\alpha) p_e E_e] + \frac{\partial}{\partial z} [\chi w E_g + (1-\chi) w E_e] \\ &= \frac{\varphi(z,t)}{A_c} - \frac{\partial}{\partial z} (\alpha p u_g + (1-\alpha) p u_e) - \frac{F_f}{A_c} \end{aligned} \quad (A.11)$$

upon using $E_g = h_g - p \pi_g + 1/2 \frac{u_g^2}{g}$
 $E_e = h_e - p \pi_e + 1/2 \frac{u_e^2}{g}$

$$\begin{aligned} \alpha p_g &= \frac{\chi w}{u_g} & \alpha u_g &= \frac{\chi w}{p_g} \\ (1-\alpha) p_e &= (1-\chi) \frac{w}{u_e} & (1-\alpha) u_e &= \frac{(1-\chi) w}{p_e} \end{aligned}$$

Equation (A.11) becomes

$$\begin{aligned} & \frac{\partial}{\partial t} \left[\frac{\chi w}{u_g} \left(h_g - p \pi_g + \frac{1}{2} \frac{u_g^2}{g} \right) \right] + \frac{\partial}{\partial t} \left[\frac{(1-\chi) w}{u_e} \left(h_e - p \pi_e + \frac{1}{2} \frac{u_e^2}{g} \right) \right] \\ &+ \frac{\partial}{\partial z} \left[\chi w \left(h_g - p \pi_g + \frac{1}{2} \frac{u_g^2}{g} \right) \right] + \frac{\partial}{\partial z} \left[(1-\chi) w \left(h_e - p \pi_e + \frac{1}{2} \frac{u_e^2}{g} \right) \right] \\ &= \frac{\varphi(z,t)}{A_c} - \frac{\partial}{\partial z} (\chi w p \pi_g + (1-\chi) w p \pi_e) - \frac{F_f}{A_c} \end{aligned} \quad (A.12)$$

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The terms involving $\frac{\partial}{\partial z}(\rho r)$ cancel. Neglecting the kinetic energy terms the remainder is

$$\begin{aligned} & \frac{\partial}{\partial t} \left[\frac{\chi \omega}{u_g} h_g + (1-\chi) \frac{\omega h_e}{u_e} \right] + \frac{\partial}{\partial z} \left[\chi \omega h_g + (1-\chi) \omega h_e \right] \\ & - \frac{\partial}{\partial t} \left[\frac{\chi \omega \rho r_g}{u_g} + (1-\chi) \frac{\omega \rho r_e}{u_e} \right] = \frac{\varphi(z,t)}{A_e} - \frac{F_f}{A_e} \end{aligned} \quad (A.13)$$

Putting $h_g = h_e + h_{fg}$ and assuming h_e is a constant (A.13) may be reduced as follows:

$$\frac{\partial}{\partial z} \left[\chi \omega h_e + \chi \omega h_{fg} + (1-\chi) \omega h_e \right] = \frac{\partial}{\partial z} (\omega h_e) + \frac{\partial}{\partial z} (\chi \omega h_{fg})$$

and

$$\begin{aligned} & \frac{\partial}{\partial t} \left[\frac{\chi \omega h_e}{u_g} + \frac{\chi \omega h_{fg}}{u_g} + (1-\chi) \frac{\omega h_e}{u_e} \right] = \frac{\partial}{\partial t} \left(\frac{\chi \omega h_{fg}}{u_g} \right) \\ & + h_e \frac{\partial}{\partial t} \left[\frac{\chi \omega}{u_g} + (1-\chi) \frac{\omega}{u_e} \right] = \frac{\partial}{\partial t} \left(\frac{\chi \omega h_{fg}}{u_g} \right) - h_e \frac{\partial \omega}{\partial z} \end{aligned}$$

from the continuity Equation (A.9). Therefore the sum of the first two terms in (A.13) is

$$\frac{\partial}{\partial t} \left(\frac{\chi \omega h_{fg}}{u_g} \right) + \frac{\partial}{\partial z} (\chi \omega h_{fg})$$

In the last term on the left of (A.13) since

$$\frac{\chi \omega \rho r_g}{u_g} = \frac{\chi \omega \rho}{\rho_g u_g} = \frac{\chi \omega \rho}{\frac{\rho \chi}{\alpha}} = \alpha \rho$$

and

$$(1-\chi) \frac{\omega \rho r_e}{u_e} = (1-\alpha) \rho$$

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this term reduces to $-\frac{\partial P}{\partial t}$.

