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NUCLEAR STRUCTURE CALCULATIONS

by

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NUCLEAR STRUCTURE CALCULATIONS[†]

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Abstract

The structure of the shell-model wave functions for low-lying states of even-even nuclei is examined and it is shown why one might hope to reproduce energies and some transition rates using the BCS and two-quasiparticle approximations to describe single-closed-shell nuclei. It is demonstrated, by comparison with exact calculations on the nickel isotopes, that if the physical quantity one wants to calculate is not symmetric about $\bar{N}$ (where $\bar{N}$ is the number of nucleons in the physical system one wants to describe) large errors result when the BCS and two-quasiparticle wave functions are used. To remedy this deficiency, it is necessary to project out of these wave functions the part describing a system with $\bar{N}$ particles, normalize this part, and then use the normalized projected eigenfunction to calculate the physical quantity of interest. It is also shown that if one does only a two-quasiparticle calculation for the low-lying excited states, one can only reliably predict the $B(E2)$ between the ground state and first 2^+ nuclear level.

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If one attempts to make nuclear structure calculations on the basis of the nuclear shell model, one rapidly becomes aware of the fact that even when all the approximations that go into the shell model are made the problem is still so large that it is beyond the bounds of present-day computers. Thus one must look for approximate ways of making nuclear structure studies — and the BCS wave function, which was originally introduced as a calculational tool in the theory of superconductivity, provides one way of approximating the nuclear problem. In this paper I would first like to illustrate the particular qualities of nuclear states that make them amenable to the BCS approximation and second to discuss to what accuracy the nuclear BCS calculations reproduce the results of certain exact calculations that have been made.

The exact calculations that I would like to discuss are those concerning the nickel isotopes.⁽¹⁾ The nickel isotopes are nuclei with 28 protons and anywhere from 28 to 38 neutrons. According to the shell model, the 28 protons and first 28 neutrons form an inert closed shell — that is, they provide the average shell-model potential in which

Fig. 1. Observed energy levels below 3 MeV in ${}^{57}\text{Ni}$.

be taken into account in discussing the structure of the low-lying states of the nickel nuclei. Thus the model we will take is one in which the $N = Z = 28$ core is inert and neutrons are filling the $2p_{3/2}$, $1f_{5/2}$, and $2p_{1/2}$ single-particle levels whose experimental energies are taken to be 0, 0.78 MeV, and 1.08 MeV, respectively.⁽²⁾ At about 2.5 MeV in $^{57}_{28}\text{Ni}$, a state with spin $\frac{7}{2}$ is seen in pickup experiments.⁽³⁾ This is interpreted as corresponding to an excitation out of the closed $f_{7/2}$ shell and immediately tells us that our simple model, in which we assume the core is inert, should not be taken seriously for states above about 2.5 MeV excitation in odd-A nuclei.

Using these single-particle basis states, one must now diagonalize the nuclear Hamiltonian

$$H = H_0 + V, \quad (1)$$

where H_0 is the independent-particle Hamiltonian which should give rise to single-particle states with energies similar to those observed in $^{57}_{28}\text{Ni}$, and V is the residual two-body interaction between the valence nucleons. In view of the violent truncation that has been made, it is not likely that V can be taken as the free nucleon-nucleon interaction. An attempt to derive the effective interaction needed in the truncated ($2p_{3/2}$, $1f_{5/2}$, $2p_{1/2}$) configuration space from the free nucleon-nucleon interaction led to results qualitatively similar to those presented here.⁽⁴⁾ However, the actual calculations I will discuss at this time are those in which the nuclear wave functions were taken to be those of a harmonic oscillator and V was assumed to have a simple Gaussian dependence with a central, a tensor, and a two-body spin-orbit part. To obtain agreement with the experimental level scheme, the interaction energies when the two interacting neutrons were in relative s states had to be taken as parameters also.⁽¹⁾ The range of the interaction was fixed to be similar to that of the free nucleon-nucleon force and then the eight remaining parameters characterizing the potential were varied until a best fit to the data on all the nickel isotopes was obtained. Approximately 40 energy levels were

Fig. 2. Comparison of experimental and theoretical energy levels in ^{58}Ni .

fitted in this way. A typical example of the quality of agreement between theory and experiment is shown in Fig. 2 for ^{58}Ni .

