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Betatron Frequency Shifts Due to Image Fields

Robert L. Gluckstern
University of Massachusetts

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ABSTRACT

The variation of betatron frequency across a beam due to image fields is calculated for both circular and plane parallel beam pipes. An approximate interpolated expression is then constructed which agrees with both limits and with the result for a slightly elliptical pipe. The effect appears to be more serious for the ISR than for the design parameters of the ISA.

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Robert L. Gluckstern
University of Massachusetts

I. INTRODUCTION

For high circulating currents, image effects in the electric and magnetic boundaries cause large shifts of the operating point in the v_x, v_y space. Laslett⁽¹⁾ has treated the case of an elliptical electric boundary and has derived expressions for the fields in terms of complicated elliptic functions and their derivatives. These expressions have been expanded to give image effects in concise form only at the center of the beam pipe. In addition Laslett⁽²⁾ has performed some mesh computation for beams and pipes of different shapes and sizes.

Our goal here is to obtain the variation of betatron frequency within the beam where this effect may be important. Rather than repeat or extend Laslett's calculations and computations, we have instead considered two limiting cases. An interpolated expression is then constructed which satisfied both limits and is likely to give a reasonable approximation between these limits. The two limiting cases are a) a circular pipe with an eccentric beam and b) parallel planes. These are considered to be limits of the following geometry.

Specifically, for the circular case we will take $a = b$ and for the parallel plane case we will take $a = \infty$. In addition we will check this interpolated

result by comparing it to a calculation for a slightly elliptical geometry.

At extremely relativistic velocities the magnetic and electric image effects cancel to within a factor γ^{-2} for all except the static image, where the static electric and magnetic images occur in different structures. Our calculation assumes a distant static magnetic boundary and therefore is restricted only to the static electric case.

II. CIRCULAR CASE ($a = b$)

The image of a line charge in a circular boundary is simply an equal and oppositely charged line located on the same azimuth at a radius

$$R = b^2 / \sqrt{p^2 + q^2} \quad (1)$$

with electrostatic potential given by

$$\varphi_{\text{image}} = \frac{\tau}{4\pi\epsilon_0} \ln \left[\left(\frac{b^2 p}{p^2 + q^2} - x \right)^2 + \left(\frac{b^2 q}{p^2 + q^2} - y \right)^2 \right] \quad (2)$$

We will express our changes in betatron frequency in terms of the two parameters

$$A_1(x) = \frac{4\pi\epsilon_0}{2\tau} \left. \frac{\delta^2 \varphi_{\text{image}}}{\delta y^2} \right|_{y=0, q=0}, \quad \frac{4\pi\epsilon_0}{2\tau} \left. \frac{\delta^2 \varphi_{\text{image}}}{\delta y \delta q} \right|_{y=0, q=0}$$

The corresponding change in the betatron frequency is then given by

$$2\nu_y \Delta\nu_y = K A_{1,2}(x) \quad (4)$$

where

$$K = \frac{NRr_p}{\pi\beta^2\gamma} = \frac{60 \text{ (ohms)} eIR^2}{\beta^2 pc} \quad (5)$$

with N being the number of circulating protons, I the circulating current, $2\pi R$ the circumference of the ring, and r_p the "proton radius" ($= e^2/4\pi\epsilon_0 Mc^2$)

One can readily obtain $A_{1,2}(x)$ from (2). The variation of betatron frequency with position within the beam is most easily described by obtaining the values and derivatives of $A_{1,2}(x)$ at the beam center, $x = p, q = 0$. Specifically, for the

circular pipe,

$$A_1(p) = \frac{p^2}{(b^2 - p^2)^2}, \quad A_2(p) = -\frac{b^2}{(b^2 - p^2)^2} \quad (6)$$

$$A_1'(p) = \frac{2p^3}{(b^2 - p^2)^3}, \quad A_2'(p) = -\frac{2b^2 p}{(b^2 - p^2)^3} \quad (7)$$

$$A_1''(p) = \frac{6p^4}{(b^2 - p^2)^4}, \quad A_2''(p) = -\frac{6b^2 p^2}{(b^2 - p^2)^4} \quad (8)$$

