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AN EVALUATION OF THE DISPERSION RELATIONS OF PHOTOPRODUCTION

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ABSTRACT

A modification of the Omnès method is used to solve the singular integral equations for the 3-3 partial wave amplitudes of photoproduction. The effects of multi-pion production are assumed to be negligible. The method requires a knowledge of the phase at all energies. Consequently, it is necessary to treat the corresponding pion-nucleon scattering problem to determine the effect of the high-energy behavior of the phase on the solution for the scattering amplitude at low energies. The sharply resonant nature of the problem suggests an approximation in the form of solution, rather than in the Born terms, which leads to relatively simple expressions for the ratios of the 3-3 photoproduction amplitudes to the scattering amplitude and for integrals involving the 3-3 amplitudes. In addition, a modified Chew-Low formula can be derived which should satisfactorily represent the 3-3 phase shift throughout the resonance region. Finally, the cross sections are calculated in the 3-3 approximation and the results compared with experiment.

I. INTRODUCTION

Considerable attention has been directed toward the determination of the amplitudes for photoproduction of pions from nucleons by the technique of dispersion relations. The formulation of the dispersion relations for this process, and the first attempts to evaluate them, were made by Chew, Goldberger, Low, and Nambu[1] (hereafter referred to as CGLN). These authors obtained the integral equations for the photoproduction partial wave amplitudes from the connection between the phases of the photoproduction and pion nucleon scattering amplitudes provided by unitarity.[2] Only those contributions from the resonant 3-3 phase shift were retained under the integrals and each contribution was expanded in inverse powers of the nucleon mass M. In the static limit ($1/M \to 0$), the P-wave total magnetic moment amplitudes were determined by a comparison with the corresponding static limit equations for the pion nucleon scattering amplitudes, whereas all charge terms were evaluated, approximately, on the basis of the cutoff model.[3]

The various attempts to improve upon the CGLN results for the (3-3) amplitudes have met with only qualified success. These attempts invariably employ, with CGLN, the assumptions that multi-pion production effects may be neglected and that the 3-3 resonance exhausts the dispersion integrals. In addition to these assumptions, however, these treatments also involve either some assumption about the ratios of the photoproduction to the scattering amplitudes[4,5] or some type of approximation for the inhomogeneous terms in the dispersion relations. Examples of the latter approach are 1/M expansions in the static limit[6] and pole representations for the Born term.[5] However, in spite of the efforts of the several authors, the situation with regard to the 3-3 amplitudes has not been clarified. On the one hand, there is lack of qualitative agreement between the results of Refs. 4 and 5 for the energy dependence of the ratio of the magnetic dipole amplitude generated by the total vector magnetic moment to the scattering amplitude, while on the other hand, the CGLN results for the 3-3 charge amplitudes, which vanish at resonance, must be at least quantitatively incorrect.

The present investigation is an attempt to resolve these uncertainties which surround the 3-3 photoproduction amplitudes and thus to improve the calculation of the dispersion relation predictions in the range of energies for which the 3-3 state is dominant.

The present approach is fundamentally different from the methods of most other authors in that we shall attempt to solve the equations for the 3-3 photoproduction amplitudes by analytical means. No approximations will be made for the Born terms in these equations. Furthermore, no assumptions will be made concerning photoproduction-to-scattering-amplitude ratios, although we will derive from our solutions relatively simple expressions for these ratios. The two assumptions we shall make (which have already been used by the previous authors) are (1) that the equality of the 3-3 photoproduction and scattering phases, provided by unitarity at low energies, may be extended to all energies and (2) that only the 3-3 state contributes appreciably to the dispersion integrals for the low energy amplitudes. The method to be used is a modification of the Omnès[7] solution of singular integral equations for functions whose phase is known on the interval of singularity. Because the singular interval in the case of photoproduction extends to energy regions in which the phase is not known, it will be necessary to treat the corresponding problem for pion nucleon scattering in order to determine the effect of the unknown, "high" energy behavior of the phase on the solution for the 3-3 scattering amplitude in the energy region from threshold through the 3-3 resonance.

The photoproduction dispersion relations are given in Sec. II, together with a brief review of kinematics. The Omnès method is described in Sec. III and a somewhat modified form of solution is derived. Section IV contains a discussion of the meaning of the phase in the Omnès method. In Sec. V, the dispersion relation for the 3-3 scattering amplitude is discussed in order to obtain a representation for the high energy behavior of the 3-3 phase. The results of Sec. V are applied, in Sec. VI, to the determination of the 3-3 photoproduction amplitudes. In Sec. VII the dispersion relations are evaluated in the 3-3 approximation and the results compared with experiment. Finally, in Sec. VIII, our results and conclusion are discussed.

II. PHOTOPRODUCTION KINEMATICS

We will follow, as closely as possible, the notation of CGLN.[1] Furthermore, all kinematic quantities will refer, throughout, to the barycentric system. In this system the differential cross section may be written

$$\frac{d\sigma}{d\Omega} = \frac{q}{k} \left| \langle f | \mathcal{F}(\underset{\sim}{\sigma}) | i \rangle \right|^2, \tag{2.1}$$

where the decomposition of the total amplitude into the usual Pauli spin matrices is given by

$$\begin{aligned} \mathcal{F}(\underset{\sim}{\sigma}) = i\underset{\sim}{\sigma}\cdot\underset{\sim}{\varepsilon}\, \mathcal{F}_1 + \underset{\sim}{\sigma}\cdot\underset{\sim}{q}\, \underset{\sim}{\sigma}\cdot(\underset{\sim}{k} \times \underset{\sim}{\varepsilon})\, \mathcal{F}_2/qk \\ + i\underset{\sim}{\sigma}\cdot\underset{\sim}{k}\, \underset{\sim}{q}\cdot\underset{\sim}{\varepsilon}\, \mathcal{F}_3/qk + i\underset{\sim}{\sigma}\cdot\underset{\sim}{q}\, \underset{\sim}{q}\cdot\underset{\sim}{\varepsilon}\, \mathcal{F}_4/q^2 \quad . \end{aligned} \tag{2.2}$$

In these expressions, $\underset{\sim}{\varepsilon}$ is the photon polarization vector and $\underset{\sim}{q}$ and $\underset{\sim}{k}$ are the barycentric momenta of the pion and photon, respectively. The magnitudes of these momenta, together with the pion energy ω_q and the initial and final nucleon energies E_1 and E are related to the total barycentric energy W by the expressions[8]

$$\begin{aligned} k = (W^2 - M^2)/2W, \quad \omega_q = k + 1/2W, \quad q = (\omega_q{}^2 - 1)^{1/2}, \\ E_1 = W - k, \quad E = W - \omega_q, \end{aligned} \tag{2.3}$$

where M is the nucleon mass. If $\mathcal{F}$ is decomposed into linearly independent isotopic matrices according to

$$\mathcal{F} = \mathcal{F}^{+}\,\delta_{\beta 3} + \mathcal{F}^{-}\,\frac{1}{2}\left[\tau_\beta, \tau_3\right] + \mathcal{F}^{0}\,\tau_\beta \;.$$

then the connection between the $\mathcal{F}^{\alpha}$ (α = +, -, 0) and the amplitudes of the four possible charge configurations are given by

$$\begin{aligned}
\mathcal{F}(\gamma p \to \pi^0 p) &= \mathcal{F}^{+} + \mathcal{F}^{0} \\
\mathcal{F}(\gamma n \to \pi^0 n) &= \mathcal{F}^{+} - \mathcal{F}^{0} \\
\mathcal{F}(\gamma p \to \pi^+ n) &= \sqrt{2}\,(\mathcal{F}^{0} + \mathcal{F}^{-}) \\
\mathcal{F}(\gamma n \to \pi^- p) &= \sqrt{2}\,(\mathcal{F}^{0} - \mathcal{F}^{-})
\end{aligned} \qquad (2.4)$$

The $\mathcal{F}_i$ ($i = 1,2,3,4$) satisfy coupled dispersion relations which have a somewhat complicated structure. The simplest expression of these integral representations is in terms of the invariant amplitudes A_j, which are related to the $\mathcal{F}_i$ by

$$\begin{aligned}
\mathcal{F}_1/h &= A_1 + (W - M)A_4 + 2M\nu_1(A_3 - A_4)/(W - M) \\
(E + M)\;\mathcal{F}_2/qh &= -A_1 + (W + M)A_4 + 2M\nu_1(A_3 - A_4)/(W + M) \\
\mathcal{F}_3/qh &= (W - M)A_2 + (A_3 + A_4) \\
(E + M)\;\mathcal{F}_4/q^2h &= -\,(W + M)A_2 + (A_3 - A_4),
\end{aligned} \qquad (2.5)$$

where

$$h(W) = (W - M)(E_1 + M)^{1/2}\,(E + M)^{1/2}/2W, \qquad (2.6)$$

and where the connection between the momentum transfer variable ν_1 and the barycentric production angle θ of the pion is given by

$$2M\nu_1 = k(\omega_q - q\cos\theta)\;.$$

The dispersion relations satisfied by the A_i have the form

$$A_j(W,\nu_1) = B_j(W,\nu_1) + \frac{1}{\pi}\int_{(M+1)^2}^{\infty} d(W')^2 \, \mathrm{Im}\, A_j(W',\nu_1) \left[\frac{1}{(W')^2-W^2} \pm \frac{1}{(W')^2+W^2-2M\nu_1-2M^2}\right] \tag{2.7}$$

with

$$B_j(W,\nu_1) = R_j(\nu_1)\left[\frac{1}{W^2-M^2} \pm \frac{1}{W^2-2M\nu_1-M^2}\right] . \tag{2.8}$$

The upper signs refer to the amplitudes $A_{1,2,4}^{(+,0)}$ and $A_3^{(-)}$. It is convenient to consider separately those parts of the Born terms generated by the nucleon total magnetic moment and charge, which we denote by the superscripts μ and e, respectively. With this notation, one finds for the residues R_j the expressions

$$\begin{aligned}
R_1^{\mu(+,-,0)} &= R_2^{\mu(+,-,0)} = 0 \\
R_3^{\mu(+,-)} &= R_4^{\mu(+,-)} = 1/2\ (\mu_p - \mu_n)g \\
R_3^{\mu(0)} &= R_4^{\mu(0)} = 1/2\ (\mu_p + \mu_n)g \\
R_1^{(+,-,0)} &= 2M\nu_1\ R_2^{e(+,-,0)} = 2R_3^{e(+,-,0)} = 2R_4^{e(+,-,0)} = -\tfrac{1}{2}\, eg
\end{aligned} \tag{2.8'}$$

where g is the unrationalized renormalized pion-nucleon coupling constant

$$g^2 = 4M^2f^2 \ , \quad f^2 \cong 0.08 \ ,$$

and $\mu_p(\mu_n)$ is the total magnetic moment of the proton (neutron)

$$\mu_p \cong 2.79\ e/2M, \quad \mu_n = -\ 1.91\ e/2M, \quad e^2 = 1/137 \ .$$

To make use of the unitarity condition, one must decompose the photoproduction amplitudes into photon multipole eigenamplitudes which correspond to transitions into eigenstates of the final pion nucleon system with a definite total angular momentum J, isotopic spin T, and parity. The correspondence between the isospin (+,-) amplitudes and the eigenamplitudes with eigenvalues T = 3/2 and T = 1/2 is given by

$$\mathcal{F}^{3/2} = \mathcal{F}^{+} - \mathcal{F}^{-} , \quad \mathcal{F}^{1/2} = \mathcal{F}^{+} + 2\mathcal{F}^{-} \tag{2.9}$$

while the amplitude $\mathcal{F}^{0}$ corresponds only to the value T = 1/2. The complete angular momentum decomposition of $\mathcal{F}$ into photon multipole eigenamplitudes was given by CGLN. Here, however, we are concerned only with the J = 3/2, even parity part of the amplitude $\mathcal{F}^{3/2}$, defined by[9]

$$\mathcal{F}^{33} = \frac{1}{4\pi} \int \frac{d\Omega_{q'}}{qq'} \left(3\, \underset{\sim}{q}\cdot\underset{\sim}{q}' - \underset{\sim}{\sigma}\cdot\underset{\sim}{q}\; \underset{\sim}{\sigma}\cdot\underset{\sim}{q}'\right) \mathcal{F}^{3/2} (\underset{\sim}{q}', \underset{\sim}{k}) \tag{2.10}$$

The amplitude $\mathcal{F}^{33}$ may be expressed in terms of the CGLN multipole amplitudes $M_{1+}^{3/2}$ and $E_{1+}^{3/2}$, which correspond to transitions induced by magnetic dipole and electric quadrupole radiation, respectively. We will find it convenient to deal not with these multipole amplitudes but with the linear combinations

$$\begin{aligned} \phi_1 &= (M_{1+}^{3/2} - E_{1+}^{3/2})/qh \\ \phi_2 &= (M_{1+}^{3/2} + E_{1+}^{3/2})/qhk \quad . \end{aligned} \tag{2.11}$$

In terms of these amplitudes the parts of $\mathcal{F}^{3/2}$ which, by unitarity, have the 3-3 phase are given by

$$\begin{aligned} \mathcal{F}_1^{33} &= 3 \cos \theta \, (M_{1+}^{3/2} + E_{1+}^{3/2}) = 3\, qhk\, \phi_2 \cos \theta \\ \mathcal{F}_2^{33} &= 2M_{1+}^{3/2} = qh\, (k\phi_2 + \phi_1) \\ \mathcal{F}_3^{33} &= -\, 3(M_{1+}^{3/2} - E_{1+}^{3/2}) = -\, 3\, qh\phi_1 \\ \mathcal{F}_4^{33} &= 0 \quad . \end{aligned} \tag{2.12}$$