Let us now turn our attention to the structure of the wave functions that describe these nuclear states. It is important to note that the lowest state of the system is a level with angular momentum $J = 0$. This corresponds to the case in which the spins of the two nucleons are paired off. Such a state, in which there are no unpaired particles, is said to be a seniority-zero state* and can be pictorially represented as

$$\uparrow \downarrow \quad J = 0, \quad v = 0.$$

This state lies approximately 1.5 MeV below all other states in the ^{58}Ni nucleus. Thus when the nuclear spins pair off, one gets a bonus of about 1.5 MeV. The other states are those in which the spins do not pair off, that is, do not have $J = 0$. These are called seniority-two states, corresponding to the fact that they contain two unpaired nucleons. Again pictorially they can be represented as

$$\uparrow \uparrow \quad J \neq 0, \quad v = 2.$$

If we now turn to the multiparticle system with an even number of particles, it is logical to expect that the lowest state of the nucleus will be the one in which all possible spins are paired off.

Pictorially one would expect the lowest state to be of the form

$$\uparrow \downarrow \uparrow \downarrow \cdots \uparrow \downarrow \quad J = 0, \quad v = 0.$$

* Throughout this discussion we shall define the seniority of the multi-level system as the sum of the seniorities of the individual configurations. That is, if the seniorities of the three configurations $(f_{5/2})^{n_1}$, $(p_{3/2})^{n_2}$, and $(p_{1/2})^{n_3}$ are v_1 , v_2 , and v_3 , respectively, then the seniority of the N -particle basis state $(f_{5/2})^{n_1}(p_{3/2})^{n_2}(p_{1/2})^{n_3}$, where $N = n_1 + n_2 + n_3$, is $v = v_1 + v_2 + v_3$.

The low-lying states in such a nucleus should be those in which only one pair is broken—that is, the seniority-two levels. Diagrammatically these would be represented by

$$\uparrow \uparrow \uparrow \downarrow \cdot \cdot \cdot \uparrow \downarrow \qquad J \neq 0, \quad v = 2,$$

and these should lie at excitation energies between 1.5 and 2.5 MeV. In a similar manner, the next group of states should be the seniority-four levels corresponding to four unpaired particles represented by

$$\uparrow \uparrow \uparrow \uparrow \uparrow \downarrow \cdot \cdot \cdot \uparrow \downarrow \qquad J, \quad v = 4$$

and these should describe states whose excitation energies are of the order of 3 MeV or greater. Thus one would expect that states below about 2.5 MeV in an even-even nucleus should have wave functions that are dominantly either seniority-zero or seniority-two. Table I shows the seniority decomposition of the low-lying states in the 6-particle nucleus ${}^{62}_{28}\text{Ni}$, and from this table it is apparent that the above surmise is true for the states of even angular momentum.

In view of this one would expect that if one made a seniority truncation and considered only the states of lowest seniority in carrying out the shell-model calculation for even- J states, the energy predictions emerging from this should be close to those that arise from the exact diagonalization. That this is so is illustrated in Table II, where the results of the two calculations are compared.⁽⁵⁾

Thus if one wants to describe even-angular-momentum states with excitation energies below about 3 MeV, a suitable way of cutting down the size of the energy matrices that must be constructed and diagonalized is to use only seniority-zero eigenfunctions for the low-lying $J = 0$ states and only seniority-two wave functions for the remaining even- J states. In the case of ${}^{62}_{28}\text{Ni}$ the saving is quite considerable. For example, there are 14 possible $J = 0$ states, only 6 of which have seniority

TABLE I. Seniority decomposition of the low-lying states in $^{62}_{28}\text{Ni}$. The lowest state of given spin has a subscript 1 and the second state of that spin has a subscript 2. The lowest possible seniority for the various states is either zero or two and is listed in column 2. The percentage of the state with this seniority is listed in column 3.

