THE NONPLANAR DUAL LOOP
IN THE VARIABLES s, t, u .⁺

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ABSTRACT

Starting from the Lovelace form a closed expression is derived for the four particle nonplanar dual loop amplitude in terms of the Mandelstam variables s , t and u . The knowledge of this form is essential for the evaluation of the asymptotic and analytic properties in these variables.

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Introduction

Ever since the nonplanar dual loop amplitude had been given by GNSS¹ one has tried and failed to evaluate its asymptotic behaviour in the Mandelstam variables s , t and u . The main cliff was that s appears in $\exp(-s f(x_i))$ along with a factor which changes sign in the integration region of the x_i . Although Kikkawa² has given a procedure how to tackle this problem it cannot be applied here since $f(x_i)$ is considerably more complicated than Kikkawa's expression.

For the same reason one hasn't yet been able to evaluate the analytic properties of this loop amplitude which one needs in the estimate of inclusive reactions from dual models³. One assumes that the diffractive part of the reactions is supplied by this loop expression.

As the first and crucial step towards the solution of these two problems we show in the present paper that the Lovelace form^{4,5} of the loop amplitude allows one to derive a neat and compact expression also in the variables s , t and u . In fact, we again find Abelian integrals, though with a different "basis" (see later).

Non planar dual loop amplitude

The shape of the corresponding duality and Feynman like diagram is given in fig. 1. It corresponds to a pomeron in the u-channel. According to Lovelace⁵, Alessandrini⁶ and Amati⁷ we can construct it from a 6 reggeon vertex (fig. 2)

$$V_6 = \int \prod_{i=1}^6 \frac{dz_i}{z_{i+1}-z_i} \frac{(z_a-z_b)(z_b-z_c)(z_c-z_a)}{dz_a dz_b dz_c} \left| \frac{z_i-z_{i+2}}{z_i-z_{i+1}} \right|^{\alpha_0} \exp \left(-\frac{1}{2} \langle \alpha_i | U_i V_j | \alpha_j \rangle \right) \quad (1)$$

by welding a twisted propagator⁸ between the legs q_2 and q_5 which evidently yields the loop of fig. 1a. Its mathematical representation is (with the same integration measure as in (1))

$$F(s,t) = \int_0^1 dx x^{-\alpha_0-1} (1-x)^{\alpha_0-1} \int d^3 \tau \prod_{i=1}^6 \left| \frac{z_i-z_{i+2}}{z_i-z_{i+1}} \right|^{\alpha_0} \prod \frac{(z_k-z_{k+1})(z_k-z_{k-1})}{z_k(z_{k-1}-z_{k+1})} \frac{1}{(\text{Re } B)^2 \prod_{n=1}^{\infty} (1-K^n)^4} \quad (2)$$

$$\exp \left[-\frac{1}{2} \sum_{i,j \in N'} q_i q_j V(z_i, z_j) \right]$$

where N' are the labels 1, 3, 4 and 6 of the unsewn legs q_i and

$$V(z_i, z_j) = \text{Re} \Omega_{z_i z_0}(z_j) - \text{Re} \Omega_{z_i z_0}(z'_0) - \log \left| 1 - \frac{z_i}{z_j} \right| \delta_{ij} \quad (3)$$

$\Omega_{z_i z_0}(z_j)$ is the 3rd Abelian integral in the real normalization which is defined in terms of that with complex normalization,

$\omega_{z_i z_0}(z_j)$ and in terms of the first Abelian integral $\phi(z)$ as⁹

$$\Omega_{z_1 z_0}(z_j) = \omega_{z_1 z_0}(z_j) + 2 \operatorname{Re}[\phi(z_1) - \phi(z_0)] \frac{1}{\operatorname{Re} B} \phi(z_j). \quad (4)$$

Here and in (2) B is the period matrix which is defined by

$$B = \phi(S(z)) - \phi(z) \quad (5)$$

where

$$S(z) = \frac{\alpha z + \beta}{\gamma z + \delta} \quad (6)$$

is the characteristic Mobius transformation for the nonplanar loop; K (of eq. 2) is its multiplier. The expressions for ω and ϕ , given by Burnside¹⁰, read in terms of the cross ratios

$$[x, y/z, u] \equiv \frac{x-y}{x-z} \frac{z-u}{y-u} \quad (7)$$

$$\phi(z) = -\frac{1}{2} \log \prod_{\rho=-\infty}^{+\infty} (S^\rho(z) - \frac{\alpha}{\gamma}) / (S^\rho(c) - \frac{\alpha}{\gamma}) \quad (8)$$

$$\omega_{z_1 z_0}(z) = \log \prod_{\rho=-\infty}^{+\infty} [S^\rho(z), a_1/z_0, S^\rho(c)] \quad (9)$$

$$B = -\frac{1}{2} \log \prod_{\rho=-\infty}^{+\infty} [S^\rho(b), a/S^{-1}(a), S^\rho(S^{-1}(b))] \quad (10)$$

where B is independent of a and b and $S^0(z) = z$.

By definition ϕ and ω are integrals¹¹ over harmonic functions in a certain domain (or Riemann surface) with z as upper and c as lower boundary. We shall call c the "basis" of the Abelian integrals.