The projections for the Born parts of $\mathcal{F}^{33}$ have been given by Gartenhaus and Blankenbecler[9] in terms of total magnetic moment and charge contributions. The results of these authors may be written in terms of the Born projections for ϕ_1 and ϕ_2, which are given by

$$\phi^{\mu}_{1B} = \frac{g(\mu_p-\mu_n)}{kq}\left[\frac{1}{3}\, Q_{02}\left(\frac{E}{q}\right) - \frac{1}{5}\left(\frac{q}{E+M}\right) Q_{13}\left(\frac{E}{q}\right)\right]$$

$$\phi^{\mu}_{2B} = \frac{g(\mu_p-\mu_n)}{(kq)^2}\left[MQ_1\left(\frac{E}{q}\right) - \frac{qM}{E+M}\, Q_2\left(\frac{E}{q}\right) + \frac{q^2}{5(E+M)}\, Q_{13}\left(\frac{E}{q}\right)\right]$$

$$\phi^{e}_{1B} = \frac{eg}{kq}\left[\frac{1}{3(W+M)}\, Q_{02}\left(\frac{\omega_q}{q}\right) - \frac{q}{5(E+M)(W-M)}\, Q_{13}\left(\frac{\omega_q}{q}\right)\right]$$
$$+ \frac{eg}{2Mkq}\left[-\frac{(W-M)}{2(W+M)}\, Q_{02}\left(\frac{E}{q}\right) + \frac{q}{5(E+M)}\,\frac{(W+M)}{(W-M)}\, Q_{13}\left(\frac{E}{q}\right)\right] \qquad (2.13)$$

$$\phi^{e}_{B} = \frac{eg}{k^2}\,\frac{1}{5(W-M)(E+M)}\, Q_{13}\left(\frac{\omega_q}{q}\right) - \frac{eg}{2Mk^2}\,\frac{(W+M)}{5(W-M)(E+M)}\, Q_{13}\left(\frac{E}{q}\right)$$

$$Q_{mn} = Q_m - Q_n\ ,$$

where the Q_m are the Legendre functions of the second kind:

$$Q_0(a) = \frac{1}{2}\,\ln\left(\frac{a+1}{a-1}\right)$$

$$Q_1(a) = aQ_0(a)-1$$

$$(m+1)Q_{m+1}(a) = (2m+1)a\, Q_m(a) - m\, Q_{m-1}(a) \qquad m = 1,2,3,\ \ldots$$

The dispersion relations for the ϕ_i $(i = 1,2)$ may be found by projection, by means of Eq. (2.10) and with the help of expressions (2.2) and (2.5), from those satisfied by the A_j $(j = 1,2,3,4)$. The contributions to the dispersion integrals for the A_j, in Eq. (2.7) may be separated into those parts which involve the denominators $(W')^2-W^2$ and into those parts which arise from the "left hand" or crossed cut. If the latter contributions are denoted by the subscript L, then one finds, by carrying out the projection outlined above, that the ϕ_i satisfy the expressions

$$\phi_i(\omega) = \phi_{iB}(\omega) + \frac{1}{\pi}\int_1^\infty d\omega' \frac{\mathrm{Im}\phi_i(\omega')}{\omega'-\omega} + (\delta_{i,j})\frac{\beta}{\omega} + \phi_{iL}(\omega)\ , \quad i = 1,2$$

$$\beta = \frac{1}{\pi}\int_1^\infty \frac{d\omega'}{\omega'}\,\mathrm{Im}\phi_1(\omega') \qquad (2.14)$$

where $\delta_{i,j}$ is the Kronecker delta and where the Born terms ϕ_{iB} are given by Eq. (2.13).

The left hand cut terms ϕ_{iL} which appear in Eq. (2.14) are rather complicated, even when only the contributions from the 3-3 state are retained under the integrals defining them. Except at energies beyond the range of present interest, however, these terms are small in comparison to the Born terms. It is therefore consistent with our approach, insofar as we have already neglected the contributions to these integrals from other states, to retain only the static limit of these terms. The results of the 1/M expansion for the ϕ_{iL}, in the limit $M \to \infty$ are given by

$$\phi_{1L} = \frac{1}{9\pi}\int_1^\infty d\omega' \frac{\mathrm{Im}(2k'\phi_2' - \phi_1')}{\omega'+\omega}$$

$$\phi_{2L} = \frac{1}{9\pi}\int_1^\infty d\omega' \left[\frac{\mathrm{Im}(\phi_2' - 2\phi_1'/k)}{\omega'+\omega} + \frac{\mathrm{Im}(2\phi_2' - \phi_1'/k')}{\omega}\right] \qquad (2.15)$$

Equations (2.15) are analogous to the dispersion relations of CGLN (see Eqs. (11.1-11.5) of Ref. 1) if only the 3-3 contributions are retained in the latter.

In fact, if the replacement $h \rightarrow k$ is made in Eq. (2.14) and if the static limit of the Born terms is taken, then the two sets of equations become identical. It must be emphasized, however, that by the neglect of all but the static limit of the small left cut terms, we have thus far introduced only a negligible error in our Eqs. (2.13-2.15).

III. THE OMNÈS METHOD

The partial wave amplitudes for photoproduction and pion-nucleon scattering satisfy dispersion relations, in a complex variable $z = x+iy$, of the form

$$A(z) = B(z) + \frac{1}{\pi}\int_{x_0}^{\infty} dx' \frac{\mathrm{Im}A(x')}{x'-z} \tag{3.1}$$

where the inhomogeneous term B is to be regarded as a known function which is real and regular on the infinite cut (x_0,∞) of A. It is clear from Eq. (3.1) that in addition to this cut, across which A has the discontinuity 2i Im A, the amplitude A has all the singularities of B. It will be assumed, of course, that all integrals in our expressions exist. It was shown by Omnès[7] that if one knows the phase δ of the amplitude A, then a solution to Eq. (3.1) is given by

$$A(z) = B(z) + e^{\Delta(z)} \frac{1}{\pi}\int_{x_0}^{\infty} dx' e^{-\rho(x')} \sin\,\delta(x') \frac{B(x')}{x'-z} \tag{3.2}$$

Here, Δ is a function which is constructed from the known phase δ according to the prescription

$$\Delta(z) = \frac{1}{\pi}\int_{x_0}^{\infty} dx' \frac{\delta(x')}{(x'-z)} \tag{3.3}$$

and ρ is given by

$$\rho = \lim_{\varepsilon \to 0+} \Delta(x \pm i\varepsilon) \mp i\delta(x) \equiv \frac{P}{\pi} \int_{x_0}^{\infty} dx' \, \frac{\delta(x')}{(x'-x)} \tag{3.4}$$

where P stands for principal value. The conditions on δ for Eq. (3.2) to be a solution of Eq. (3.1) are, according to Omnès, that δ be continuous and that $\delta(\infty) = 0$.

Expression (3.2) for A is a solution of the original equation in that it has only those singularities prescribed by that equation and it has the correct phase. That the solution does have the correct phase may be seen by taking the limit $z \to x + i\varepsilon$, $x > x_0$, and by then recombining the singular part of the integral with the inhomogeneous term. It is clear, however, that it is possible to add to the result (3.2) any solution of the homogeneous counterpart of Eq. (3.1) which has the correct phase on the cut. These additions take the form of polynomials to be added to the coefficient of e^{Δ} in the solution. Such appendages, of course, are not spurious; the correct polynomial must be determined from a consideration of the expected behavior of the amplitude at large values of its argument.

It was indicated above that Eq. (3.2) is a possible solution if the phase is continuous. There are, however, physically admissible situations in which the amplitude phase may be discontinuous. Leaving the detailed discussion of the phase to the following section, we proceed now to construct a solution which admits this possibility.

Let A, B, and Δ be the same as before, with the understanding that a subtracted form of Eq. (3.3) must be used if δ does not vanish at infinity. If A is to be analytic on the cut plane, then B must have the Cauchy integral representation

$$B(z) = \frac{1}{2\pi i} \int_{C_B} dz' \, \frac{B(z')}{(z'-z)} \tag{3.5}$$

where the contour C_B does not cross the cut (x_0,∞). We will assume, as is usually the case, that B satisfies

$$B(z) \to 0 \text{ as } |z| \to \infty \quad .$$

Then the contour can be chosen so that it encloses only the singularities of B and is taken in a counterclockwise sense about them. The integrals of B taken about the singularities <u>via</u> C_B can now be expressed as an integral of the discontinuities of B along the singularity curves (denoted by S_B):

$$B(z) = \frac{1}{2\pi i} \int_{S_B} dz' \; \frac{\text{disc. } B(z')}{z'-z} \tag{3.6}$$

Let us define an auxiliary function G by means of

$$e^{\Delta(z)}\, G(z) = A(z) \tag{3.7}$$

so that, from Eq. (3.1) we also have

$$e^{\Delta(z)}\, G(z) = B(z) + \frac{1}{\pi} \int_{x_0}^{\infty} dx' \; \frac{\mathrm{Im}A(x')}{(x'-z)} \tag{3.8}$$

In the cases of physical interest, A has no poles x_0 and $\delta(x_0) = 0$ so that G, from its definition, also has no poles at x_0. That G has no singularities on the cut (x_0,∞) can be shown explicitly as follows: If we denote the limit of $G(x \pm i\varepsilon)$ as $\varepsilon \to 0^+$ by $G_\pm(x)$ and evaluate Eq. (3.7) as the cut is approached from above, we find that the limit is given by

$$e^{\rho}\, G_+ e^{i\delta} = A_+ \equiv |A_+| e^{i\delta} \quad .$$

From Eq. (3.8) it follows that the discontinuity of $e^{\Delta}G$ across the cut may be written

$$e^{\rho}\left[G_+ e^{i\delta} - G_- e^{-i\delta}\right] = 2i \;\mathrm{Im}A_+ = 2i|A_+|\sin\delta$$

If we now eliminate $|A_+|$ between these last two expressions, we find the result

$$e^{\rho} (G_+ - G_-) = 0$$

and it follows that G has no singularities on (x_0,∞) except, perhaps, at points where e^{ρ} vanishes. The discussion of such zeros in e^{ρ} is given in the following section. There it will be seen that for the physical partial wave amplitudes with which we are concerned, e^{ρ} can vanish only at the zeros of A. It then follows, from Eq. (3.7), that G must also be regular at such isolated points of (x_0,∞).

The determination of the function G is now straightforward. The singularity curve S_B of the inhomogeneous term B cannot intersect that cut (x_0,∞). Because e^{Δ} is regular and nonvanishing in the vicinity of S_B and because G is analytic on (x_0,∞), it follows from Eq. (3.8) that G is analytic everywhere except on the curves S_B. It further follows that the discontinuity of G across S_B is given by

$$\text{disc. } G = e^{-\Delta} \text{ disc. } B$$

By the same arguments which led to the result (3.6) for B, we may write the Cauchy integral formula for G in the form

$$G(z) \equiv \frac{1}{2\pi i}\int_{S_B} dz' \, \frac{\text{disc. } G(z')}{z'-z} = \int_{S_B} dz' \, \frac{e^{-\Delta(z')} \text{ disc. } B(z')}{z'-z} \tag{3.9}$$

This result may also be written as

$$G(z) = \int_{C_B} dz' \, \frac{e^{-\Delta(z')} \, B(z')}{z'-z} \tag{3.10}$$

where we recall that C_B is the contour which encloses the singularity curve S_B of the inhomogeneous term. Finally, we have for the solution of Eq. (3.1) for a function A with known phase δ the result

$$A(z) = e^{\Delta(z)} \frac{1}{2\pi i} \int_{S_B} dz' \frac{e^{-\Delta(z')} \text{disc. } B(z')}{z'-z} \tag{3.11}$$

It is in this form that the Omnès method will be applied to the integral equations of photoproduction.

Expression (3.11) for A is not unique, as was the case for the Omnès solution in its original form (3.2). It is possible to add to the result (3.9) for G some polynomial in z, the form of which must be determined by the behavior of A(z) as $|z| \to \infty$. For the amplitudes of present interest, however, no such polynomial will be needed. The question on uniqueness is discussed in some detail in Ref. 10.

The relatively simple form for the general solution (3.9) becomes even simpler when the inhomogeneous term B consists of poles. In such a case the discontinuities of B are Dirac delta functions or derivatives thereof. Thus, if B in Eq. (3.1) has the form

$$B(z) = \frac{\gamma}{(z-\xi)^{n+1}} \equiv \frac{\gamma}{n!} \frac{d^n}{d\xi^n} \left(\frac{1}{z-\xi}\right) , \quad n = 0,1,2, \ldots$$

where γ and ξ are parameters, then the solution (3.10) for A assumes the form

$$A(z) = e^{\Delta(z)} \frac{\gamma}{n!} \frac{d^n}{d\xi^n} \left[\frac{e^{-\Delta(\xi)}}{z-\xi} \right] .$$

This result will be used to some extent in the following sections.

IV. REMARK ON THE PHASE

An inherent feature of the solution given by Eq. (3.11) is that the phase δ referred to in the definition of Δ is the amplitude phase of A; that is, the real phase δ is defined by

$$A(x) = |A(x)|e^{i\delta(x)} \tag{4.1}$$

A physical partial wave scattering amplitude f_s, however, is ordinarily expressed in terms of the <u>phase shift</u> δ_s by

$$f_s = \frac{\left(e^{i\delta_s} \sin \delta_s\right)}{q} \tag{4.2}$$

where q is a positive momentum. In the general case, wherein inelastic channels are open, the amplitude may be expressed in terms of a complex phase shift with a non negative imaginary part. It is evident from (4.2) that neither a complex phase shift nor its real part is the amplitude phase referred to in (4.1).

Even in pure elastic scattering (e.g., potential scattering), the amplitude phase is not necessarily equal to the real phase shift. In fact, the amplitude phase must always lie between 0 and π (mod 2π) whereas there is no such restriction on the phase shift. This restriction on the amplitude phase follows from Eq. (4.2), which implies that the imaginary part of f_s is always positive or zero.

The distinction between the amplitude phase and a real phase shift is evidenced, for example, when the phase shift $\delta_s(x)$ in Eq. (4.2) passes through π at some point x_π. In this case the amplitude phase δ, which must be equal to δ_s (mod 2π) up to x_π, must be discontinuous at that point. An illustration of such a possibility is shown in Fig. 1. In Fig. 1(a), our hypothetical phase shift $\delta_s(x)$ vanishes below some threshold x_0 and passes linearly through π at x_π. Fig. 1(b) displays the corresponding amplitude phase $\delta(x)$ when its discontinuity at x_π is $-\pi$. If we now assume that f_s satisfies an equation of the type (3.1), then the Omnès solution for f_s has the form (3.9) and is thus proportional to e^{Δ}. It follows from Eq. (3.3), however, that if δ has a discontinuity $-\pi$ at x_π, then $e^{\Delta(x)}$ is proportional to $(x-x_\pi)$ for $x \approx x$. This result is appropriate because the amplitude f_s, which by definition is proportional to $e^{i\delta_s} \sin \delta_s$, must have a linear zero at x_π. Thus we may be assured that the choice of amplitude phase in Fig. 2(b) leads to the correct behavior of the solution in the vicinity of a linear zero in the amplitude. In general, a discontinuity of $-n\pi$ in δ, $n = 0,1,2, \ldots$, will lead to a zero of order n in the solution whereas positive discontinuities, which would give rise to poles in the physical region, must not occur.

In the course of our derivation of the Omnès solution (3.9), we found that the auxiliary function G has no singularities on the interval (x_0,∞) except possibly at the zeros of e^Δ. The above considerations show, however, that the order of a xero of e^Δ can always be chosen equal to the order of the zero in the total amplitude A. It then follows from Eq. (3.7) that G has no singularities on (x_0,∞).