$^{62}_{28}\text{Ni}$ state	Lowest seniority	Percentage
0_1^+	0	99.7
0_2^+	0	87.3
2_1^+	2	99.5
2_2^+	2	89.0
4_1^+	2	92.9
1_1^+	2	23.7
3_1^+	2	40.6

zero. In the case of $J = 2$ there are 33 states that can be constructed from the entire $(2p_{3/2}, 1f_{5/2}, 2p_{1/2})$ configuration whereas there are only 15 states if we limit ourselves to seniority two. The saving is, therefore, appreciable—but still not enough. To illustrate the inadequacy, consider the case of $^{116}_{50}\text{Sn}$. To describe this nucleus we would have to consider 16 neutrons outside the closed $N = Z = 50$ shell and these neutrons would have to be distributed over the five single-particle levels $2d_{5/2}$, $1g_{7/2}$, $1h_{11/2}$, $2d_{3/2}$, and $3s_{1/2}$. Even if we consider only the lowest seniority, there are 110 $J = 0$ states and approximately 1000 $J = 2$ basis states. Thus even

though the seniority truncation provides an appreciable saving, it is not enough and hence additional approximations have to be introduced.

TABLE II. Energies of the various states in ${}^{62}_{28}\text{Ni}$ obtained when all states of the $(2p_{3/2}, 1f_{5/2}, 2p_{1/2})$ configuration are used and those obtained when only seniority-zero eigenfunctions are used for the $J = 0$ states and seniority-two wave functions are used for the $J \neq 0$ states. The tabulated ground-state energy is the total interaction energy of the six extra-core nucleons, whereas the other energies tabulated are excitation energies relative to the ground state.

J	Exact calculation (MeV)	Lowest seniority only (MeV)
0_1^+	(0.448)	(0.463)
2_1^+	1.533	1.534
0_2^+	2.013	2.130
4_1^+	2.201	2.278
2_2^+	2.253	2.358

To see how the BCS approximation can be used in nuclear structure calculations,⁽⁶⁾ let us define the Fermion creation operator a_{jm}^+ and the destruction operator a_{jm} which respectively create and destroy a particle with angular momentum j and z -component of angular momentum m . Since these are Fermion operators, they obey the anticommutation relationship

$$\{a_{jm}^+, a_{j'm'}\} = \delta_{jj'} \delta_{mm'}.$$

In terms of these operators, the $J = 0$ seniority-zero state of two particles can easily be written as

$$\uparrow \downarrow \quad J = 0, \quad v = 0 \implies \sum_{m>0} (-1)^{j-m} a_{jm}^+ a_{j-m}^+ |0\rangle,$$

where the phase factor $(-1)^{j-m}$ is proportional to the Clebsch-Gordan coefficient that ensures that the two particles couple their angular momentum to zero, and $|0\rangle$ is the vacuum. For the N -particle system, the seniority-zero state is simply written as

$$\uparrow \downarrow \uparrow \downarrow \cdots \uparrow \downarrow \implies \left(\sum_{m>0} (-1)^{j-m} a_{jm}^+ a_{j-m}^+ \right)^{N/2} |0\rangle.$$