The final result can therefore be written in the form

$$\frac{\varphi(z,t)}{A_c} = \frac{\partial}{\partial z} (\chi w h_{fg}) + \frac{\partial}{\partial t} \left(\frac{\chi w h_{fg}}{u_g} \right) - \frac{\partial P}{\partial t} + \frac{F_f}{A_c} \quad (A.14)$$

The merit of Equation (A.14) is that it contains only the quality and no mixed terms in quality and void. It refers to the gas phase and this can surely be treated as incompressible if the heat addition rate does not change too rapidly. Since in a reactor core it is the void which is of interest, an auxiliary relation is needed expressing the void in terms of quality. Such an expression is the Bankoff relation which has been used in this paper.

The crucial assumption made in deriving (A.14) is that the velocity distribution is nearly flat, that is very near to plug flow. In a small boiling channel this is probably as good an assumption as any other.

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APPENDIX II

The results of several typical runs are enclosed. The figures are Nyquist plots as constructed by TRAFIC.

Figures 1 and 2 are the results of runs with negative feedback from both void and rod temperature but for different powers. The P printed on the plots is the power in Btu/sec. The parameter Q depends on both power and coolant flow rate. The figures shown are only representative of the numerous cases studied.

Figures 3 and 4 are runs made with both low power and low coolant flow rate.

Figures 5 and 6 were made at fixed power and flow rate, but with a negative fuel temperature coefficient (positive temperature feedback). In each successive run the magnitude of the coefficient was increased ten fold. It can be seen that the system becomes unstable for a relatively small value of the fuel coefficient.

The stability criterion is the usual one. One clockwise encirclement of the (-1) point indicates instability. The plots are traversed in a clockwise manner beginning at $w = 0$ and going through the origin ($w = \infty$) to $w = (-0)$. The curve must then be closed so that the enclosed area lies to the right as one goes round the contour in a clockwise direction. Therefore Figures 1,2,3,4 and 5 are closed to the right but Figure 6 must be closed to the left of the -1 point. The latter case is therefore the only one which indicates instability.

FIGURE 1

Power - 5 megawatts
Flow - 110 lbs/sec
 α_T - 0.0025%/°F
 α_v - 47.0%/ft³

FIGURE 2

Power - 10.5 megawatts
Flow - 110 lbs/sec
 α_T - 0.0025 $^{\circ}\text{F}$
 α_v - 47.0 $^{\circ}\text{ft}^3$

FIGURE 3

Power - 0.5 megawatt
Flow - 10 lbs/sec
 α_T - 0.0025 $^{\circ}\text{F}$
 α_V - 47.0 $^{\circ}\text{F}/\text{ft}^3$

FIGURE 4

Power - 1 megawatt
Flow - 10 lbs/sec
 α_T - 0.0025 $^{\circ}\text{F}$
 α_v - 47.0 $^{\circ}\text{F}/\text{ft}^3$

FIGURE 5

Power - 2.5 megawatts
Flow - 110 lbs/sec
 α_T - (-0.0025) $^{\circ}\text{F}$
 α_V - 47.0 $^{\circ}\text{F}/\text{ft}^3$

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FIGURE 6

Power - 2.5 megawatts
Flow - 110 lbs/sec
 α_T - (-0.025) $^{\circ}\text{F}$
 α_V - 47.0 $^{\circ}\text{F}/\text{ft}^3$