Generating Functions and KNO Scaling Models*

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ABSTRACT

We obtain the generating functions of empirically motivated KNO scaling models. The generating functions are used to obtain the asymptotic correlation function $f_k \sim \langle n \rangle^k$ ($\langle n \rangle$ is the average charged multiplicity). This is compared with various models of production mechanisms.

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I. INTRODUCTION

At presently attainable energies, there is at best tenuous theoretical justification for Koba, Nielsen, Olesen¹ (KNO) scaling. The phenomenological observation that ad hoc KNO scaling models give remarkable fits is the main motivation for further theoretical investigation. The ramifications of the hypothesis of KNO scaling may yield improved insights into properties of hadronic matter and production mechanisms.

In this paper, we attempt to use the generating function as the medium in which to study simple KNO scaling functions of the type introduced by Slattery,² and by Weisberg.³

In Sec.II, the basic formulas involving the generating function are presented. Section III deals with the relation between KNO scaling functions that fit the data and generating functions. These phenomenological models are compared with various models of production mechanisms in Sec.IV.

II. THE GENERATING FUNCTION

The generating function is defined by the equation

$$I(z) = \sum \sigma_n z^n / \sigma_{in}, \quad (1)$$

where σ_n is the topological cross section for n charged particles, $\sigma_{in} = \sum \sigma_n$ is the total inelastic cross section, and z may be interpreted in some Regge-type models as the square of the coupling constant. A model is said to exhibit KNO scaling if it is possible to define a KNO scaling function, ψ , so that

$$\langle n \rangle \sigma_n / \sigma_{in} = \psi(x) \quad (2)$$

is independent of energy for fixed $x = n/\langle n \rangle$. Any ψ must give the correct zeroth and first moments; a concrete test of KNO scaling properly focuses on whether the higher moments of $\psi(x)$ are independent of energy. A model thus exhibits KNO scaling if the coefficients

$$C_g = \langle n^g \rangle / \langle n \rangle^g \quad (3)$$

are energy independent.

A hypothesis for the form of $\psi(x)$ gives the form of the corresponding generating function $I(z)$.

$$I(z) = \sum \langle n \rangle^{-1} \psi(n/\langle n \rangle) z^n, \quad (4)$$

still satisfying $I(1) = 1$.

The binomial moments are obtained from

$$\langle n(n-1) \dots (n-l+1) \rangle = \left. \frac{\partial^l I(z)}{\partial z^l} \right|_{z=1}, \quad (5)$$

the topological cross sections σ_l recovered by

$$\frac{\sigma_l}{\sigma_{in}} = \frac{1}{l!} \left. \frac{\partial^l I(z)}{\partial z^l} \right|_{z=0} \quad (6)$$

The n particle correlation function f_k may be obtained from

$$\ln I(z) = \sum_{k=1}^{\infty} h^k f_k / k! \quad (7)$$

where $h = z - 1$. The correlation function f_k can be found from the binomial coefficients⁴ A_k :

$$f_k = k! \sum_{p=1}^k (-1)^{p+1} (p-1)! \sum_{a_1, a_2, \dots} \frac{A_1^{a_1} A_2^{a_2} \dots A_k^{a_k}}{(1!)^{a_1} a_1! (2!)^{a_2} a_2! \dots (k!)^{a_k} a_k!} \quad (8)$$

$$A_k = \langle n(n-1) \dots (n-k+1) \rangle ,$$

$$a_1 + 2a_2 + \dots + k a_k = k ,$$

$$a_1 + a_2 + \dots + a_k = p .$$

(9)

The coefficients C_q are given by

$$C_q = \left(z \frac{\partial}{\partial z} \right)^q I(z) \Big|_{z=1} / \left[\left(z \frac{\partial}{\partial z} \right) I(z) \Big|_{z=1} \right]^q .$$

(10)

III. SCALING FUNCTIONS

Slattery² has proposed a KNO scaling function to fit the high energy $p p$ collision data:

$$\psi(x) = \sum_{\substack{l=1 \\ l \text{ odd}}}^7 N_l x^l e^{-ax} .$$

(11)

Here $x = n/\langle n \rangle$; $a = 3.04$; and $N_1 = 3.79$, $N_3 = 33.7$, $N_5 = -6.64$, $N_7 = 0.332$.