III. PLANE PARALLEL CASE ($a = \infty$)

The image of a line charge between two parallel plates is a succession of line charges of alternating sign located at $x = b$ and $y = \pm 2nb + (-1)^n q$. This leads to expressions involving zeta functions which are written in closed form ($\pi^2/12$, $\pi^4/96$, etc.) only for $q = 0$. To within an accuracy of about 20% one can neglect all terms except the images at $y = \pm 2b - q$ in order to obtain simple expressions for the change in frequency and its derivatives. Specifically one obtains

$$A_1(p) = A_2(p) = -\frac{1}{2b^2} \quad (9)$$

$$A_1'(p) = A_2'(p) = 0 \quad (10)$$

$$A_1''(p) = A_2''(p) = \frac{3}{4b^4} \quad (11)$$

IV. INTERMEDIATE GEOMETRY

It is possible to construct simple expressions which reduce to (6) and (9) in the appropriate limits. Specifically

$$A_1(p) \approx \frac{-\frac{1}{2b^2} + \frac{1}{2a^2} + \frac{p^2}{a^2 b^2}}{\left(1 - \frac{p^2}{a^2}\right)^2} \quad (12)$$

$$A_2(p) \approx \frac{-\frac{1}{2b^2} - \frac{1}{2a^2}}{(1 - \frac{p^2}{a^2})^2} \quad (13)$$

reduce to (6) and (9) in the limits $a \rightarrow b$ and $a \rightarrow \infty$ respectively. Similar expressions can also be constructed for $A'_{1,2}(p)$ and $A''_{1,2}(p)$ if desired.

Equation (12) also agrees closely with the result near $b = a$ which can be derived to first order in $a-b$ by requiring that the potential vanish on the approximately elliptical surface given in polar coordinates by

$$r = \frac{a+b}{2} + \frac{a-b}{2} \cos 2\theta = R(1 + \epsilon \cos 2\theta) \quad (14)$$

where

$$R = (a+b)/2, \quad \epsilon = (a-b)/2.$$

The potential for the line charge τ at $x = p, y = 0$ in free space is readily expanded as

$$\varphi_{\text{direct}} = \frac{2\tau}{4\pi\epsilon_0} \left[\ln r - \sum_{n=1}^{\infty} \frac{p^n}{nr^n} \cos n\theta \right] \quad (15)$$

The image potential is regular at $r = 0$ and therefore can be written as

$$\varphi_{\text{image}} = \frac{2\tau}{4\pi\epsilon_0} \left[B_0 + \sum_{n=1}^{\infty} B_n \frac{r^n}{nR^n} \cos n\theta \right] \quad (16)$$

If one expands to first order in ϵ , the requirement that $\varphi_{\text{total}} = 0$ on the boundary given by (14) reduces to

$$\begin{aligned} \ln R - \epsilon \cos 2\theta + \sum_{n=1}^{\infty} \frac{p^n}{R^n} \left(\frac{\cos n\theta}{n} - \epsilon \cos 2\theta \cos n\theta \right) \\ = B_0 + \sum_{n=1}^{\infty} \frac{B_n \cos n\theta}{n} + \cos 2\theta \cos n\theta \end{aligned} \quad (17)$$

To lowest order in ϵ , one finds the usual result

$$B_n = \frac{p^n}{R^n}, \quad n \geq 1. \quad (18)$$

To first order in ϵ , one then has

$$B_1 = \frac{p}{R} - \epsilon \frac{p}{R} - \epsilon \frac{p^3}{R^3} \quad (19)$$

$$B_n = \left(\frac{p}{R}\right)^n - n\epsilon \left(\frac{p}{R}\right)^{n-2} - n\epsilon \left(\frac{p}{R}\right)^{n+2}, \quad n \geq 2. \quad (20)$$

Since only $n \geq 2$ terms enter into the betatron frequency calculation, one can write, apart from a constant term and a uniform field term,

$$\varphi_{\text{image}} = \frac{2\tau}{4\pi\epsilon_0} \left[\left(\frac{rp}{R^2}\right)^n \cos n\theta \left(\frac{1}{n} - \epsilon \frac{p^2}{R^2} - \epsilon \frac{R^2}{p^2} \right) \right] \quad (21)$$

$$\approx - \frac{2\tau}{4\pi\epsilon_0} \left[\frac{rp}{R^2 \left(1 + \epsilon \frac{p^2}{R^2} + \epsilon \frac{R^2}{p^2}\right)} \right]^n \frac{\cos n\theta}{n} \quad (22)$$

corresponding to a displacement of the image charge from $R_1 = R^2/p$ to

$$R'_1 = \frac{R^2}{p} \left(1 + \epsilon \frac{p^2}{R^2} + \epsilon \frac{R^2}{p^2}\right). \quad (23)$$