V. THE 3-3 PHASE AND SCATTERING AMPLITUDE

It is clear from the foregoing that in order to apply the Omnès method to the equations for the 3-3 multipole amplitudes, we first must know the 3-3 phase at all energies. Unfortunately, such complete information about the phase is not available. If we assume, however, that the Omnès method yields physically meaningful solutions, then we may hope to learn something of the unknown portion of the phase from a consideration of the scattering amplitude, which satisfies a dispersion relation similar to those for the photoproduction amplitudes. Our rational is the following: If we can use the Omnès method to "solve" the scattering equations--that is, to determine the function Δ in Eq. (3.3)--then, by unitarity, we can use the same Δ to evaluate the photoproduction amplitudes. The problem here, then, is to construct as much of the function Δ as is possible from the experimentally known values of the phase and to construct that part of Δ which arises from the unknown high energy behavior of the phase in such a way that the solution reproduces the known, low energy, scattering amplitude.

A. The 3-3 Phase

Below the resonance energy ω_r, the 3-3 phase shift δ_{33} is well represented by the Chew-Low effective range formula[3]

$$q^3 \cot \delta_{33} = \omega(\omega_r-\omega)/\left(\frac{4}{3} f^2 \omega_r\right) , \omega<\omega_r \tag{5.1}$$

The currently accepted value for the coupling constant is $f^2 \approx 0.08$. The values used by various authors for the resonance position range from $\omega_r = 2.06$ (Ball[11]) to the value $\omega_r = 2.14$, which follows from McKinley's three-parameter fit[4] to $q^3 \cot \delta_{33}$. Throughout this investigation, we will use the values

$$\omega_r = 2.07 , f^2 = 0.082 . \tag{5.2}$$

Some typical experimental values of the phase above resonance are shown in Table I, where it can be seen that the values of the phase approach π as the energy increases. One can readily see this tendency of the phase from the plot of the function $\xi(\omega) = \omega(\omega_r-\omega)/(\omega_r q^3 \cot \delta_{33})$ shown in Fig. 2. According to Eq. (5.1), $\xi(\omega)$ should have the constant value $4f/3^2$. The values of $\xi(\omega)$ that correspond to the phases of Table I, however, show a marked downward trend toward the value zero. Should this trend persist, the point at which the function ξ vanished would correspond to the point where the phase shift passes through π. A zero in $\xi(\omega)$ somewhere in the interval $5 \leq \omega \leq 10$ is seen to be consistent with the data of Table I.

Because inelastic effects will certainly be important at very high energies, no assumption that we make about the asymptotic behavior of the phase shift can have any a priori justification. Insofar as we are assuming that the phase shift is everywhere real, we may further assume, as Levinson's theorem for potential scattering suggests,[12] that the phase shift asymptotically approaches some integral multiple of π. In particular, the simplest assumption that can be made about the high energy behavior of the phase shift δ_{33} are (1) that δ_{33} approaches π from below (2) that δ_{33} drops rapidly and approaches zero from above, and (3) that $\delta_{33}(\omega)$ passes through π at some point ω_m and then asymptotically approaches π from above. We will find it unnecessary to treat assumption (1) separately, because it is a special case of assumption (3)(i.e., the limit $\omega_m \to \infty$).

Assumptions (2) and (3) have consequences which differ little from one another--at least insofar as they affect the construction of the solution (3.11) in the low energy region. That this is so follows from the arguments of Sec. IV, the conclusions of which are summarized in Fig. 3. The solid curve in this figure represents the amplitude phase in case (3), the dotted line is the extension of the phase shift in case (3), and the dashed line is the phase in case (2). In both cases, ω_r is the resonance position and ω_m can be regarded as the position at which the amplitude phase passes downward through $\pi/2$. In both cases, again, there will be a local minimum in the vicinity of ω_m; for case (2) however the minumum will occur below ω_m and will be followed immediately by a resonance. It follows that, except in the immediate vicinity of ω_m, assumptions (2) and (3) lead to similar results for the function Δ. Since we are only interested in the determination of the amplitude in the region of known phase, we can concentrate our attention on case (3).

It is not to be expected that assumption (3) correctly describes the high energy behavior of the phase but it should be sufficient to describe the effect of that behavior on the solution at low energy. Also, we have seen in Fig. 2 that this assumption is in accord with the experimental values of the phase above resonance. Finally, assumption (3) has the advantage that we know beforehand just what its major effect on the solution will be. That is, we know from Sec. IV that the function $e^{\Delta(\omega)}/(\omega_m-\omega)$ is roughly independent of ω_m and depends, therefore, primarily on the known phase shift in the vicinity of the resonance.

To represent the phase above resonance for use in our computations, we adopt the convenient form

$$\delta(\omega) = \begin{cases} \pi \left[1 - \frac{1}{2}\left(\frac{\omega_m-\omega}{\omega_m-\omega_r}\right) \left(\frac{\omega_r-\alpha}{\omega-\alpha}\right)\right] & ; \omega_r \leq \omega \leq \omega_m \\ 0 \; ; \; \omega>\omega_m \end{cases} \quad (5.3)$$

This form ensures that δ has the values $\pi/2$ and π at ω_r and ω_m, respectively. The number α is to be found from the condition $\delta(2.74) = 0.74\pi$. This condition was chosen in agreement with the 310-MeV phase shift of Table I.

B. Equation for the Scattering Amplitude

The 3-3 partial wave scattering amplitude $f_{1+}^{3/2}$ is given in terms of the 3-3 phase shift δ_{33} by

$$f_{1+}^{3/2} = e^{i\delta_{33}} \frac{\sin \delta_{33}}{q}$$

where q and all other kinematic quantities to be used in this section have the same meaning as Sec. 2. The analytic structure of this amplitude has already been investigated in great detail by several authors.[13,14] The results of these analyses may be condensed into the statement that the amplitude Ψ, defined by

$$\Psi = \frac{2W}{(E+M)} \frac{f_{1+}^{3/2}}{q^2} = \frac{2W}{E+M} \frac{e^{i\delta_{33}} \sin \delta_{33}}{q^3} \tag{5.4}$$

has no kinematic singularities in the $\omega = W-M$ plane relative to the invariant amplitudes which satisfy the Mandelstam representation. If follows that Ψ satisfies a dispersion relation of the form

$$\Psi(\omega) = \Psi_B^{(\omega)} + \frac{1}{\pi} \int_1^\infty d\omega' \frac{\text{Im } \Psi(\omega')}{\omega'-\omega} + \Psi_L(\omega) \tag{5.5}$$

where Ψ_L is the contribution of the left hand of "crossed" cut and Ψ_B is the Born term. Equation (5.5) for Ψ represents the 3-3 projection of the fixed-momentum-transfer dispersion relations for the scattering amplitudes (see Eqs. (3.3) and (3.4) of Ref. 15). The Born contribution Ψ_B is given by[10]

$$\Psi_B = \left(\frac{-g^2}{q^4}\right) \left[(W-M)\alpha(W) - \frac{1}{2} \frac{(E-M)(W+M)\gamma(W)}{E+M} \right] \tag{5.6}$$

where g is the unrationalized renormalized pion-nucleon coupling constant and where α and γ are given by

$$\alpha = 1 - \frac{a}{2} \ln \left(\frac{a+1}{a-1}\right)$$

$$\gamma = 3a + \frac{1}{2} (1-3a^2) \ln \left(\frac{a+1}{a-1}\right)$$

$$a = \frac{(E\omega_q - \frac{1}{2})}{q^2}$$

The left cut term Ψ_L involves contributions to Ψ from all angular momentum states but we are interested here only in the contributions of the 3-3 state.

The resulting expression for Ψ_L is of the same relative order of magnitude as the corresponding contributions to the photoproduction amplitudes. In accordance with our treatment of the latter, we will retain only the static limit of the 3-3 contributions to Ψ_L. In this limit we find the relatively simple expression

$$\Psi_L = \frac{1}{9\pi} \int_1^{\infty} d\omega' \; \frac{\text{Im } \Psi(\omega')}{\omega'+\omega} \tag{5.7}$$

According to the result (3.11), the equation for the scattering amplitude has an Omnès solution of the form

$$\Psi(\omega) = \frac{e^{\Delta(\omega)}}{2\pi i} \int_{C_I} d\omega' \; \frac{e^{-\Delta(\omega')} \; \Psi_I(\omega')}{\omega'-\omega} \tag{5.8}$$

where $\Psi_I = \Psi_B + \Psi_L$ is the resultant inhomogeneous term in the equation for Ψ. Although we have dwelt at great length on the manner in which the high energy behavior of the phase will be represented, we have not yet specified how the parameter ω_m is to be determined. The solution (5.8) for Ψ depends on ω_m through the function Δ, given by

$$\Delta(\omega;\omega_m) = \frac{1}{\pi} \int_1^{\omega_m} d\omega' \; \frac{\delta(\omega;\omega_m)}{\omega'-\omega} \quad , \tag{5.9}$$

where $\delta(\omega,\omega_m)$ is constructed according to Eqs. (5.1-5,3). We may determine ω_m by normalizing the solution to the known resonance value of the amplitude (5.4): That is, the condition

$$|\Psi(\omega_r)| = \frac{2W_r}{(E_r+M)q_r^3} \tag{5.10}$$

is imposed on the solution (5.8). This condition is sufficient to determine a unique value for ω_m.

The normalization procedure just described requires some justification. If one allows ω_m to vary, one finds that the solution at low energies is quite sensitive to this parameter. The important point here is that the phases of the scattering amplitude at low and high energies are interrelated by the conditions of unitarity and analyticity. The latter condition is expressed by the dispersion relation for Ψ, while the unitarity condition for the scattering amplitude is expressed by the restriction (5.4) on the form of Ψ. Thus, if the phase were known for all energies, the Omnès method would automatically give the solution for Ψ which satisfies unitarity. If, however, one uses an arbitrary ω_m together with the experimentally known phases to form the Omnès solution, then the result is just a function which satisfies the dispersion relation and has the correct phase at low energies. Our procedure of normalizing the solution for Ψ at resonance gives us the "correct" ω_m--that is, the correct representation for the effect on the solution of the unknown part of the phase. We see, therefore, that there is only one value for ω_m that is consistent with unitarity.

The actual evaluation of ω_m by means of Eqs. (5.8-5.10), while straightforward, necessitates a detailed investigation of the singularity structure of the inhomogeneous term Ψ_I, coupled with the evaluation of a complicated integral about the contour C_I. In some cases this precise but somewhat tedious method will be unavoidable. For the 3-3 amplitudes, however, the peculiar nature of the sharp resonance admits of a considerable simplification in the solution by which both of the above complications may be circumvented. This simplification, which is described below, takes the form of an approximation in the structure of the solution.

C. The Relativistic Approximation

It can be seen from Fig. 2 that, according to our assumptions concerning the parameter ω_m, the 3-3 amplitude phase δ has the following behavior: δ is small below resonance but rises rapidly through $\pi/2$ and toward π in the vicinity of ω_r; above resonance, the phase remains near π up to the position of the amplitude zero ω_m, at which point δ drops discontinuously to zero. Thus, in the limit of a sharp resonance at ω_r, the phase would assume the form:

$$\delta \approx \begin{cases} \pi \ ; \ \omega_r < \omega < \omega_m \\ 0 \ ; \ \omega < \omega_r, \ \omega > \omega_m \end{cases} \tag{5.11}$$

Except in the physical region where the detailed behavior of the phase is important, the function Δ of Eq. (5.9) depends only on the gross features of the phase. That is, one can compute $\Delta(\omega)$ approximately by using Eq. (5.11) in the defining integral:

$$\Delta(\omega) = \frac{1}{\pi}\int_1^{\omega_m} d\omega' \frac{\delta(\omega')}{\omega'-\omega} \quad \int_{\omega_r}^{\omega_m} \frac{d\omega'}{\omega'-\omega} = \ln\left(\frac{\omega_m-\omega}{\omega_r-\omega}\right)$$

or

$$e^{-\Delta(\omega)} = \left(\frac{\omega_m-\omega}{\omega_r-\omega}\right) \quad . \tag{5.12}$$

This form for $e^{-\Delta}$ is approximately valid for all complex ω sufficiently far from ω_r. One can check the validity of this approximation by computing Δ "exactly" from the phase of Eqs. (5.1-5.3) for various values of ω_m. At the singularity of the inhomogeneous term nearest the physical region, i.e., at $\omega \approx 0$, the approximation (5.12) differs from the exact $e^{-\Delta}$ by about 3%; the error decreases as ω recedes from the physical region. In other words, Eq. (5.12) is correct to 3% throughout the singularity region of the inhomogeneous term.

The approximation just described enables us to calculate, in closed form, the contour integral that appears in the Solution (5.8): We find the result

$$\frac{1}{2\pi i}\int_{C_I} d\omega' e^{-\Delta(\omega')} \frac{\Psi_I(\omega')}{\omega'-\omega} \approx \frac{1}{2\pi i}\int_{C_I} d\omega' \left(\frac{\omega_r-\omega'}{\omega_m-\omega'}\right) \frac{\Psi_I(\omega')}{\omega'-\omega}$$

$$= \left[(\omega_r-\omega)\ \Psi_I(\omega) + (\omega_m-\omega_r)\ \Psi_I(\omega_m)\right] \Big/ (\omega_m-\omega) \quad .$$

The last expression follows immediately from the Cauchy integral formula for Ψ_I. Thus, the solution for the scattering amplitude may be written in the form

$$\Psi(\omega) \equiv \frac{2W}{E+M} \frac{e^{i\delta_{33}} \sin \delta_{33}}{q^3} = e^{\Delta(\omega)} \left[(\omega_r-\omega)\ \Psi_I(\omega) + (\omega_m-\omega_r)\ \Psi_I(\omega_m)\right] \Big/ (\omega_m-\omega) \quad . \tag{5.13}$$

While this approximation is relativistic in the sense that the inhomogeneous terms may be treated exactly, we cannot expect it to be correct at very high energies where the phase is unknown. The principal advantage of Eq. (5.13) is, of course, that the solution in the physical region involves the inhomogeneous term only in the physical region.