The particular values of N_ℓ and the absence of even values of ℓ were determined by the data.

We are led, by Eqs. (2) and (11), to consider a model for σ_n of the form

$$\sigma_n = \sigma_{in} N_\ell n^\ell e^{-\beta n} / \langle n \rangle^{\ell+1}, \quad (12)$$

where β in (12) corresponds to $a/\langle n \rangle$ in (11), and ℓ is an arbitrary positive integer. The generating function is, from (1):

$$I(z) = N_\ell \langle n \rangle^{-(\ell+1)} \sum_n n^\ell (z e^{-\beta})^n = N_\ell \langle n \rangle^{-(\ell+1)} \left(-\frac{\partial}{\partial \beta} \right)^\ell (1 - z e^{-\beta})^{-1} \quad (13)$$

The condition $I(1) = 1$ determines N_ℓ so that

$$I(z) = \left(-\frac{\partial}{\partial \beta} \right)^\ell (1 - z e^{-\beta})^{-1} / \left(-\frac{\partial}{\partial \beta} \right)^\ell (1 - e^{-\beta})^{-1} \quad (14)$$

For example, for $\ell = 1$, (14) reads

$$I(z) = z(1 - e^{-\beta})^2 / (1 - z e^{-\beta})^2 \quad (15)$$

The various binomial moments can be obtained by insertion of (14) in (5),

$$\langle n(n-1)\dots(n-\ell+1) \rangle = \frac{\left(-\frac{\partial}{\partial \beta} \right)^\ell \prod_{j=1}^{\ell} \left(-\frac{\partial}{\partial \beta} + 1 - j \right) (1 - e^{-\beta})^{-1}}{\left(-\frac{\partial}{\partial \beta} \right)^\ell (1 - e^{-\beta})^{-1}} \quad (16)$$

The relation

$$(1 - e^{-\beta})^{-1} = \sum_{n=0}^{\infty} B_n (-1)^n \beta^{n-1} / n! , \quad (17)$$

where B_n are the Bernoulli numbers, facilitates the computation of the asymptotic correlation functions f_k from (16)', because $\beta \rightarrow 0$ as $\langle n \rangle \rightarrow \infty$.

Since $B_0 = 1$, (17) implies

$$\begin{aligned} \left(-\frac{\partial}{\partial \beta}\right)^k (1 - e^{-\beta})^{-1} &= \sum_{n=0}^{\infty} (-1)^{n+k} B_n \left(\frac{\partial}{\partial \beta}\right)^k \beta^{n-1} / n! \\ &= k! \beta^{-(k+1)} + \sum_{n=0}^{\infty} (-1)^{n+1} B_{n+k+1} \beta^n / [(n+k+1)n!] \\ &= k! \beta^{-(k+1)} + O(1) . \end{aligned} \quad (18)$$

The correlation function f_1 determines β :

$$f_1 = \langle n \rangle = \left(-\frac{\partial}{\partial \beta}\right)^{l+1} (1 - e^{-\beta})^{-1} / \left(-\frac{\partial}{\partial \beta}\right)^l (1 - e^{-\beta})^{-1} \simeq (l+1) / \beta . \quad (19)$$

It thus follows that

$$\beta \langle n \rangle = a \simeq l+1 . \quad (20)$$

The calculation of a can be carried out exactly. For $l = 1$, one obtains

$$a(l=1) = \langle n \rangle \ln \frac{\langle n \rangle + 1}{\langle n \rangle - 1} \xrightarrow{\langle n \rangle \rightarrow \infty} 2 + \frac{2}{3} \langle n \rangle^{-2}$$

(21)

In general, the corrections to (20) for arbitrary positive l are of order $\langle n \rangle^{-2}$.