One can now easily see that the BCS wave function

$$\psi = \prod_{j, m>0} (u_j + (-1)^{j-m} v_j a_{jm}^+ a_{j-m}^+) |0\rangle \quad (2)$$

describes a system in which pairs of particles always couple their spins to zero. For example, if we consider the simple case in which the only j of interest is $j = \frac{3}{2}$, the BCS eigenfunction can be expanded and written as

$$\begin{aligned} \psi &= (u + v a_{3/2 \ 3/2}^+ a_{3/2 \ -3/2}^+) (u - v a_{3/2 \ 1/2}^+ a_{3/2 \ -1/2}^+) |0\rangle \\ &= \{u^2 + uv(a_{3/2 \ 3/2}^+ a_{3/2 \ -3/2}^+ - a_{3/2 \ 1/2}^+ a_{3/2 \ -1/2}^+) \\ &\quad - v^2 a_{3/2 \ 3/2}^+ a_{3/2 \ -3/2}^+ a_{3/2 \ 1/2}^+ a_{3/2 \ -1/2}^+\} |0\rangle \\ &= \{u^2 + uv \sum_{m>0} (-1)^{3/2-m} a_{3/2 \ m}^+ a_{3/2 \ -m}^+ \\ &\quad + \frac{v^2}{2} \left(\sum_{m>0} (-1)^{3/2-m} a_{3/2 \ m}^+ a_{3/2 \ -m}^+ \right)^2\} |0\rangle. \quad (3) \end{aligned}$$

Inspection of Eq. (3) shows that each term in the expanded wave function describes zero-coupled pairs and consequently the BCS wave function is a linear combination of seniority-zero eigenfunctions. This wave function is automatically normalized provided that

$$u_j^2 + v_j^2 = 1 \quad (4)$$

and v_j^2 represents the probability that the state (jm) is occupied.

The BCS wave function obviously does not describe a system with a fixed number of particles. For the simple case described by Eq. (3) there is a finite probability u^4 that the wave function describes a state with no particles, a probability $2u^2v^2$ that there are two particles, and a probability v^4 that the wave function corresponds to a system with four particles. Thus one cannot use ψ to minimize the Hamiltonian of Eq. (1) directly. One must minimize H subject to the constraint that the expectation value of the number operator

$$N = \sum_{jm} a_{jm}^+ a_{jm} \quad (5)$$

corresponds to the number $\bar{N}$ of particles that are actually in the physical system we wish to describe. That is, the quantity

$$\mathcal{H} = \langle \psi | H - \lambda N | \psi \rangle, \quad (6)$$

where λ is a Lagrange multiplier, is minimized as a function of the v_j . Therefore, the equations that replace the diagonalization of the shell-model energy matrix are

$$\frac{\partial \mathcal{H}}{\partial v_j} = 0 \quad (7)$$

and the constraint equation

$$\langle \psi | N | \psi \rangle = \langle \psi | \sum_{jm} a_{jm}^+ a_{jm} | \psi \rangle = \sum_j (2j + 1) v_j^2 = \bar{N}. \quad (8)$$

To illustrate the calculational simplicity that the BCS approximation introduces, consider the example of $^{116}_{50}\text{Sn}$ discussed previously. Equations (7) and (8) with $\bar{N} = 16$ provide six equations for the six unknowns (five v_j corresponding to the five single-particle levels used in the calculation, and the multiplier λ). These six equations replace the 110×110 energy matrix that would have to be constructed and diagonalized if all possible seniority-zero eigenfunctions were used. The BCS approximation attains this simplicity by imposing a relationship between the probabilities that the configuration $(j_1)^{n_1}(j_2)^{n_2} \dots (j_m)^{n_m}$ and the configuration $(j_1)^{n'_1}(j_2)^{n'_2} \dots (j_m)^{n'_m}$ occur in the wave function. Therefore, the BCS wave function provides a particular linear combination of seniority-zero wave functions that can be used to make the variational calculation.

As stated before, v_j^2 represents the probability that the state (jm) is occupied. Figure 3 is a plot of this quantity as a function of the single-particle energy. For states well within the Fermi sea, the state is definitely occupied and v_j^2 is certainly unity; for states far above the Fermi sea, there is zero probability of occupation; and for states

Fig. 3. The value of v_j^2 as a function of single-particle energy.

near the Fermi surface, the probability of occupation is somewhere between zero and one.