There is another advantage to the approximate Solution (5.13). It is possible to derive from this solution an approximate formula by which certain non-singular integrals involving the 3-3 amplitudes can be evaluated in closed form. Some of these integrals, such as the expression (5.7) for the small left cut terms, have the form

$$\frac{1}{\pi}\int_1^\infty d\omega' \frac{\mathrm{Im}\ \Psi(\omega')}{\omega'-\sigma} \quad ,$$

where σ is not on the cut $(1,\infty)$. That is, σ is in a region where, to a good approximation, the expression

$$e^{\Delta(\sigma)} = \frac{\omega_m - \sigma}{\omega_r - \sigma}$$

is valid. The integral, however, is just $\Psi(\sigma) - \Psi_I(\sigma)$. If one uses the above expression for $e^{\Delta(\sigma)}$ in the solution (5.13) to evaluate $\Psi(\sigma)$, one finds the result

$$\frac{1}{\pi}\int_1^\infty d\omega' \, \frac{\mathrm{Im}\,\Psi(\omega')}{\omega' - \sigma} \equiv \Psi(\sigma) - \Psi_I(\sigma) = \frac{\omega_m - \omega_r}{\omega_r - \sigma}\,\Psi_I(\omega_m) \qquad (5.14)$$

Now let G by any function analytic on $(1,\infty)$ and consider the integral

$$I_G = \frac{1}{\pi}\int_1^\infty d\omega' \, G(\omega') \, \mathrm{Im}\,\Psi(\omega') \quad .$$

For G we have the integral representation

$$G(\omega') = \frac{1}{2\pi i}\int_{C_G} d\sigma \, \frac{G(\sigma)}{\sigma - \omega'}$$

where the contour C_G encloses all the singularities of G. With the help of Eq. (5.14) we can rewrite the integral in the form

$$I_G = \frac{1}{\pi}\int_1^\infty d\omega' \, \frac{1}{2\pi i}\int_{C_G} d\sigma \, \frac{G(\sigma)}{\sigma - \omega'} \, \mathrm{Im}\,\Psi(\omega')$$

$$= \frac{1}{2\pi i}\int_{C_G} d\sigma \, G(\sigma) \, \frac{1}{\pi}\int_1^\infty d\omega \, \frac{\mathrm{Im}\,\Psi(\omega')}{\sigma - \omega'}$$

$$= (\omega_m - \omega_r) \, \Psi_I \, (\omega_m) \, \frac{1}{2\pi i}\int_{C_G} d\sigma \, \frac{G(\sigma)}{\sigma - \omega_r}$$

$$= (\omega_m - \omega_r) \, \Psi_I \, (\omega_m) \, G(\omega_r)$$

Thus, the closed form approximation for all nonsingular integrals is given by

$$\frac{1}{\pi}\int_{1}^{\infty} d\omega'\ G(\omega')\ \text{Im}\ \Psi(\omega') = (\omega_m - \omega_r)\ \Psi_I\ (\omega_m)\ G(\omega_r) \qquad (5.15)$$

The appearance of $G(\omega_r)$ in this result clearly illustrates the sharp resonance nature of our approximation.

In view of the approximations that we had made in previous sections--such as the neglect of the contributions of states other than the 3-3 state to Ψ_L and the neglect of inelastic effects--it would be superfluous to attempt to find corrections to the solution (5.13) and the approximate formula (5.15). This solution, as we have seen, should be correct to a few percent. Furthermore, this small error in the solution for the scattering amplitude will be absorbed, for the most part, in the parameter ω_m by the normalization procedure described above. It is now a simple matter to carry out this procedure and evaluate ω_m.

The inhomogeneous term Ψ_I in Eq. (5.13) is given by

$$\Psi_I = \Psi_B + \Psi_L$$

where the Born term Ψ_B is given by Eq. (5.6). The integral (5.7) for the left cut term Ψ_L can be evaluated formally with the help of Eq. (5.15); the result is

$$\Psi_L(\omega) = \frac{1}{9}\ \left(\frac{\omega_m - \omega_r}{\omega + \omega_r}\right)\ \left[\ \Psi_B(\omega_m) + \Psi_L(\omega_m)\right] \qquad . \qquad (5.16)$$

We can evaluate this result at $\omega = \omega_m$ and we find the expression

$$\Psi_L(\omega_m) = \frac{1}{9}\ \left(\frac{\omega_m - \omega_r}{\omega_m + \omega_r}\right)\ \left[1 - \frac{1}{9}\ \left(\frac{\omega_m - \omega_r}{\omega_m + \omega_r}\right)\right]^{-1}\ \Psi_B(\omega_m) \qquad (5.17)$$

If we now impose the normalization condition (5.10) on the Solution (5.13), we are led to a unique value for ω_m. We find, in fact, that ω_m is determined from the expression

$$\left[(\omega_m-\omega_r)\ \Psi_B(\omega_m)\right]^{-1} = \frac{e^{\rho(\omega_r)}}{\omega_m-\omega_r}\ \frac{q_r^3(E_r+M)}{2W_r}\left[1+\frac{(\frac{1}{9})(\omega_m-\omega_r)/(\omega_m+\omega_r)}{1-(\frac{1}{9})(\omega_m-\omega_r)/(\omega_m+\omega_r)}\right] \tag{5.18}$$

where the principal integral (3.4) for $\rho = \Delta - i\delta$ is calculated with the phase δ of Eqs. (5.1-5.3). The above equation for ω_m was solved graphically, by plotting the left and right hand sides as functions of ω_m. For the present choice of resonance position and coupling constant, namely $\omega_r = 2.07$ and $f^2 = 0.082$, the intersection of the two curves occurs at $\omega_m = 6.38$. With the parameter ω_m thus determined, the left cut term Ψ_L can be computed from Eqs. (5.16-5.17); this result is tabulated, together with the total inhomogeneous term at ω_m, in Table II.

Before we apply the result $\omega_m = 6.38$ to the determination of the 3-3 photoproduction amplitudes, let us indicate how one may derive from Eq. (5.13) a generalization of the Chew-Low effective-range formula. If we define a function η according to

$$\mathrm{Re}\ e^{-\Delta(\omega)} \equiv e^{-\rho}\cos\delta = \left(\frac{\omega_r-\omega}{\omega_m-\omega}\right)\eta(\omega)\quad,\quad 1 < \omega < \omega_m \tag{5.19}$$

then, because the zeros of e^{ρ} and $\cos\delta$ have been built in to the right-hand side of the last expression, we can expect that η is a smooth and slowly varying function of ω. Furthermore, because expression (5.12) is valid for $\omega < 1$ and $\omega > \omega_m$, we can also expect to have $\eta \approx 1$. When the representation (5.19) is used to eliminate $e^{-\rho}$ from the solution (5.13), the resulting expression is the formula

$$q^3\cot\delta_{33} = \left(\frac{2W}{E+M}\right)\frac{\eta(\omega)(\omega_r-\omega)}{\left[(\omega_r-\omega)\ \Psi_I(\omega) + (\omega_m-\omega_r)\ \Psi_I(\omega_m)\right]} \tag{5.20}$$

To compute $\eta(\omega)$ one can use the phase of Eqs. (5.1-5.3) together with the definition (5.19). It is found that η can be represented to an accuracy of 2%

by the (non-unique) expression

$$\eta(\omega) = \left[1.245 + \frac{(\omega-1)(\omega_m-1-\omega)}{70}\right]\left(\frac{E+M}{2W}\right) \tag{5.21}$$

The result (5.20-5.21) is compared with the experimental data in Fig. 4.

Although we have required an approximate knowledge of the 3-3 phase in order to arrive at the result (5.20), it may yet be useful in some future analysis of the scattering data. This is so because the approximations used to derive (5.20) are roughly independent of the precise values of ω_r and ω_m. Thus, Eq. (5.20) may be fit to the experimental data as a three-parameter formula. However, because of the fact that from the present point of view one of the parameters--the zero-position ω_m--is related to the other two by the normalization procedure, a more profitable approach would be to treat the combined expressions (5.18) and (5.20) as a two parameter representation for δ_{33} in terms of ω_r and f^2. In this way one might obtain not only a satisfactory representation for the phase shift but also a more precise determination of the resonance position and coupling constant.

VI. THE 3-3 PHOTOPRODUCTION AMPLITUDES

A. Solutions in the Relativistic Approximation

We are now in a position to derive the solution for the 3-3 photoproduction amplitudes ϕ_i, i = 1,2, which are defined by Eqs. (2.11-2.12). These amplitudes satisfy the dispersion relations (2.14), in which the inhomogeneous terms consist of the pole in ϕ_2 at $\omega = 0$, the approximations (2.15) for the small left cut terms ϕ_{iL}, and the relativistic expressions (2.13) for the Born terms ϕ_{iB}.

According to our assumption, the 3-3 photoproduction and scattering amplitudes have the same phase and thus are associated with the same function Δ. It follows that the solution (5.13) and the formula (5.15) for the evaluation of non-singular integrals will apply equally well to the photoproduction case if we replace the scattering amplitude Ψ everywhere in these expressions by either of the amplitudes ϕ_i. Thus, the solutions for the photoproduction amplitudes are given by

$$\phi_i = e^{\Delta(\omega)} \quad \frac{\left[(\omega_r - \omega)\ \phi_{iI}(\omega) + (\omega_m - \omega_r)\ \phi_{iI}(\omega_m) \right]}{\omega_m - \omega} \quad , \tag{6.1}$$

and all nonsingular integrals involving these amplitudes may be evaluated according to the prescription

$$\frac{1}{\pi} \int_1^\infty d\omega'\ G(\omega')\ \mathrm{Im}\ \phi_i(\omega') = (\omega_m - \omega_r)\ G(\omega_r)\ \phi_{iI}(\omega_m) \quad , \tag{6.2}$$

where ϕ_{iI} is the resultant inhomogeneous term in the equation for ϕ_i .

We may use the result (6.2) to evaluate formally the integrals for the left cut terms in (2.15) and the residue β of the pole in ϕ_2; we find the expressions

$$\phi_{1L} = \frac{1}{9}\ (\omega_m - \omega_r) \ \left(\frac{2k_r \phi_{2I}(\omega_m) - \phi_{1I}(\omega_m)}{\omega_r + \omega} \right)$$

$$\phi_{2L} = \frac{1}{9}\ (\omega_m - \omega_r) \ \left\{ \frac{\phi_{2I}(\omega_m) - 2\phi_{1I}(\omega_m)/k_r}{\omega_r + \omega} + \frac{2\phi_{2I}(\omega_m) - \phi_{1I}(\omega_m)/k_r}{\omega} \right\}$$

$$\beta = \frac{(\omega_m - \omega_r)}{\omega_r}\ \phi_{1I}(\omega_m) \quad . \tag{6.3}$$

When these representations for the non-Born inhomogeneous terms are inserted into the definitions of the $\phi_{iI}(\omega_m)$, i.e., into the expressions

$$\phi_{iI}(\omega_m) = \phi_{iB}(\omega_m) + \phi_{iL}(\omega_m) + (\delta_{i,2})\ \beta/\omega_m$$

there results a pair of coupled linear equations for the $\phi_{iI}(\omega_m)$ in terms of the known $\phi_{iB}(\omega_m)$. The solution of these equations for a given ω_m and resonance position ω_r is straightforward and the results--for the values $\omega_m = 6.38$ and $\omega_r = 2.07$--are tabulated in Table II.

While the Solution (6.1) for the ϕ_i is satisfactory as it stands, it is more instructive and more convenient in practice to work with the ratios of the photoproduction amplitudes to the scattering amplitude. We can readily construct these ratios from Eqs. (5.13) and (6.1) and we find the result

$$\frac{\phi_i}{\Psi} = \frac{\phi_I}{\left(\frac{2W}{E+M}\right) e^{i\delta_{33}} \sin \delta_{33}} = \frac{(\omega_r-\omega)\ \phi_{iI}(\omega) + (\omega_m-\omega_r)\ \phi_{iI}(\omega_m)}{(\omega_r-\omega)\ \Psi_I(\omega) + (\omega_m-\omega_r)\ \Psi_I(\omega_m)} . \qquad (6.4)$$

The advantage of a closed form such as (6.4) for the amplitude ratios is that the dependence of these ratios on the parameters ω_r, ω_m, and the coupling constant is made explicit. The ratios predicted by Eq. (6.4) for the amplitudes generated by the nucleon total magnetic moment (μ) and charge (e) are shown in Figs. 5 and 6, respectively. Also shown in these figures are the corresponding predictions of CGLN[1]; however, the ratios for the CGLN charge terms are shown only up to ω_r, where they vanish.

B. Connection with the CGLN 3-3 Charge Terms

It can be shown that the CGLN prescription for the determination of the 3-3 charge amplitudes is a special case of the Solution (6.1). This solution may be rewritten in the form

$$\phi_i^e = e^{\rho+i\delta}\left[\left(\frac{\omega_r-\omega}{\omega_m-\omega}\right)\ \phi_{iI}^e(\omega) + \left(\frac{\omega_m-\omega_r}{\omega_m-\omega}\right)\ \phi_{iI}^e(\omega_m)\right]$$

$$= e^{\rho+i\delta}\left[(e^{-\rho}\cos\delta)\ \phi_{iI}^e(\omega) + (e^{-\rho}\cos\delta)\ \left(\frac{\omega_m-\omega_r}{\omega_r-\omega}\right)\phi_{iI}^e(\omega_m)\right]$$

$$= e^{i\delta}\cos\delta\ \phi_{iI}^e(\omega) + e^{i\delta}\left(\frac{\cos\delta}{\omega_r-\omega}\right)\cdot(\omega_m-\omega_r)\ \phi_{iI}^e(\omega_m) \qquad (6.5)$$

where we have used Eq. (5.19) with $\eta(\omega) = 1$. The second term in the last expression is finite at resonance because cos δ vanishes there. In the limit $\omega_m \to \infty$, Eq. (6.5) reduces to the expression

$$\phi_i^e = e^{i\delta}\cos\delta\ \phi_{iB}^e \quad ; \quad i = 1,2 \qquad (6.6)$$

This result follows from the fact that the Born terms $\phi_{iB}^e(\omega)$ approach zero faster than $1/\omega$ as $\omega \to \infty$; consequently, all of the non-Born inhomogeneous terms evaluated by the procedure of preceding section vanish in the limit $\omega_m \to \infty$. The result (6.6), however, is just the CGLN expression for the charge terms and the difference between expressions (6.5) and (6.6) represents the present corrections to these terms.