We obtain the correlation functions f_k by use of Eqs. (16), (18) and (20),

$$\begin{aligned} f_2 &\equiv \langle n(n-1) \rangle - \langle n \rangle^2 \\ &= \frac{(l+2)(l+1)}{\beta^2} - \frac{(l+1)^2}{\beta^2} - \frac{l+1}{\beta} + O(\beta^l) \\ &= \frac{\langle n \rangle^2}{l+1} - \langle n \rangle + O(1), \end{aligned} \quad (22)$$

$$\begin{aligned} f_3 &\equiv \langle n(n-1)(n-2) \rangle - 3\langle n \rangle \langle n(n-1) \rangle + 2\langle n \rangle^3 \\ &= \frac{(l+3)(l+2)(l+1)}{\beta^3} - \frac{3(l+2)(l+1)^2}{\beta^3} + \frac{2(l+1)^3}{\beta^3} \\ &\quad + \frac{3(l+1)^2}{\beta^2} - \frac{3(l+2)(l+1)}{\beta^2} + \frac{2(l+1)}{\beta} + O(\beta^{l-2}), \\ &= \frac{2\langle n \rangle^3}{(l+1)^2} - \frac{3\langle n \rangle^2}{(l+1)} + O(\langle n \rangle). \end{aligned} \quad (23)$$

In general, one can show, assuming $\langle n \rangle \sim Y$, that

$$f_k = \frac{(k-1)! \langle n \rangle^k}{(l+1)^k} + O(\langle n \rangle^{k-1}) \sim Y^k, \quad (24)$$

where $Y = \ln(s/s_0)$ and s is the square of the center of mass

energy.⁵

In similar fashion, one finds with the aid of (3),

(16) and (18)

$$C_8 \simeq (l+8)! / [l! (l+1)^8] \quad (25)$$

For $l = 1$, the exact answer for C_2 is for comparison,

$$C_2 = 3/2 - \frac{1}{2} \langle n \rangle^{-2}.$$

Topological cross sections may be recovered from (6) and (14).

Another possible KNO scaling function $\psi(x)$ was used by Weisberg³ to fit the pp data:

$$\psi(x) = N_m x^m e^{-\alpha x^2} \quad (26)$$

where m and α are parameters and $x = n/\langle n \rangle$. Equation (2) implies

$$I(z) = N_m \langle n \rangle^{-(m+1)} \sum_{\text{even } n} n^m z^n e^{-\alpha n^2 / \langle n \rangle^2} \quad (27)$$

Since it is not possible to sum series (27) exactly, the sum is converted to an integral.⁶ The condition $I(1) = 1$ and the requirement that $\langle n \rangle$ be reproduced determines

$$N_m = 4 \alpha^{(m+1)/2} / \Gamma[(m+1)/2]$$

$$\alpha^{1/2} = \Gamma[(m+2)/2] / \Gamma[(m+1)/2]$$

(28)

leaving but one adjustable parameter, m , to determine the fit to the experimental data. Then

$$\begin{aligned} I(z) &= \exp \left[\langle n \rangle^2 (\ln z)^2 / 8\alpha \right] U(m + \frac{1}{2}, -\langle n \rangle \ln z / (2\alpha^{\frac{1}{2}})) / U(m + \frac{1}{2}, 0) \\ &= \Phi \left[(m + \frac{1}{2}), \frac{1}{2}; (\langle n \rangle \ln z / 2\alpha^{\frac{1}{2}})^2 \right] \\ &\quad + \langle n \rangle \ln z \Phi \left[(m + \frac{3}{2}), \frac{3}{2}; (\langle n \rangle \ln z / 2\alpha^{\frac{1}{2}})^2 \right], \end{aligned} \quad (29)$$

where U is a parabolic cylinder function,⁷ and Φ is the confluent hypergeometric function.⁸ Using Eq. (26) in Eq. (3), the C_q are

$$C_q = \left\{ \Gamma[(m+1)/2] \right\}^{q-1} \Gamma[(m+q+1)/2] / \left\{ \Gamma[(m+2)/2] \right\}^q. \quad (30)$$

The asymptotic values of the correlation function may be obtained by use of Eqs. (3) and (30),

$$f_k \sim \langle n \rangle^k \sum_{p=1}^k (-1)^{p+1} (p-1)! \sum_{a_1, a_2, \dots} \frac{k! C_1^{a_1} C_2^{a_2} \dots C_k^{a_k}}{(1!)^{a_1} a_1! (2!)^{a_2} a_2! \dots (k!)^{a_k} a_k!} \quad (31)$$