Since the BCS eigenfunction represents the lowest state of the system, one can consider it to be a vacuum state. Let us, therefore, look for an operator which when acting on the BCS wave function gives zero. If one finds such an

operator, one can say that the BCS state represents the vacuum state for entities created by this operator. It is apparent that applying

$$a_{jm} = u_j a_{jm} - (-1)^{j-m} v_j a_{j-m}^+ \quad (9)$$

to the BCS wave function gives zero. This follows because a_{jm} commutes with all terms in the product wave function except the term in Π in which $(jm) = (j_1 m_1)$, i. e.,

$$\begin{aligned} a_{jm} \psi &= \prod'_{j_1 m_1 > 0} (u_{j_1} + (-1)^{j_1 - m_1} v_{j_1} a_{j_1 m_1}^+ a_{j_1 - m_1}^+) \\ &\times (u_j a_{jm} - (-1)^{j-m} v_j a_{j-m}^+) (u_j + (-1)^{j-m} v_j a_{jm}^+ a_{j-m}^+) |0\rangle \\ &\equiv 0, \end{aligned} \quad (10)$$

where the prime on Π means that the term in the product in which $(j_1 m_1) = (jm)$ is left out. The Hermitian adjoint of a_{jm} is

$$a_{jm}^+ = u_j a_{jm}^+ - (-1)^{j-m} v_j a_{j-m}; \quad (11)$$

and since $u_j^2 + v_j^2 = 1$, it follows that

$$\{a_{jm}^+, a_{j'm'}\} = \delta_{jj'} \delta_{mm'} \quad (12)$$

Thus the operators a_{jm}^+ and a_{jm} act like Fermion creation and annihilation operators, respectively. The entities that they create and destroy are called quasi-particles and the BCS state is the state in which there are no quasi-particles present. If the level (jm) is far inside the Fermi sea, then it is apparent from Fig. 3 that $v_j^2 = 1$ and the quasi-particle creation

operator creates a hole in the sea. For (jm) considerably above the Fermi surface, $u_j^2 = 1$ and a_{jm}^+ produces an ordinary particle. Near the Fermi surface the quasi-particle is partly particle and partly hole.

If we now look at the form of the one-quasiparticle state, it is easy to show that it is identical to an ordinary seniority-one shell-model wave function. This follows because

$$\begin{aligned}
 a_{jm}^+ \psi &= \prod'_{j_1 m_1 > 0} (\quad) [(u_j a_{jm}^+ - (-1)^{j-m} v_j a_{j-m}) (u_j + (-1)^{j-m} v_j a_{jm}^+ a_{j-m}^+)] |0\rangle \\
 &= \prod'_{j_1 m_1} (\quad) (u_j^2 + v_j^2) a_{jm}^+ |0\rangle \\
 &= \prod'_{j_1 m_1} (\quad) \frac{1}{u_j} (u_j + (-1)^{j-m} v_j a_{jm}^+ a_{j-m}^+) a_{jm}^+ |0\rangle \\
 &= \frac{1}{u_j} \left[\prod_{j_1 m_1 > 0} (u_{j_1} + (-1)^{j_1-m_1} v_{j_1} a_{j_1 m_1}^+ a_{j_1-m_1}^+) \right] a_{jm}^+ |0\rangle, \tag{13}
 \end{aligned}$$

where the $(\quad)$ immediately after $\prod'$ stands for that part of the product BCS wave function in which $(j_1 m_1) \neq (jm)$. Thus the one-quasiparticle state is related to the situation in which all but one of the particles are paired to angular momentum zero; and this is precisely the definition of a seniority-one level. In a similar way, the two-quasiparticle states can be shown to be equivalent to seniority-two eigenfunctions, i. e.,