VII. CROSS SECTIONS

The method we will use to obtain the photoproduction cross sections from the 3-3 amplitudes is similar to that of Gartenhaus and Blankenbecler.[9] It was shown by Ball that the differential cross section, summed over final nucleon spins and averaged over initial spins and photon polarizations can be written[11]

$$\frac{d\sigma}{d\Omega} = \frac{q}{k} \Big\{ |\mathcal{F}_1|^2 + |\mathcal{F}_2|^2 - 2 \cos\theta \, \mathrm{Re}\, \mathcal{F}_1^* \mathcal{F}_2$$

$$+ \frac{1}{2}(1 - \cos^2\theta) \Big[|\mathcal{F}_3|^2 + |\mathcal{F}_4|^2 + 2 \cos\theta \, \mathrm{Re}\, \mathcal{F}_1^* \mathcal{F}_4$$

$$+ 2\, \mathrm{Re}\, (\mathcal{F}_1^* \mathcal{F}_4 + \mathcal{F}_2^* \mathcal{F}_3) \Big] \Big\}$$

According to Eqs. (2.4) and (2.9), the amplitudes for photoproduction from protons are given by

$$\mathcal{F}(\gamma p \to \pi^o p) = \frac{1}{3}\, \mathcal{F}^{1/2} + \frac{2}{3}\, \mathcal{F}^{3/2} + \mathcal{F}^o$$

$$\mathcal{F}(\gamma p \to \pi^+ n) = \frac{\sqrt{2}}{3}\, (\mathcal{F}^{1/2} - \mathcal{F}^{3/2}) + \sqrt{2}\, \mathcal{F}^o$$

The resultant amplitudes $\mathcal{F}_i$, i = 1,2,3,4, can be evaluated from Eqs. (2.5-2.7) under the assumption that the contributions from the 3-3 state exhaust the dispersion integrals. The present scheme for evaluating the resultant amplitudes in the following. We may use the relation (2.5) between the $\mathcal{F}_i$ and the invariant amplitudes A_j to separate $\mathcal{F}$ into parts which arise from the Born terms (B), the right hand cut terms (R) associated with the denominators $(W')^2 - W^2$, and the remaining left hand cut terms (L) of the dispersion relations (2.7):

$$\mathcal{F} = \mathcal{F}_B + \mathcal{F}_R + \mathcal{F}_L$$

We subtract from these contributions to $\mathcal{F}$ the corresponding 3-3 projections (2.10) and finally add the solution functions by means of Eqs. (2.12) and (6.4). Thus, if P^{33} is the 3-3 projection operator on the integral expression for $\mathcal{F}$ and $\mathcal{F}^{33}$ is the corresponding solutions of the equations for $P^{33}\mathcal{F}$, then our resultant $\mathcal{F}$ is given by

$$\mathcal{F} = \mathcal{F}^{33} + (1-P^{33})\mathcal{F} \quad .$$

The general formulae resulting from this approach are given in the Appendix. Finally, $\mathcal{F}^{33}$ may be expressed in terms of the 3-3 phase shift by means of Eq. (6.4) while the nonsingular integrals defining $(1-P^{33})(\mathcal{F}_R + \mathcal{F}_L)$ can be evaluated with the help of Eq. (5.15) and Table II.

The coupling parameters that were used in the present calculation are the same as in Eq. (2.8'). The nucleon mass M was taken to be 6.73/R, where R is the ratio of the mass of the photoproduction pion to that of the charged pion ($R = 1$ for $\gamma p \to \pi^+ n$, $R = 0.967$ for $\gamma p \to \pi^0 p$). The positions of the resonance and amplitude zero must be similarly modified: The values used were $\omega_r = 2.07/R$ and $\omega_m = 6.38/R$. This procedure ensures, for example, that the total barycentric energy (in MeV) of the resonance is the same for all charge configurations. The conversion factor that gives the proper units to the cross section is

$$\left(\frac{\hbar}{m_\pi c}\right)^2 = (19.96/R^2) \times 10^{-30}\ \mathrm{cm}^2$$

where m_π is the pion mass. Finally, we remark that the numerical values in Table II were derived for the case of charged pion photoproduction; the different powers of R which appear there were determined from the requirement that each contribution to Ψ, ϕ_1, and $k\phi_2$ behaves like (unitless energy)$^{-3}$ in accordance with the behavior of the respective Born terms.

The results of the present calculations are compared with experiment in Figs. 7-11. These results are in satisfactory agreement with the main features of the data. Because of the large discrepancies among the results of the various experimental groups, however, it is impossible to draw from this comparison any conclusions about the quantitative accuracy of the present predictions for the 3-3 amplitudes. For the case of charged pion photoproduction, we show the matrix element squared at 90° (Fig. 7) and the differential cross section at 260 MeV (Fig. 8). It can be seen that the theoretical curves tend to agree best with the higher data in the vicinity of resonance.

The cross sections for the process $\gamma p \to \pi^0 p$ are reported in terms of the coefficients in the expansion

$$\frac{d\sigma^{\pi^0}}{d\Omega} = A + B \cos \theta + C \cos^2\theta + \ldots$$

These coefficients are shown in Figs. 9-11. The predictions for the coefficient A in Fig. 9 appear to be too large near resonance.

VIII. DISCUSSION AND CONCLUSIONS

Our primary aim has been to improve the calculation of the 3-3 photoproduction amplitudes from the CGLN dispersion relations. Except for a slight modification, the method used is that of the Omnès. To carry out this program we have made three basic assumptions, each of which compensates for some aspect of our present lack of knowledge of the photoproduction amplitudes. These assumptions are (1) that only the 3-3 amplitudes contribute appreciably to the dispersion integrals at energies below and in the vicinity of the 3-3 resonance, (2) that the phase of the 3-3 photoproduction amplitudes is the same as that of the 3-3 scattering amplitude for all physical energies, and (3) that the effect of the unknown high energy behavior of the phase on the solution for the 3-3 scattering amplitude can be represented by a zero in that amplitude at some energy ω_m in the physical region. The parameter ω_m in the last assumption is not arbitrary but is determined from the unitarity condition at resonance.

The principal result of this investigation is contained in the expression (6.4) for the ratios of the 3-3 photoproduction amplitudes to the scattering amplitude. The present predictions for these ratios are compared with those of CGLN in Figs. 5 and 6. The significant features that one observes from this comparison are (1) that the ratio associated with the magnetic dipole amplitude $M_{1+\mu}^{3/2}$ (half the sum of the solid curves in Fig. 5) is smaller than the corresponding CGLN result by 9% at resonance and decreases relative to the CGLN result as the energy increases[15] and (2) that the charge amplitudes do not vanish at resonance--although the amplitude $E_{1+e}^{3/2}$ does have a zero just above resonance.

A secondary result is the generalized effective range formula (5.20) for the 3-3 phase shift. As we indicated at the end of Sec. V, this formula can be used not only as a representation of this phase shift that is valid over a wide range of energies but also as a means for a precise determination of the resonance position and the pion-nucleon coupling constant.

The cross sections shown in Figs. 7-11 were calculated under the assumption that the only important singularities of the photoproduction amplitudes are the Born terms and the 3-3 contributions to the dispersion integrals. Examples of other singularities which may be important in the region of the 3-3 resonance are the contributions to the dispersion integrals of the higher energy pion nucleon resonances and effects due to the exchange of rho-mesons. While rho exchange contributes directly only to the isoscalar (0) photoproduction amplitudes, it also contributes to the isospin 3/2 scattering amplitude and thus will affect the present results for the 3-3 amplitude ratios. The contribution of the rho-meson to the 3-3 scattering amplitude, which has been treated by Frautschi and Walecka, has an energy dependence and sign consistent with a decrease in the 3-3 amplitude ratios below resonance.[14] This contribution can readily be incorporated into the present expressions for the scattering amplitude and photoproduction amplitude ratios, provided that a redetermination of the parameter ω_m is carried out.

APPENDIX

The decomposition of the total amplitude $\mathcal{F}$ into Born terms and terms arising from the right and left hand cuts of the CGLN dispersion relations is discussed in Sec. VII. The following expressions for these contributions are valid only under the assumption that the 3-3 resonance exhausts the dispersion integrals. The definitions of all quantities which appear below can be found in Sec. II. For the right hand cut contributions we have

$$\frac{1}{h}\,\mathcal{F}^{3/2}_{1R} = 3qk\cos\theta\left[\phi_2 - \phi_{2B} - \phi_{2L}\right]$$

$$+\frac{1}{\pi}\int_1^\infty d\omega'\left\{\left(\frac{3}{2WW'}\left[(\omega_q' + k)(W'W+M^2)-k\right] - \frac{k'(E'+M)}{W'+W}\right)\,\mathrm{Im}\phi_2'\right.$$

$$\left. - \left(\frac{E'+M}{W'+W} + \frac{3\omega_q(W+M)}{2\omega'W}\right)\,\mathrm{Im}\phi_1'\right\}$$

$$\frac{E+M}{qh}\,\mathcal{F}^{3/2}_{2R} = (E+M)\left[k(\phi_2 - \phi_{2B} - \phi_{2L}) + (\phi_1 - \phi_{1B} - \phi_{1L})\right]$$

$$+\frac{1}{\pi}\int_1^\infty d\omega'\left\{\left(\frac{(E+M)(WW'+M^2)}{2WW'} + \frac{k'(W'W-M^2+1)}{2W'W} - \frac{3\omega_q'k'-2M\nu_1)}{W'+W}\right)\,\mathrm{Im}\phi_2'\right.$$

$$\left. - \left(\frac{k(E+M)}{\omega\omega'} - \frac{W'W-M^2+1}{2W'W} + \frac{6M\nu_1}{\omega'(W'+M)}\right)\,\mathrm{Im}\phi_1'\right\}$$

$$\frac{1}{qh}\,\mathcal{F}^{3/2}_{3R} = -3\left[\phi_1 - \phi_{1B} - \phi_{1L}\right]$$

$$\frac{(E+M)}{q^2h}\,\mathcal{F}^{3/2}_{4R} = -\frac{1}{\pi}\int_1^\infty \frac{d\omega'}{W'+W}\,\mathrm{Im}\phi_1'$$

$$\mathcal{F}^{1/2}_{iR} = \mathcal{F}^{0}_{iR} = 0\;;\; i = 1,2,3,4\;.$$

The left hand cut contributions are even more involved than those given above. In order to condense the formulae somewhat, we introduce the following functions:

$$B(W';\nu_1) = (3(W'+M)(\omega_q'k'-2M\nu_1) - \omega'(E'+M)k')\ \text{Im}\phi_2'$$

$$- (\omega'(E'+M) - 12M^2\nu_1/\omega')\ \text{Im}\phi_1'$$

$$C(W';\nu_1) = (3(\omega_q'k'-2M\nu_1) + k'(E'+M))\ \text{Im}\phi_2' + ((E'+M) + 6M\nu_1/(W'+M))\ \text{Im}\phi_1'$$

$$D(W';W,\nu_1) = 3((W')^2 + W^2-2M^2-4M\nu_1) \quad .$$

The contributions from the left hand cuts can now be written as

$$\frac{1}{h}\ \mathcal{F}_{1L}^{3/2} = \frac{1}{\pi}\int_1^{\infty} \frac{d\omega'}{\omega D}\left\{\omega B + (\omega^2-4M\nu_1)\ C + 6M\nu_1\ (W'+M)\ \text{Im}\phi_1'\right\}$$

$$\frac{(E+M)}{qh}\ \mathcal{F}_{2L}^{3/2} = \frac{1}{\pi}\int_1^{\infty} \frac{d\omega'}{(W+M)D}\left\{-(W+M)\ B + ((W+M)^2-4M\nu_1)\ C + 6M\nu_1(W'+M)\ \text{Im}\phi_1'\right\}$$

$$\frac{1}{qh}\ \mathcal{F}_{3L}^{3/2} = \frac{1}{\pi}\int_1^{\infty} \frac{d\omega'}{D}\left\{3(W'-W+2M)\ \text{Im}\phi_1'-2C\right\}$$

$$\frac{(E+M)}{q^2h}\ \mathcal{F}_{4L}^{3/2} = \frac{1}{\pi}\int_1^{\infty} \frac{d\omega'}{D}\left\{3(W'+W+2M)\ \text{Im}\phi_1'-2C\right\}$$

$$\left.\begin{aligned}\mathcal{F}_{iL}^{1/2} &= 4\,\mathcal{F}_{iL}^{3/2}\\ \mathcal{F}_{iL}^{0} &= 0\end{aligned}\right\} \quad i = 1,2,3,4 \quad .$$

Finally, the corresponding expressions for the Born terms $\mathcal{F}_{iB}^{3/2,1/2,0}$ (i = 1,2,3,4) can be obtained readily from Eqs. (2.5-2.9).

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FOOTNOTES

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TABLE I

3-3 Phase Shift Above Resonance

Lab Pion Energy (MeV)	ω	Phase Shift δ_{33} (Degrees)
220	2.27	111.5 ± 5 [a]
294	2.66	128 ± 6 [b]
307	2.72	132.6 ± 2 [a]
310	2.74	133.2 ± 1.7 [c]
370	3.04	147.7 ± 3 [d]
395	3.16	147 ± 6 [b]
430	3.33	150.6 ± 3 [d]
460	3.47	160 ± 2 [d]
525	3.78	155.3 ± 2.5 [e]
600	4.11	157 ± 5 [d]

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TABLE II. Results for the 3-3 Amplitudes

Amplitude	Left Cut Contribution	β (Residue of Pole at ω=0)	Resultant Inhomogeneous Term at ω_m
$\psi \Big/ \left(\frac{g^2}{3M^2}\right)$	$\frac{0.1090}{\omega_r+\omega}$	-	0.2262 R
$\phi_1^\mu \Big/ \left(\frac{g(\mu_p-\mu_n)}{3M}\right)$	$\frac{0.0820}{\omega_r+\omega}$	-	0.1848 R
$\phi_2^\mu \Big/ \left(\frac{g(\mu_p-\mu_n)}{3M}\right)$	$\left(\frac{-0.0400}{\omega_r+\omega} + \frac{0.0479}{\omega}\right) R$	0.3845 R	$0.0973\ R^2$
$\phi_1^e \Big/ \left(\frac{eg}{6M}\right)$	$\frac{0.0054}{\omega_r+\omega}\ R$	-	$0.0242\ R^2$
$\phi_2^e \Big/ \left(\frac{eg}{6M}\right)$	$\left(\frac{-0.0066}{\omega_r+\omega} + \frac{0.0037}{\omega}\right) R$	$0.0505\ R^2$	$0.0097\ R^3$

Notes: Quoted results are for the case f^2-0.082, ω_r=2.07, and ω_m=6.38 .

The correction factor $R = \frac{\text{mass of final state pion}}{\text{mass of charged pion}}$ is discussed in the text (Sec. VII).

LIST OF ILLUSTRATIONS

Figure 8. Barycentric differential cross section at 260 MeV. The data notation is the same as in Fig. 7.

Figure 9. The coefficient A in the expansion $\frac{d\sigma}{d\Omega} = A+B \cos\theta + C \cos^2\theta + \ldots$ for the process $\gamma p \to \pi^0 p$.

Figure 10. The coefficient B for the process $\gamma p \to \pi^0 p$. The data notation is the same as in Fig. 9.

