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DISPERSION RELATION FOR RELATIVISTIC STREAMS OF FINITE RADIUS

Livermore, California

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We have obtained a new dispersion relation for longitudinal plasma oscillations of a finite plasma, which includes relativistic and retardation effects, temperature, and streaming. A considerable unification of previous results[1-6] has thereby been accomplished. The equations exhibit a discontinuity when the phase velocity of the plasma wave equals the light velocity c.

The solution of these equations will be useful in the plasma oscillations of thin beams and will decide whether the two-stream instability can produce plasma radiation.[7]

The model considered is that of a cylinder of radius a and infinite length populated with charged particles and immersed in an infinitely strong magnetic field, directed along the cylinder axis, so that only longitudinal motion occurs. The magnetic field thus acts only as a constraint. Other discussions of longitudinal oscillations dispense with this artifice and consider only motion parallel to some vector $\underset{\sim}{k}_z$, which in our case is along the cylinder axis.

Each charged particle stream is assumed to be uniform in density across the cylinder and may be moving relativistically. The velocity distribution of each stream is also uniform throughout the cylinder, but is assumed sufficiently narrow so that the beam may be characterized by one value of γ (the relativistic factor). It is otherwise arbitrary. Each beam is characterized by a particle density, particle mass, charge sign, velocity distribution, and average velocity. These five quantities may differ for each beam but the radius of each beam is the same. The cylinder is isolated in space. (See Fig. 1.)

This system is in neutral equilibrium. At time $t = 0$ the system is disturbed. The subsequent time evolution of the fields and other variables is obtained by the Laplace transform and the assumption that all variables are first order small. The result may be expressed in the following form. The system behaves in time like $\sum a_n e^{-i\omega_n t}$, where the discrete ω_n are solutions of a dispersion relation. There is also a term which decays with time, and for given k becomes negligible as $t \gg 1/kc \equiv 1/\omega_k$. This term apparently establishes causality, i.e., that no signal can travel faster than c, so that at time $t = \tau$, for example, there is no disturbance further than $c\tau$ away from the edge of the plasma cylinder. This term arises from a branch cut integral and is apparently related to the branch cut integrals discussed by Brillouin[8] and Stratton[9] in their analyses of signal velocities in dispersive media. All quantities vary as e^{ikz}.

The dispersion relation obtained may be simplified in two limits. One where $\kappa a \gg 1$, $\left(\kappa a \equiv \left[1 - \left(\frac{\omega}{kc} \right)^2 \right]^{\frac{1}{2}} ka \right)$ and the other where $\kappa a \ll 1$. Thus the form of the equations depends not only on ka, but also on the phase velocity of the wave. The 'thin beam' equation ($\kappa a \ll 1$) will then be obtained even for $ka \gg 1$, if $\omega/kc \approx 1$.

The physical significance of κ when $\omega/kc < 1$ is that κ is the wave vector measured in a system moving with the wave velocity ω/kc. Only in this system can the longitudinal electric field be obtained from Poisson's equation, as the charge distribution is stationary. (The particles move, but the charge distribution in this moving frame is stationary.) Since the longitudinal electric field is Lorentz invariant, this field is the same in the lab frame and hence is the physically significant quantity.

These ideas immediately suggest the existence of two branches of the dispersion relation. One where $(\mathrm{Re}\ \omega)/kc < 1$ and the other where $\mathrm{Re}\ \omega/kc > 1$. For the former case, we have the situation just discussed, where a Lorentz transformation may be made to a system where the charges are independent of time though varying in z. For the latter case a Lorentz transformation may be made to a system where the currents are independent of z, though varying in time. These ideas are covered in more detail by Landau and Schmidt.[10]

For the case where the phase velocity of the wave is larger than c, in the lab frame, the waves will radiate by a Cerenkov-type process.[11] The two-stream instability mechanism may provide an energy source for this radiation. Unfortunately, our dispersion relation has not, as yet, been solved to see if the above qualitative arguments are true.

From our dispersion relation the works of other authors may be obtained. The results of Jackson,[2] Bludman *et al.*,[4] and others for the infinite beam case may be recovered when $\kappa a \gg 1$. As mentioned, these results are now seen to be invalid when $\omega/kc \approx 1$. It is moreover clear that a discontinuity in the equations exists when $\omega/kc \approx 1$ and that only solutions with complex ω are found if $\omega/kc > 1$. This is not found in the strictly one-dimensional case considered by Buneman,[12] where the waves are undamped.

The thin beam results of Sturrock[6] and Finkelstein and Sturrock[13] are also recovered when $\kappa a \ll 1$. It is seen that the results obtained by these authors are valid only if $\omega/kc \ll 1$.

When the temperature of the plasma is set equal to zero, the results of Budker[5] are obtained. His treatment is more restrictive than ours in that only the m = 0 angular independent perturbations are considered. The derivation of Budker's results has heretofore been unavailable in the literature.