We shall now proceed to introduce the Mandelstam variables in the exponential part of eq. (2)

$$\exp \left(- \frac{1}{2} \sum_{i,j \in N'} q_i q_j V(z_i, z_j) \right) \equiv e^{E/2}. \quad (11)$$

Let us first rename $q_1 q_3 q_4 q_6$ respectively as $p_1 p_4 p_2 p_4$ as indicated in fig. 1. Then s , t and u are introduced by

$$\begin{aligned} p_1 p_2 &= p_3 p_4 = \frac{s}{2} - m^2 \\ p_1 p_4 &= p_2 p_3 = \frac{t}{2} - m^2 \\ p_1 p_3 &= p_2 p_4 = \frac{u}{2} - m^2. \end{aligned} \quad (12)$$

We then find

$$E = - \frac{s}{2} f(z_i) - \frac{t}{2} g(z_i) - 2\alpha_0 h(z_i) \quad (13)$$

or alternatively

$$E = - \frac{s}{2} \bar{f}(z_i) - \frac{u}{2} \bar{g}(z_i) - 2\alpha_0 \bar{h}(z_i) \quad (14)$$

where

$$\begin{aligned} f &= N_{14} + N_{36} - N_{13} - N_{46} \\ g &= N_{16} + N_{34} - N_{13} - N_{46} \\ h &= -f - g - N_{13} - N_{46} + N_{11} + N_{33} + N_{44} + N_{66} \end{aligned} \quad (15)$$

and

$$\begin{aligned} N_{ij} &= V(z_i, z_j) + V(z_j, z_i) \text{ for } i \neq j \\ &= V(z_i, z_i) \text{ for } i = j. \end{aligned} \quad (16)$$

f , g and h are found from (15) by exchanging $3 \leftrightarrow 6$.

Let us now introduce the third Abelian integral Ω from (3) into (16), which consists of an ω and a logarithmic term. Treating all ω -terms first in (17) we get with ω_{io} for $\omega_{z_i z_0}$

$$f_{\omega} = \omega_{10}(z_4) + \omega_{40}(z_1) + \omega_{30}(z_6) + \omega_{60}(z_3) - \omega_{10}(z_6) - \omega_{40}(z_3) - \omega_{30}(z_4) - \omega_{60}(z_1). \quad (17)$$

Using the relation¹⁰

$$\omega_{io}(z_a) - \omega_{io}(z_b) = \omega_{ab}(z_i) - \omega_{ab}(z_o) \quad (18)$$

we can also write

$$f_{\omega} = \omega_{46}(z_1) + \omega_{13}(z_4) + \omega_{64}(z_3) + \omega_{31}(z_6) - f_o \quad (19)$$

where for f_o all arguments in the first four terms are to be replaced by z_o .

With the representation (9) for the ω 's we see that in (19) all factors of the infinite products containing $S^p(c)$ cancel out and $f_o = \log 1 = 0$. The remainder can be written as

$$f_{\omega} = \log \pi[S^p(z_1), z_4/z_6, S^p(z_3)] + \log \pi[z_1, S^p(z_4)/S^p(z_6), z_3] \quad (20)$$

which after making use of the group properties of the S^p reads

$$f_{\omega} = 2 \log \Pi[S^{\rho}(z_1), z_4/z_6, S^{\rho}(z_3)]. \quad (21)$$

This can be seen, from (9), to be again a third Abelian integral, though with respect to a different basis z_3

$$f_{\omega} = 2 \omega_{z_4 z_6}(z_1; z_3). \quad (22)$$

As for the second part of the Ω 's we note that $\text{Re } B$ is the same everywhere and that the terms containing $\Phi(z_0)$ cancel away. We therefore get

$$f_2 = 4 (\text{Re}\Phi(z_1) - \text{Re}\Phi(z_3)) (\text{Re}\Phi(z_4) - \text{Re}\Phi(z_6)). \quad (23)$$

In these differences factors containing $S^{\rho}(c)$, from the representation (8) of Φ , will cancel leaving us with

$$f_2 = 2 \cdot 2 \text{Re} (\phi(z_4; z_3) - \phi(z_6; z_3)) \text{Re}\phi(z_1; z_3) / \text{Re } B \quad (24)$$

where we have again defined the first Abelian integrals with respect to the basis z_3 . The sum of both terms is therefore a third Abelian integral in the real normalization and with the basis z_3 :

$$f(z_i) = 2 \text{Re } \omega_{z_4 z_6}(z_i; z_3). \quad (25)$$

$g(z_i)$ can be treated equivalently and in $h(z_i)$ we collect all terms with m^2 arising from the introduction of (12) as well as from those terms of E which have $i = j$. The result of these manipulations is

$$\begin{aligned}
 E &= - \sum_{i,j \in N'} q_i q_j V(z_i, z_j) \\
 &= - (s-2m^2) \text{Re } \Omega_{z_4 z_6}(z_1; z_3) \\
 &\quad - (t-2m^2) \text{Re } \Omega_{z_3 z_6}(z_1; z_4) \\
 &\quad - m^2 [\text{Re } \Omega'_{z_1 z_6}(z_1; z_6) + \text{Re } \Omega'_{z_3 z_4}(z_3; z_4)].
 \end{aligned} \tag{26}$$

where Ω' means that the vanishing factors in the infinite products are to be omitted due to the third term of eq. (3).

A similar expression holds in terms of the variables s and u . All we need is to replace t by u in (26) and interchange $z_3 \leftrightarrow z_6$.

We have therefore been able to write down the diffractive dual loop amplitude in a closed and neat form in terms of the Mandelstam variables s , t and u as well. We are now prepared to attack the problem of finding the asymptotic and analytic properties of the amplitude in these variables. This we defer, however, to a further publication.

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Figure Captions

Fig. 1: a) Quark diagram for orientable nonplanar dual loop
b) Corresponding Feynman-like diagram

Fig. 2: Quark diagram for 6-reggeon vertex before sewing.

(a)

(b)

Fig.1

Fig.2