Figure 11. Coefficient C for the process $\gamma p \to \pi^0 p$. The data notation is the same as in Fig. 9.

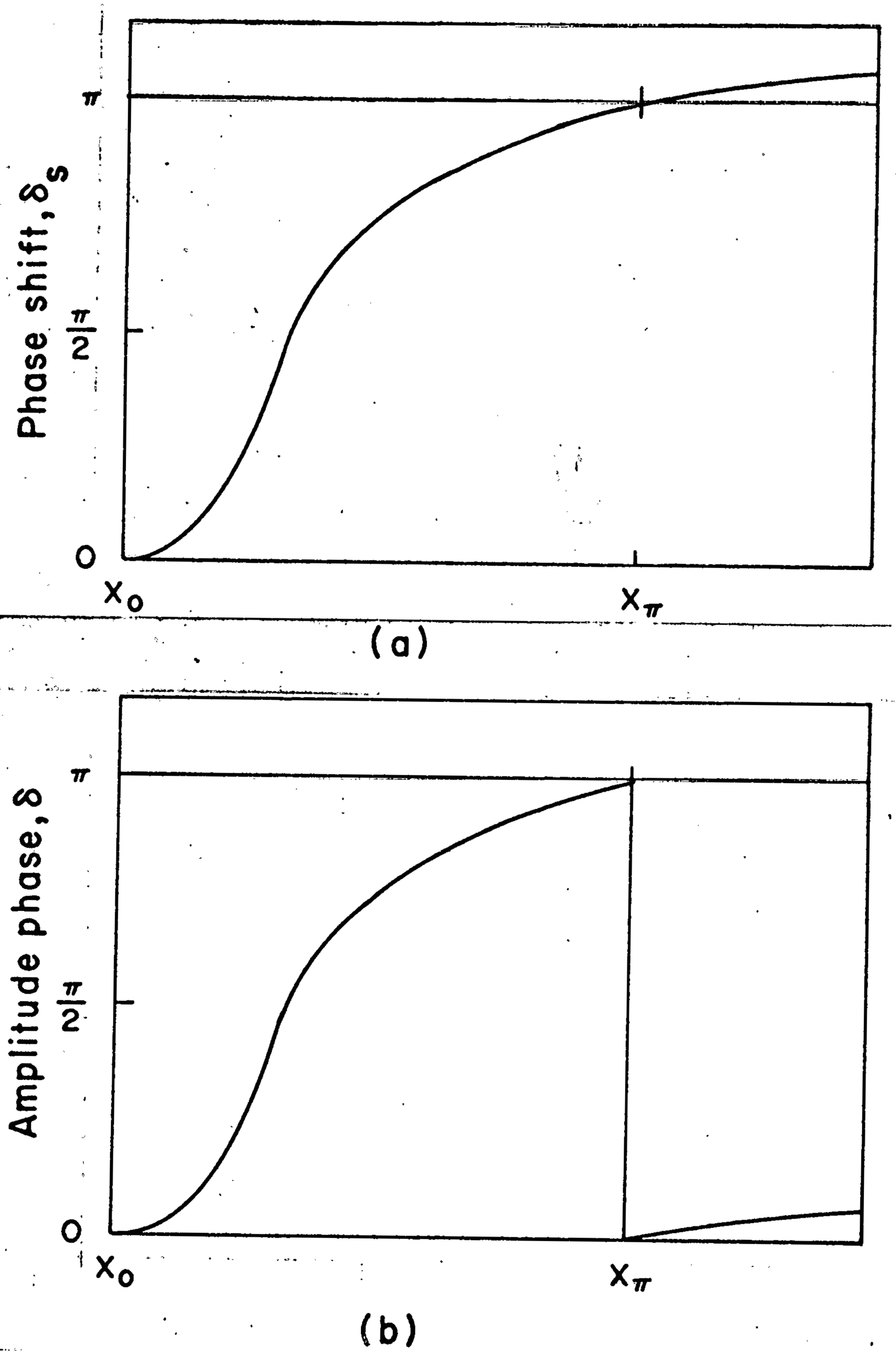

Fig. 1 Distinction between the phase shift δ_s, shown in (a) and the amplitude phase δ, shown in (b) when the former passes through π. The two phases can be taken as equal for $x < x_\pi$.

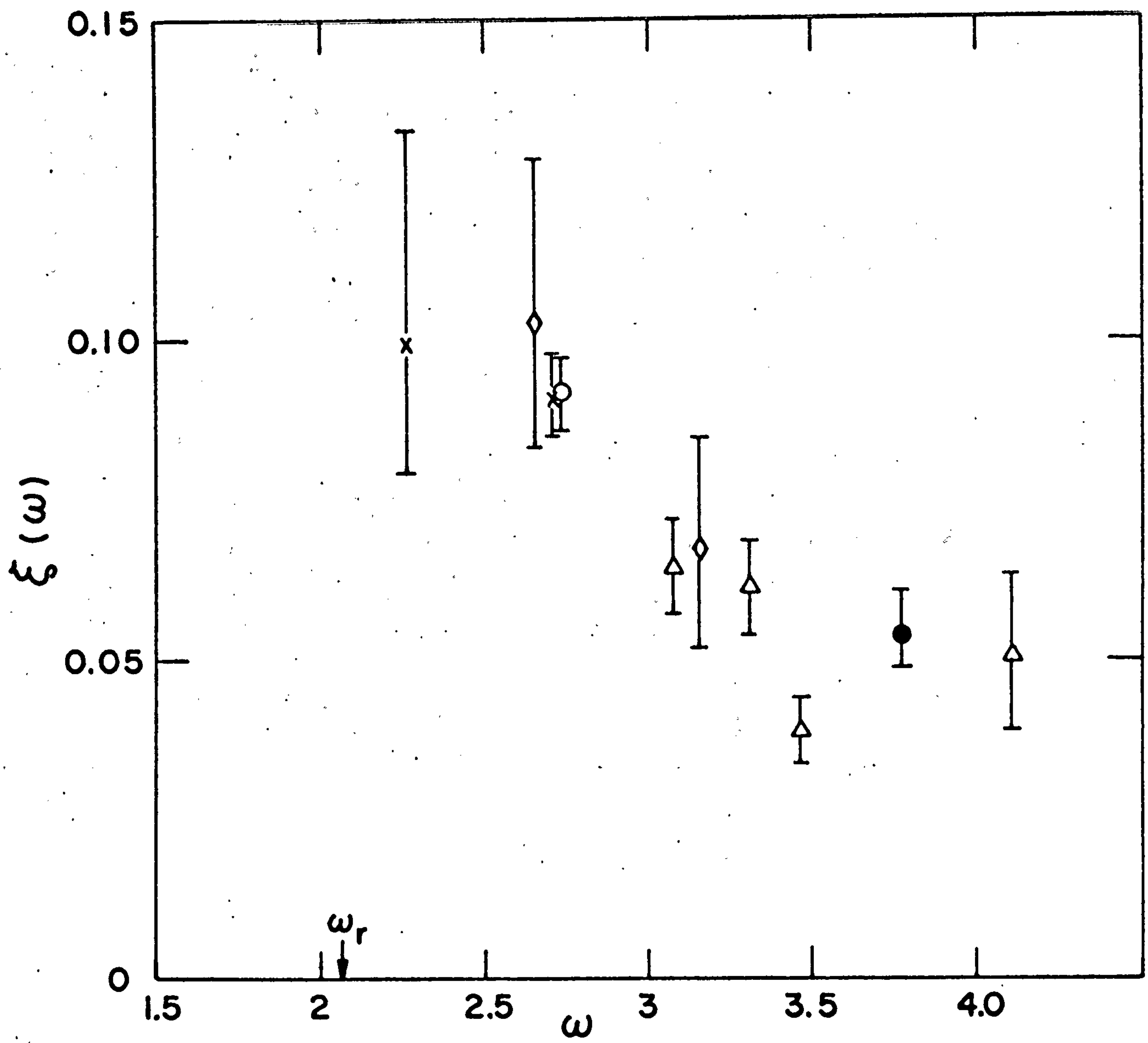

Fig. 2 Plot of $\xi(\omega) = \omega(\omega_r - \omega)/(\omega_r q^3 \cot \delta_{33})$ vs., ω. For the resonance energy we have used $\omega_r = 2.07$. The experimental points are those of Table I.

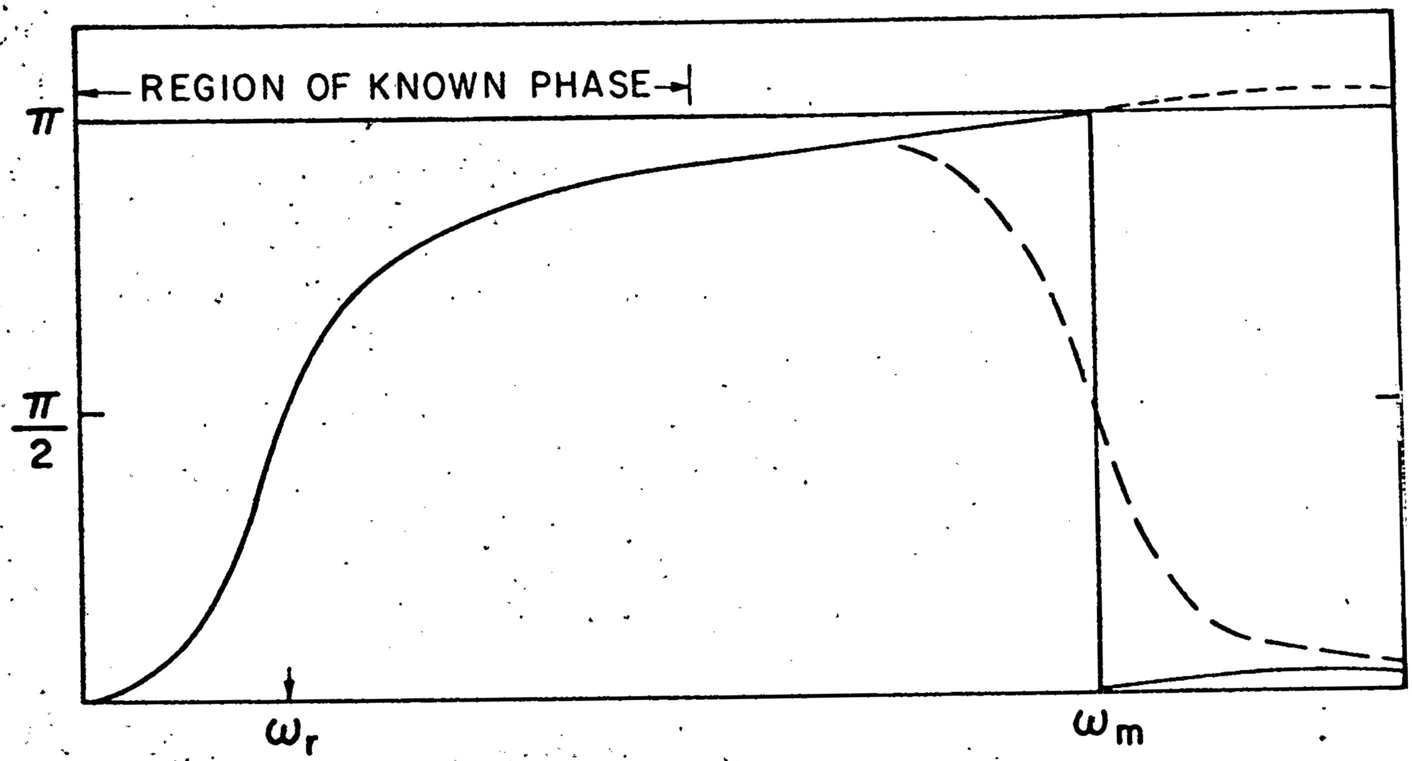

Fig. 3 Behavior of the amplitude phase above the region of known phase. The short dashed curve represents the extension of the phase shift under the assumption that it passes through π at ω_m, while the solid curve gives the corresponding amplitude phase. The dashed curve is the phase in the case where the phase shift drops rapidly toward zero.

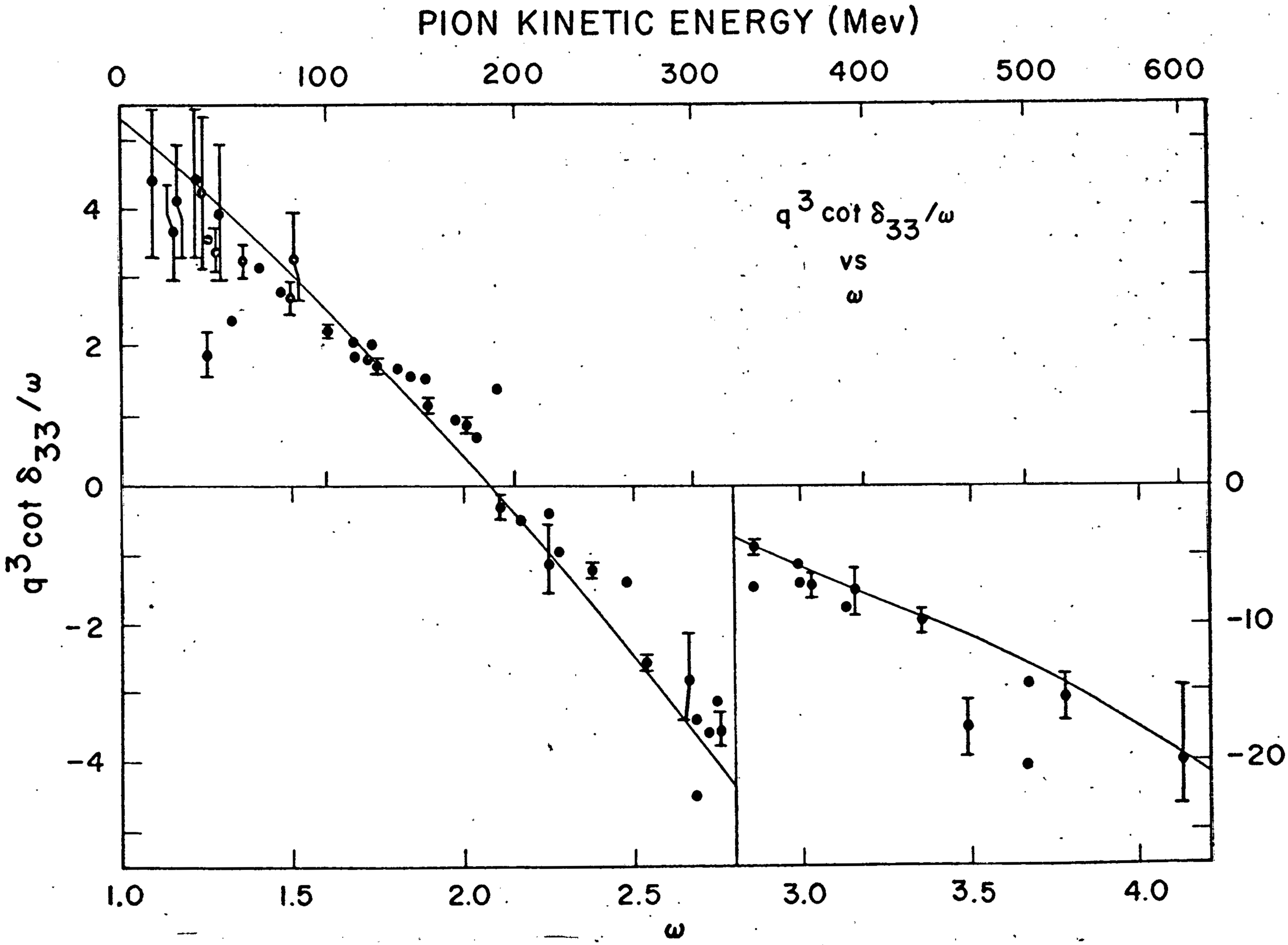

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Fig. 4 Chew-Low Plot: $q^3 \cot \delta_{33}/\omega$ vs., ω. The solid curve is the prediction of Eq. (5.20) for the values $\omega_r = 2.07$, $\omega_m = 6.38$, and $f^2 = 0.082$. The experimental points are from the compilation of McKinley.[4]