The first five f_k are

$$f_1 = \langle n \rangle$$

$$f_2 = (C_2 - 1) \langle n \rangle^2 + O(\langle n \rangle)$$

$$f_3 = (C_3 - 3C_2 + 2) \langle n \rangle^3 + O(\langle n \rangle^2)$$

$$f_4 = (C_4 - 4C_3 - 3C_2^2 + 12C_2 - 6) \langle n \rangle^4 + O(\langle n \rangle^3)$$

$$f_5 = (C_5 - 5C_4 - 10C_2C_3 + 20C_3 - 60C_2 + 30C_2^2 + 24) \langle n \rangle^5 \\ + O(\langle n \rangle^4)$$

(32)

Again, it is clear that in leading order $f_k \sim Y^k$. We explicitly verified, with the aid of (3), that the coefficients of $\langle n \rangle^k$ of (32) do not vanish, for $1 \leq m \leq 3.5$, which range covers the value $m = 1.215$ found by Weisberg.³

IV. MODELS FOR PRODUCTION MECHANISMS AND KNO SCALING

The apparent success of fits to (2) for pp cross sections has produced papers in which KNO scaling is predicted on the basis of physical arguments. These, basically, fall into two classes: the geometric, semiclassical approach exemplified by Barshay,⁹ Moreno,¹⁰ Le Bellac, et al.,¹⁰ and Neilsen and Olesen¹¹; and the explanations based on Regge-type procedures such as given by Suranyi,¹² and Naito and Odaka.¹²

The geometric approach draws heavily on the Chou-Yang¹³ idea of scattering in terms of the overlap of two matter density functions. The dynamics enters in the specific assumptions about parton-parton collisions ("parton" is taken to mean hadron constituent here) or the matter distribution (opaqueness) which are required to parametrize the number of particles produced. In all these models, the elastic scattering is shadow scattering. In Ref.9, one assumes a gaussian for $\langle n(b) \rangle \propto [1 - S(b)]$ and $\langle n(b)^q \rangle = \langle n(b) \rangle^q$, where $\langle n(b) \rangle$ and $S(b)$ are the average number of pions produced and S-matrix, respectively, both for impact parameters b . This leads to C_q values that agree with the data.²

In the Nielsen-Olesen¹¹ model, hadrons are thought to be very dilute parton clouds, so that the probability of any particular parton-parton collision is independent of any other collision. The probability that there are N parton-parton collisions is a Poisson distribution

$$P_N = C^N e^{-C} / N!$$

Each parton-parton collision produces n_1 particles, and the sole parameter c is determined by the ratio σ_{el}/σ_{in} of the elastic to the inelastic pp cross-sections. This results in reasonable C_q values for $q \leq 5$. The constancy of σ_{el}/σ_{in} leads to KNO scaling. The generating function

$$I(z) = (e^{cz^{h_1}} - 1) / (e^c - 1) \quad (34)$$

enables one to construct all interesting quantities as in Sec.II. In either of these geometrical models, the energy dependence of $\langle n \rangle$ is undetermined.

The Regge approach of Suranyi¹² derives KNO scaling from the exchange of an arbitrary number of fixed Pomeron poles. In this model, $\langle n \rangle \sim Y$. The approach of Naito and Odaka¹² derives KNO by exchanging a non-Pomeron trajectory between two novas. Their result is identical to Eq.(12) with $\ell = 3$ and $\beta = 2.2 s^{-\frac{1}{4}}$. The use of Eq.(20) leads to $\langle n \rangle \sim s^{\frac{1}{4}}$ as $s \rightarrow \infty$. It appears that the idea of KNO scaling is as yet independent of the asymptotic behavior of $\langle n \rangle$.

The models discussed in Sec.III have the property that the two particle correlation function $f_2 \sim Y^2$, assuming $\langle n \rangle \sim Y$. We now compare the consequences of this result with various production models that do not have KNO scaling.