$$\begin{aligned}
 \varphi(J) &= (a_{j_1 m_1}^+ \times a_{j_2 m_2}^+)_{JM} \prod_{jm > 0} (u_j + v_j (-1)^{j-m} a_{jm}^+ a_{j-m}^+) |0\rangle \\
 &= \frac{1}{u_{j_1} u_{j_2}} \left[\prod_{jm > 0} (u_j + (-1)^{j-m} v_j a_{jm}^+ a_{j-m}^+) \right] (a_{j_1 m_1}^+ \times a_{j_2 m_2}^+)_{JM} |0\rangle, \tag{14}
 \end{aligned}$$

where $(\times)_{JM}$ stands for the coupling of the two particles or quasiparticles to angular momentum J with z -component M .

To cut down the size of the shell-model calculation, it is now clear what should be done. For even-even single-closed-shell nuclei, one should determine v_j and λ from the variational calculation described by Eqs. (6), (7), and (8). The ground-state energy of the nucleus can then be computed from

$$E_0 = \langle \psi | H | \psi \rangle. \quad (15)$$

The values of v_j and λ are thus fixed by this calculation. One then constructs all possible two-quasiparticle states ϕ of the form given by Eq. (14) and uses these as basis states in diagonalizing the shell-model Hamiltonian. The labor saved by this approximation is immense. For example, in $^{116}_{50}\text{Sn}$ we would have had to construct and diagonalize a matrix whose size was of the order of 1000×1000 if we did the seniority-two calculation for $J = 2$. On the other hand, the number of two-quasiparticle states with $J = 2$ and positive parity is only nine.

Although this approximation saves a large amount of labor, the ultimate test of its usefulness is whether or not it yields results in reasonable agreement with exact calculations. To this end we have listed (Table III) the results that the BCS and two-quasiparticle approximations give for the case of $^{62}_{28}\text{Ni}$. A comparison of columns 2 and 3 of this table immediately shows that the BCS and two-quasiparticle approximations are not valid for this case. The reason may be understood as follows. If we examine the probability distribution for the number of particles in the BCS wave function, we find that the maximum probability is for the state with $N = \bar{N}$ (where $\bar{N}$ is the number of particles in the state of physical interest). However, this part constitutes only about 40% of the wave function. The remaining 60% describes a system with $N \neq \bar{N}$. A plot of

TABLE III. Energies of the various states in ${}^{62}_{28}\text{Ni}$. The results of the exact calculation are given in column 2. The tabulated ground-state energy is the total interaction energy of the six extra-core neutrons. Other energies tabulated are excitation energies relative to the ground state. Column 3 gives the results of the BCS calculation and column 4 lists the results obtained by use of projected BCS wave functions as described by Eq. (16).

J	Exact energy (MeV)	BCS or two- quasiparticle energy (MeV)	Projected BCS or two-quasiparticle energy (MeV)
0_1	(0.448)	(1.932)	(0.567)
2_1	1.533	1.066	1.455
2_2	2.253	1.860	2.265
4_1	2.201	1.830	2.226

this distribution (Fig. 4) shows that the distribution is more or less symmetric about $N = \bar{N}$. Thus if we calculate the expectation value of any quantity that is symmetric about $\bar{N}$, the errors introduced by nonconservation

Fig. 4. Distribution of the number of particles in the BCS wave function.

of particle number in the BCS wave function will cancel out. In the case of the nickel isotopes, the curve for the binding energy of the extra-core nucleons as a function of the number of valence nucleons is as illustrated in Fig. 5. This curve is clearly not symmetric about $\bar{N} = 6$, the number of valence nucleons in ${}^{62}_{28}\text{Ni}$, and this explains why the approximation is so bad in this case.