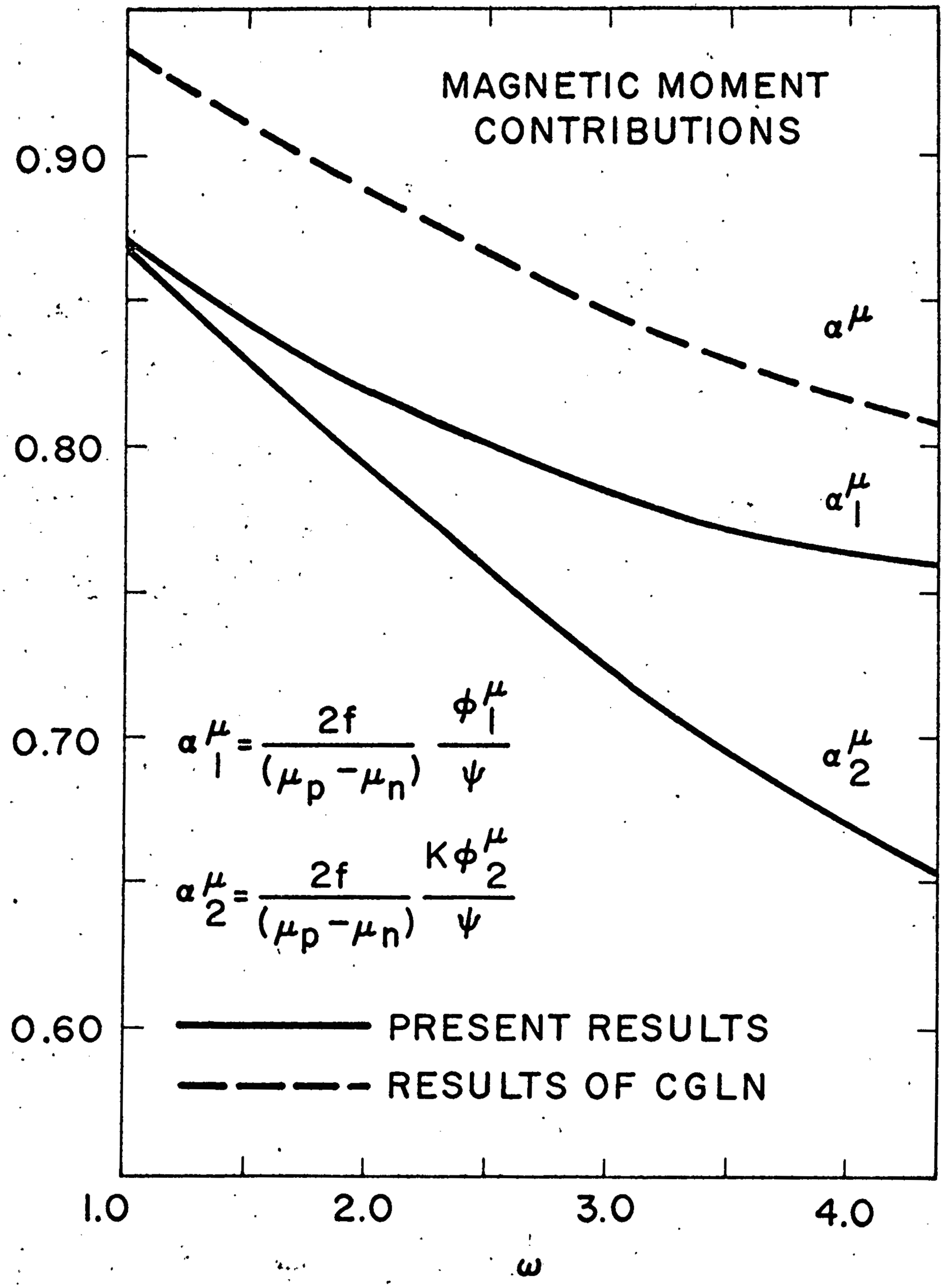

Fig. 5 Ratios of the photoproduction amplitudes generated by the total nucleon magnetic moment to the scattering amplitude. The solid curves are the predictions of Eq. (6.4). The CGLN results, which predict $\alpha_1^{\mu} = \alpha_2^{\mu}$, are

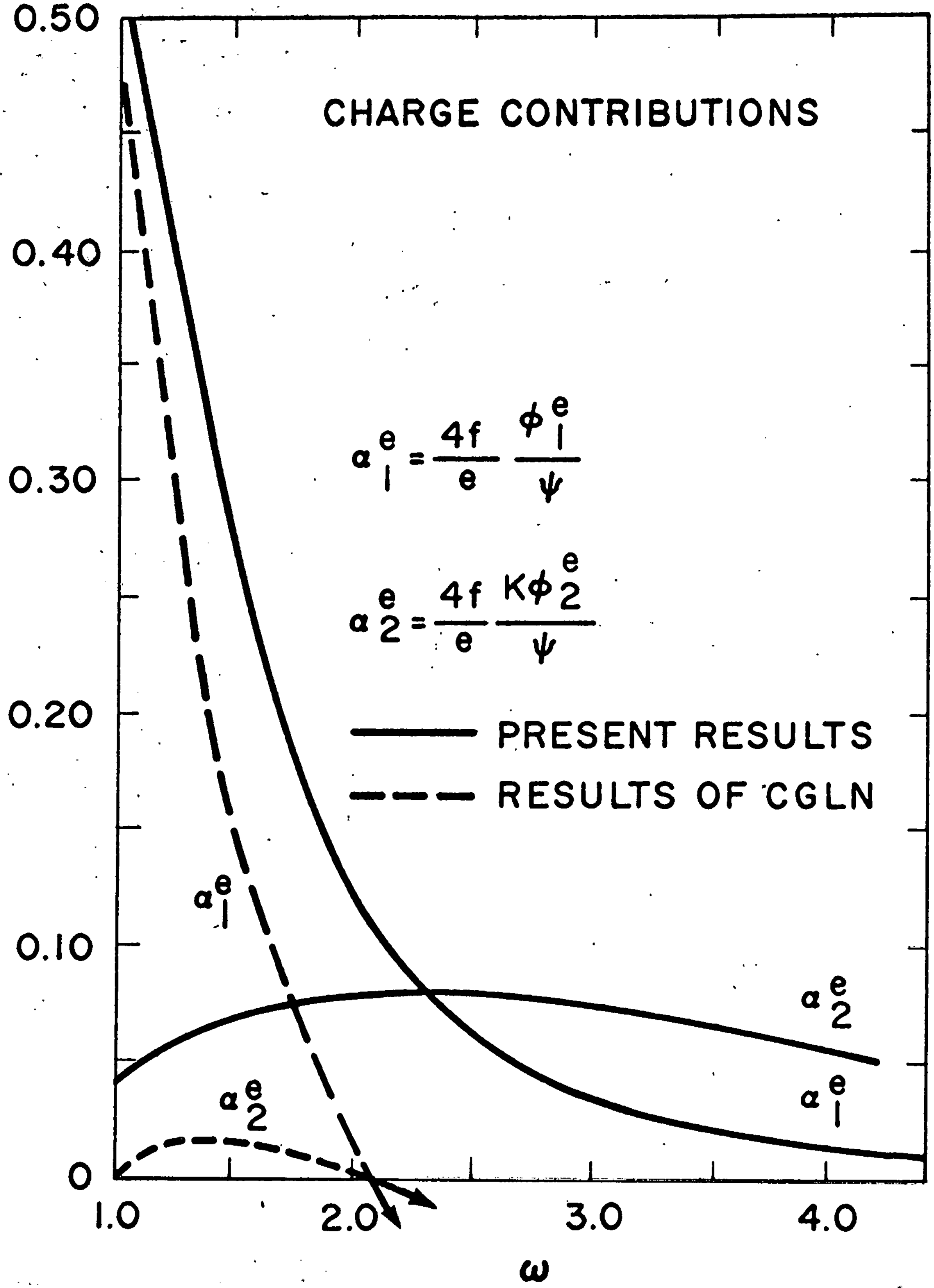

Fig. 6 Ratios of the photoproduction amplitudes generated by the nucleon charge to the scattering amplitude. The solid curves are the predictions of Eq. (6.4). The corresponding predictions of CGLN (dashed curves) are shown only up to the resonance position, where they vanish linearly.

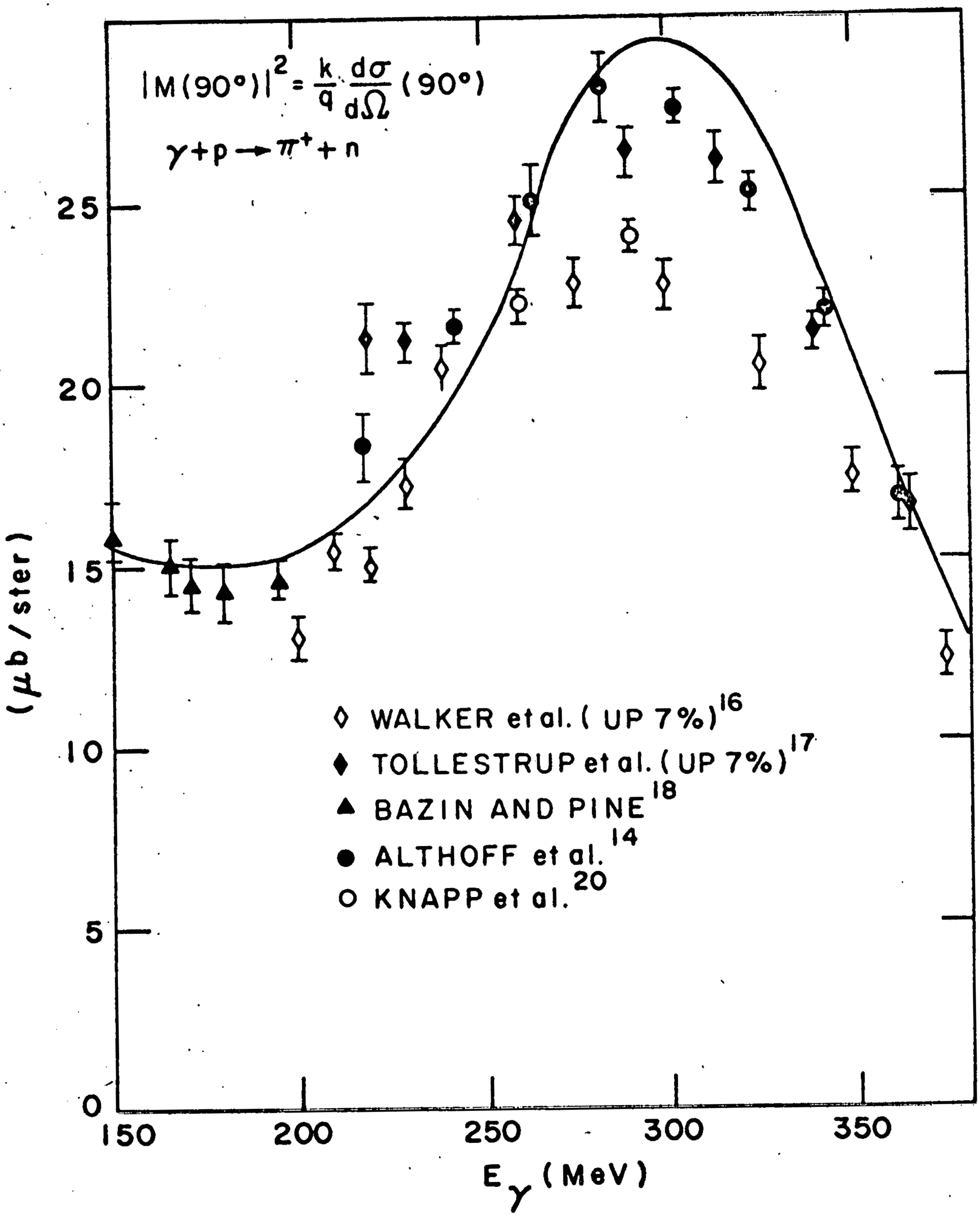

Fig. 7 Matrix element squared at 90° for π^+ photoproduction. The present predictions are compared with various experiments.