The basic equations are the kinetic equation and Maxwell's equations. The kinetic equation is written in the canonical formalism. Under the assumption of purely longitudinal motion, the kinetic equation for each specie may be linearized and put in the form,

$$\frac{\partial f_1}{\partial t} + \dot{z}_0 \frac{\partial f_1}{\partial z} + \dot{p}_z^1 \frac{\partial f_0}{\partial p} = 0 . \tag{1}$$

The coefficients $\dot{z}_0$ and $\dot{p}_z^1$, the velocity and the canonical momentum velocity, respectively, are given, to the required order by

$$\dot{z}_0 = \overline{v} + \frac{1}{\gamma^3 m_0} p ,$$

$$\dot{p}_z^1 = \frac{e\dot{z}_0}{c} \frac{\partial A_1}{\partial z} - e \frac{\partial \phi_1}{\partial z} . \tag{2}$$

The first order fields are given by the retarded potentials

$$\phi_1(\underset{\sim}{r},t) = e \int \frac{f_1\left(\underset{\sim}{r}, t - \frac{|r - r'|}{c}, p\right)}{|r - r'|} \, d\underset{\sim}{r} \, dp, \tag{3}$$

$$A_1(\underset{\sim}{r}',t) = \frac{e}{c} \int \frac{z_0 f_1 - \frac{eA_1 f_0}{c\gamma^3 m_0}}{|r - r'|} \, d\underset{\sim}{r} \, dp .$$

The integrand of A_1 is also evaluated at the retarded time. A_1 is the perturbed part of A_z. The average beam velocity is $\overline{v}$, while p differs from the linear momentum by a constant, chosen so that $\overline{p}$ for a beam is zero. The quantities f_1, A_1, ϕ_1 and $\dot{p}_z^1$ are the first order quantities. ($f_1 << f_0$).

These equations may be solved by expanding the unknown quantities f, ϕ, A and $\dot{p}_z$ in terms of a Laplace transform in time, Fourier transform in z, Fourier series in θ, and Fourier-Bessel (Dini)[14] series transform in r. A typical integrand is of the form

$$\phi_1 = \phi^\tau J_m(\lambda_i r)\, e^{-i\omega t + ikz + im\theta}, \qquad r \le a. \tag{4}$$

For ϕ one may substitute A, f, or $\dot{p}_z$. With the choice of λ_i that satisfies

$$\lambda_i a \frac{J_{m+1}(\lambda_i a)}{J_m(\lambda_i a)} = \kappa a \frac{K_{m+1}(\kappa a)}{K_m(\kappa a)}, \qquad \begin{array}{l} \mathrm{Re}\,\lambda_i > 0, \\ m = 0, \pm 1, \pm 2, \ldots, \end{array} \tag{5}$$

the various transforms ϕ^τ, A^τ, f^τ, and $\dot{p}_z^\tau$ may be obtained in terms of each other, and each transform may be obtained as a function of the initial conditions. The 'mode' given by Eq. (4) has the dependence

$$\phi_1 = \phi^\tau K_m(\kappa r)\, e^{-i\omega t + ikz + im\theta}, \qquad r \ge a, \tag{6}$$

outside the cylinder. One may also substitute A for ϕ.

A dispersion relation may be obtained by inverting the Laplace transform to obtain the time dependence. These integrals may be deformed into contour integrals in the complex ω plane, and under the assumption that the initial perturbation is analytic, the integrands may be analytically continued into the lower half complex ω plane following the procedure of Landau[1] and Jackson.[2] These integrals may be evaluated by the residue theorem and one sees that the possible frequencies of the system occur at poles in the complex ω plane, i. e., when certain denominators are zero. Setting the denominators equal to zero gives the result

$$\frac{-(\lambda^2 + \kappa^2)}{4\pi e^2} = \left[1 - \left(\frac{\omega}{kc}\right)^2\right] \sum_{i=1}^{n} \int \frac{\frac{\partial f_{0i}}{\partial p}}{\frac{\omega}{k} - v_i}\, dp\,. \tag{7}$$

These integrals are analytically continued into the lower ω plane according to the Landau,[1] Jackson[2] prescription.

In the contour integration, certain terms are multivalued and hence there are integrals along branch cuts. These are indicated in Fig. 2 and extend out to $-\infty$ and $+\infty$, respectively. These branch cut integrals may be shown to decay with time and in fact for long enough time, decay faster than any power of t. These integrals vanish in the usual infinite plasma formalism.

The quantities v and κ are defined as

$$v_i = \bar{v}_i + \frac{1}{\gamma^3 m_i} p, \quad \kappa = \left(k^2 - \frac{\omega^2}{c}\right)^{1/2},$$

$$\arg \kappa = \frac{\theta_1 + \theta_2 - \pi}{2}, \quad k \text{ real}, \tag{8}$$

and m_i is the rest mass, and γ the relativistic factor.

Equations (5), (7), and (8) are our new dispersion relations which may be solved for complex ω and λ_i. The radiating solutions are near the branch cut, where $|\omega/kc| > 1$ and the damping is very small.

A more detailed report is being prepared for publication.[15]

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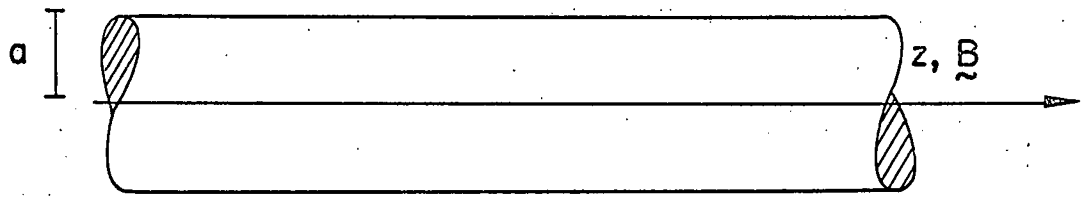

Fig. 1. Cylinder geometry.

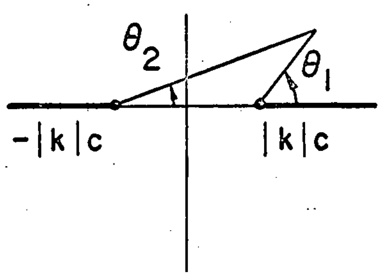

Fig. 2. Complex ω plane.