However, if one projects out of ψ the part that describes $\bar{N}$ particles and normalizes this projected wave function, then the energy

$$E_{\text{proj}} = \langle P\psi | H | P\psi \rangle \quad (16)$$

agrees extremely well with the results of the exact calculation. In the above equation, the notation $P\psi$ stands for the normalized $\bar{N}$ -particle part of ψ . The diagonalization of H by use of the two-quasiparticle wave functions ϕ of Eq. (14) leads to eigenfunctions

$$\chi_k(J) = \sum_i \beta_{ik} \phi_i(J), \quad (17)$$

where J denotes the angular momentum of the state and k labels which state with

angular momentum J we are considering. In a similar way the energies

$$E_{\text{proj}}(J, k) = \langle P\chi_k(J) | H | P\chi_k(J) \rangle \quad (18)$$

agree closely with the exact calculations. This is seen from a comparison of columns 2 and 4 of Table III.

A second quantity that is of interest in an even-even nucleus is the value of $B(E2)$ between the various nuclear states. In Table IV we list the values of this quantity for $^{62}_{28}\text{Ni}$. The results shown compare the exact $(2p_{3/2}, 1f_{5/2}, 2p_{1/2})$ shell-model calculation with the results that emerge when one makes the seniority-zero and seniority-two truncation and then does the standard shell-model calculation. It is apparent that the seniority-zero and seniority-two approximation is only sufficient to explain the transition between the first excited 2^+ level and the ground state. The other transition rates are inadequately described; the 10% seniority-four admixtures in the 4_1^+ and 2_2^+ states drastically change the values of $B(E2)$. Since the two-quasiparticle result is merely

Fig. 5. Experimental binding energy of the even-even nickel isotopes as a function of the number of valence nucleons.

TABLE IV. Values of $B(E2)$ in arbitrary units for ${}_{28}^{62}\text{Ni}$. The results tabulated in column 2 are the results of the exact shell-model calculation and those given in column 3 are the shell-model results when the ground state is taken to be seniority-zero and the excited states seniority-two.

Transition	Exact	Seniority-0 and 2 approximation
$2_1 \rightarrow 0_1$	255	262
$4_1 \rightarrow 2_1$	65	2.5
$2_2 \rightarrow 2_1$	175	45

an approximation to the seniority calculation, it is senseless to use the two-quasiparticle eigenfunctions to describe transitions other than the transition from the first 2^+ state to the ground state. However, for this latter transition the quasiparticle calculation compares quite favorably with the results of the exact calculation; i. e., where the exact calculation gives 255, the BCS and two-quasiparticle approximations give 213. If one uses the projected wave functions, the $B(E2)$ becomes 242, in excellent agreement with the shell-model result.

In summary, the results that have been presented here show:

- (1) The BCS wave function for the ground state of an even-even nucleus provides an approximation to the seniority-zero shell-model calculation, and the two-quasiparticle theory is an approximation to the seniority-two result.

- (2) As far as energies are concerned, there is some hope that the BCS and quasi-particle approximations will reproduce the results of the exact calculations for states with even values of J . However, the approximations can come close to the shell-model results only if the energy one is trying to calculate for the physical system with $\bar{N}$ particles is symmetric about $\bar{N}$. If the energy as a function of N is not symmetric about $\bar{N}$, then to get agreement with the exact calculation one must project from the BCS wave function or the two-quasiparticle eigenfunction the part describing $\bar{N}$ particles, normalize this projected wave function, and then calculate the energy by use of this projected wave function. For odd values of J , the shell-model wave functions have large seniority-four components and hence the two-quasiparticle calculation is not justified.
- (3) The two-quasiparticle approximation to the first 2^+ level and the BCS result for the ground state lead to an E2 transition rate between these levels in good agreement with the exact calculations. On the other hand, the first excited 4^+ level and the second 2^+ level have a sufficient seniority-four contaminant in their wave functions (at least for $^{62}_{28}\text{Ni}$) to give a marked difference between the truncated seniority-zero and seniority-two shell-model calculation and the result for the full configuration. Thus to study these transitions, it is necessary to take into account four-quasiparticle states in the description of the levels. The formalism for this has been set up by Pal, Gambhir, and Raz.⁽⁷⁾

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