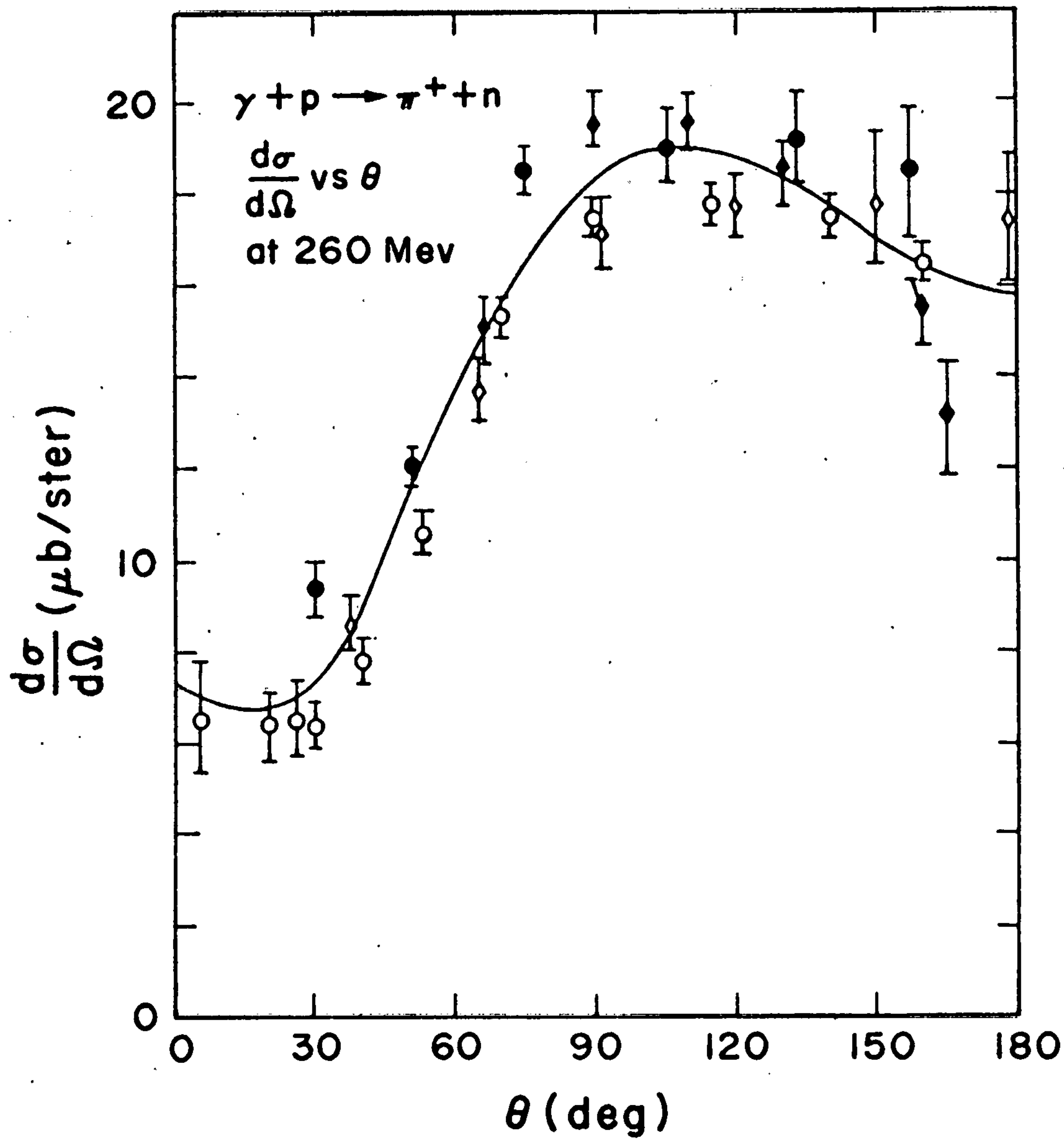

Fig. 8 Barycentric differential cross section at 260 MeV. The data notation is the same as in Fig. 7

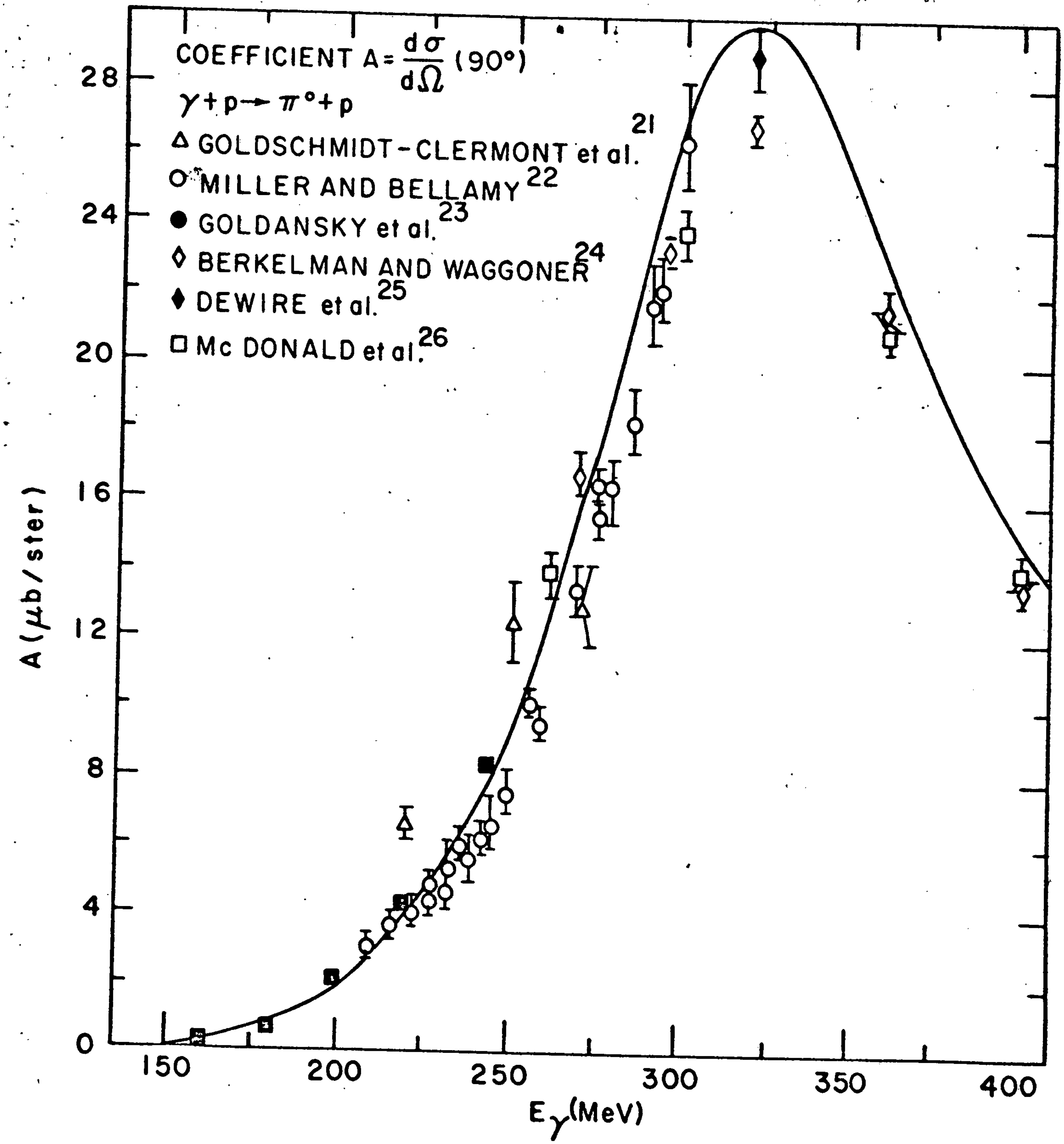

Fig. 9. The coefficient A in the expansion $\frac{d\sigma}{d\Omega} = A + B\cos\theta + C\cos^2\theta + \ldots$ for the process $\gamma p \rightarrow \pi^{0} p$.

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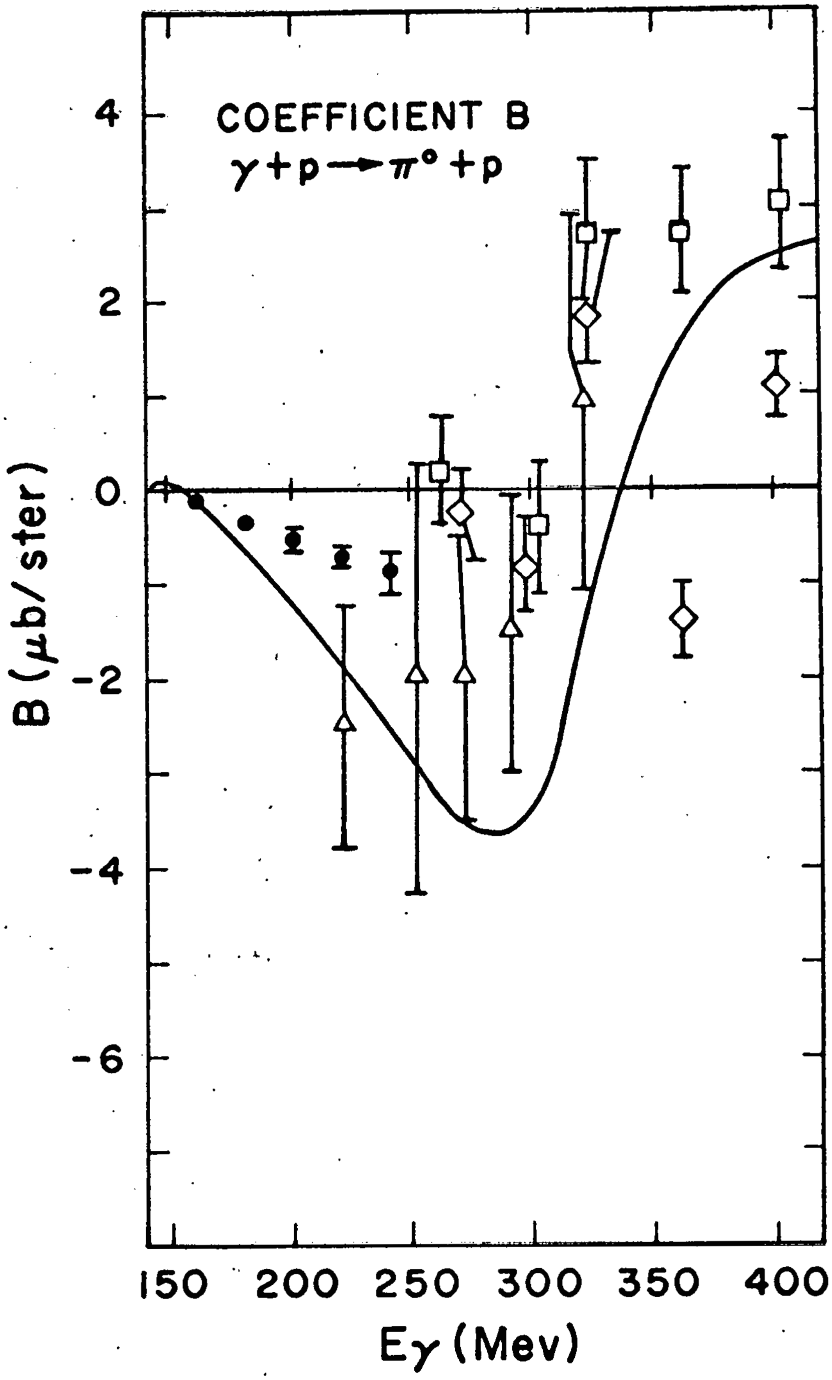

Fig. 10 The coefficient B for the process $\gamma p \to \pi^0 p$. The data notation is the same as in Fig. 9.

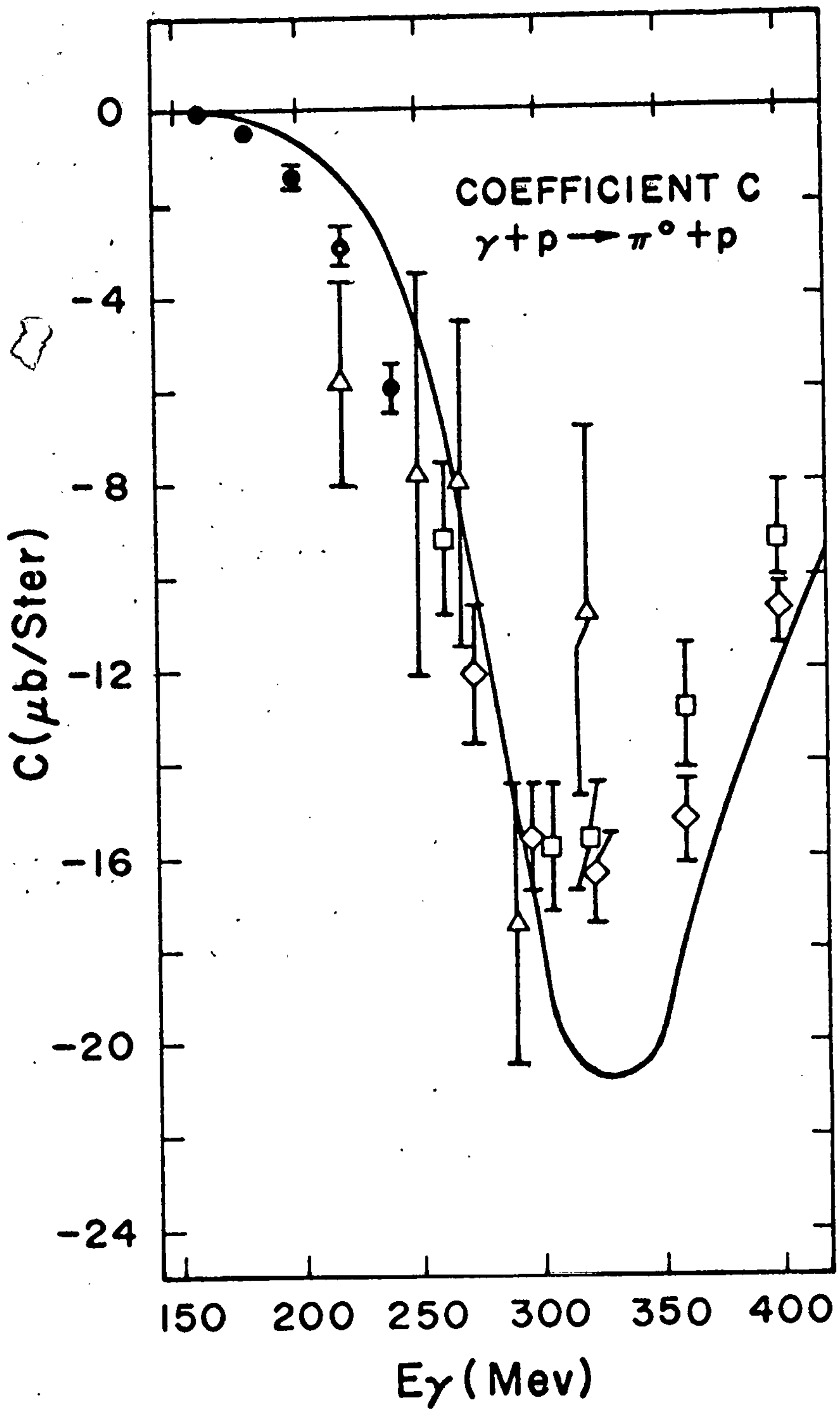

Fig. 11 Coefficient C for the process $\gamma p \rightarrow \pi^0 p$. The data notation is the same as in Fig. 9.