From (23) one finds for the equivalent betatron frequency shift at $x = p, y = 0$

$$A_1(p) = \frac{1}{(R'_1 - p)^2} \approx \frac{p^2}{(R^2 - p^2)^2} \left[1 - 2\epsilon \frac{(R^4 + p^4)}{p^2(R^2 - p^2)} \right] \quad (24)$$

Equation (12) can also be put into the form of Equation (24) by setting $q = 0$, $a = R(1+\epsilon)$, yielding exactly the same result as in (24), making it quite likely that (12) and (13) are adequate approximations for our purposes.

V. NUMERICAL VALUES

A. ISR

If one takes the following parameters

$$a = 8 \text{ cm}, \quad b = 2.5 \text{ cm}, \quad p \leq 3 \text{ cm},$$

it is clear from (12) and (13) that the parallel plane geometry dominates. In this approximation it would be a simple matter to include a finite distribution for the charge, as well as secondary images in the walls. However, one can obtain results from (9), (10) and (11) which are sufficiently accurate for our

purposes.

$$\begin{aligned}
 A_1 &= A_2 \simeq -0.08 \text{ cm}^{-2} \\
 A'_1 &= A'_2 = 0 \\
 A''_1 &= A''_2 \simeq 0.02 \text{ cm}^{-4}
 \end{aligned} \tag{25}$$

The seriousness of the values in (25) is connected with the variation in v_x and v_y across the stack. Landau damping requirements lead to a tune shift of ± 0.03 at $x = \pm 2.4$ cm at ISR. For $I = 10$ amperes at $pc = 22$ GeV, $R = 150$ meters, and $v \simeq 8.6$, one finds from (4), (5) and (25) a tune shift from center to edge given by

$$\Delta v^{\text{image}} \simeq \frac{60 \times 10 \times 2.25 \times 10^8}{22 \times 10^9} \frac{(2.4)^2}{2} \frac{1}{17.2} \times 0.02 \simeq 0.02,$$

comparable with the Landau damping tune shift.

B. ISA

The vacuum chamber in ISA is circular, and the relevant results are those contained in (6), (7), (8). For

$$b = 2.5 \text{ cm}, \quad p \leq 1 \text{ cm}$$

one finds

$$\begin{aligned}
 A_1 &\leq 0.04 \text{ cm}^{-2} & A_2 &\simeq -0.22 \text{ cm}^{-2} \\
 A'_1 &\leq 0.014 \text{ cm}^{-3} & A'_2 &\leq -0.09 \text{ cm}^{-3} \\
 A''_1 &\leq 0.008 \text{ cm}^{-4} & A''_2 &\leq -0.05 \text{ cm}^{-4}
 \end{aligned} \tag{26}$$

For $I = 15$ amperes at $pc = 30$ GeV, $R = 350$ m, $v = 18.3$, one finds a tune shift from center to edge for a beam of half width 1 cm

$$\Delta v^{\text{image}} \simeq 0.004,$$

significantly smaller than for the ISR.

If one considers the effect of narrow uniform beam of width $2f$ located at the center of a circular pipe, the image field in (2) can be readily used to obtain

$$A_1(0) \approx \frac{f^2}{3b^4}$$

$$A_1'(0) \approx 0$$

$$A_1''(0) = \frac{6}{5} \frac{f^4}{b^8}$$

For $b = 2.5$ cm, $f = 1$ cm, one finds

$$A_1'' < 10^{-3} \text{ cm}^{-4},$$

which is much smaller than the value of A_1'' in (26).

VI. ACKNOWLEDGMENTS

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References

1. L.J. Laslett, Proceedings of the 1963 Summer Study on Storage Rings at Brookhaven National Laboratory, p. 325 ff.
2. L.J. Laslett, University of California Report, UCID-10162, June 1966.