For the multiperipheral model with a factorized Pomeron exchange and other short-range correlation models,¹⁴ one has $f_2 \sim Y$, but in more general multiperipheral models, $f_2 \sim Y^2$ is possible.¹⁵ The fragmentation type models,¹⁶ diffractive excitation model¹⁷ and Nova model¹⁸ have $f_2 \sim s^{\frac{1}{2}}$.

The hydrodynamic model¹⁹ has $f_2 \sim s^{\frac{1}{4}}$. In the case of the thermodynamic model,²⁰ $f_2 \sim Y^2$ or $f_2 \approx s^{W/2} + C Y^2$ ($0 < W \leq 1$) depending on the possible choices for the energy and rapidity of the fireball mass.

It therefore appears that if the results from the phenomenological fits to the data $\langle n \rangle \sim Y$ and $f_2 \sim Y^2$ holds at higher energies, the general multiperipheral and thermodynamic models, both of which have sufficient freedom, are consistent with the data.²¹

REFERENCES

1. Z. Koba, H. B. Nielsen and P. Olesen, Nucl. Phys. B40, 317 (1972).
2. P. Slattery, Phys. Rev. Letters 29, 1624 (1972); Phys. Rev. D 7, 2073 (1973).
3. H. Weisberg, University of Pennsylvania Report No. UR-409 1972 (to be published).
4. David M. Scott, K. Tanaka and R. Torgerson, Phys. Rev., in press.
5. One can also obtain Eq. (24) by combining $A_k \approx \langle n^k \rangle \approx C_k \langle n \rangle^k$, (8), and (25);

$$f_k \approx \frac{k!}{(\ell+1)^k} \langle n \rangle^k \sum_{p=1}^k (-1)^{p+1} \frac{1}{(p-1)!} \sum_{a_1, a_2, \dots, a_p} \prod_{m=1}^p \left(\frac{(\ell+m)!}{\ell! m!} \right)^{a_m} \frac{1}{a_m!}$$

$$\approx \frac{(\frac{1}{2}k-1)!}{(\ell+1)^{k-1}} \langle n \rangle^k \sim \gamma^k$$

In order to obtain the secondary terms in addition to the leading terms, the method in the text is preferable.

6. It was found in Ref.4 that the integral technique introduces an error which is inconsequential near $z=1$.
7. M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions. The parabolic cylinder functions are found on pg. 687 ff.
8. Ref.7, pg. 555 ff.
9. S. Barshay, Phys. Lett. 42B, 457 (1972).
10. H. Moreno, SLAC Report No. SLAC-PUB-1190; M. Le Bellac, et al., Univ. of Nice Report No. N TH 73/3, 1973.

11. H. B. Nielsen and P. Olesen, Phys. Letters 43B, 37 (1973).
12. P. Suranyi, University of Cincinnati preprints (unpublished).
S. Naito and T. Odaka, Osaka City University preprint (unpublished).
13. T. T. Chou and C. N. Yang, Phys. Rev. 170, 1591 (1968).
14. K. G. Wilson, Cornell University Report No. CLNS-131, 1970;
A. Mueller, Phys. Rev. D 4, 150 (1971).
15. W. R. Frazer, et al., Rev. Mod. Phys. 44, 284 (1972); J. Ellis,
J. Finkelstein, and R. D. Peccei, SLAC Report No. SLAC-PUB-1020, 1972;
R. C. Arnold, Argonne Report ANL/HEP 7242, 1972.
16. J. Benecke, et al., Phys. Rev. 188, 2159 (1969).
17. R. Hwa, Phys. Rev. Letters 26, 1143 (1971).
18. M. Jacob and R. Slansky, Phys. Rev. D 5, 1847 (1972); E. L. Berger,
B. Y. Oh and G. A. Smith, Phys. Rev. Lett. 29, 675 (1972).
19. P. Carruthers, Cornell University Report No. CLNS-219, 1973.
20. J. Ranft and G. Ranft, CERN Report No. TH 1544, 1972.
21. The $\langle n \rangle \sim Y$ is not well established. The behavior $\langle n \rangle \sim s^{\frac{1}{4}}$ also seems to
fit the data.