A BIRDS-EYE VIEW OF S-MATRIX THEORY*

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MASTER

I. The scattering matrix, or S-matrix, provides the mathematical description of idealized scattering experiments. The general program of trying to understand the properties of the S-matrix and the relations among these properties is called "S-matrix theory". S-matrix theory is therefore not really a "Theory" in the traditional sense, but a collection of problems; this fact often causes some confusion. It would be impossible to go into any detail about the mathematical aspects of these problems in this report, so I shall only try to describe how varied these problems are.

The problems of S-matrix theory have many different facets and they are being worked on by people with many different points of view, so to give any coherent picture of S-matrix theory, it is necessary to invent some scheme of classification. My classification (Table I) is meant to be a very rough one, and is given only to emphasize the variety of the problems. I hope it does not obscure their underlying unity.

I should like, first, to distinguish between the descriptive aspects of S-matrix theory - that is, the description or enumeration of their properties - and constructive S-matrix theory, by which I mean the construction of an actual S-matrix from some simple rules. I also wish to distinguish between formal and empirical S-matrix theory. The first refers to the proving of theorems, and the second is supposed to be more directly related to the interpretation of experimental data.

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* For presentation at the Third Eastern U. S. Theoretical Physics Conference, October 31, 1964.

College Park, Md.

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TABLE I

S- Matrix Theory	Descriptive	Constructive
Formal	<u>Questions</u> Analyticity Singularity Structure Asymptotic Behavior <u>Methods</u> Axiomatic Field Theory Potential Scattering Perturbation Expansion	<u>Questions</u> Approximation Axiomatization Inversion <u>Methods</u> Successive Inclusion of Singularities
Empirical	<u>Questions</u> Parametrization and Correlation of Data Verification of Principles <u>Methods</u> Application of Semi- Empirical Formulas Dispersion Relations	<u>Questions</u> Dynamical Models Compositeness (Definition) Existence and Uniqueness <u>Methods</u> See rest of table

II. The first corner of my table (Formal-Descriptive) contains work on the deduction of mathematical theorems about the S-matrix in which one starts from some theory with local fields. This includes the work of most S-matrix theorists. The main question concerns the existence and form of dispersion relations in any of the variables one has at one's disposal. The work on "Axiomatic Field Theory" falls into this class, but this work is so hard that I am glad that it is not my job to review it. In order to meet their responsibility to provide a mathematical system for the analysis of experimental data, theorists have been forced to also consider other starting points, such as non-relativistic potential scattering and the formal perturbation expansion of the covariant theory (perhaps with some partial summations). One can deduce from these bargain-basement axioms many fascinating theorems, but it should be remembered that in the long run such results can serve only as guides to intuition.

It has been learned so far that the analyticity properties of a bargain-basement S-matrix do permit the existence of dispersion relations in any number of variables. Let us denote by $T(\mathbb{Z})$ a "transition amplitude" or connected part of the S-matrix, chosen to be free of kinematical singularities. The general form of a dispersion relation is then:

$$T(z) = \sum_{\text{poles}} \frac{R_i}{z_i - z} + \frac{1}{2\pi i} \int_{|z'| < R} \frac{\text{disc}[T(z')]}{z' - z} dz' + \overline{X}_R(z), \quad (1)$$

where z is any energy or momentum-transfer variable and $\overline{X}_R(z)$ is some function which represents the contribution of the asymptotic

region. The residues R_1 and $\text{disc}[T(z)]$ can be expressed in terms of other transition amplitudes, although in general these must be analytically continued in some way out of their physical regions.

The obvious question is, what is X ? That is, how do S -matrix elements behave for large energies and momenta? Although many people have studied this question, the situation is still somewhat confused. Some interesting results have been obtained through the use of just unitarity and some weak analyticity properties (by Froissart, Greenberg and Low, and Kinoshita and Khuri¹). However, it seems clear that potential scattering and perturbation theory will give mostly misleading information, and it may be necessary to invent new kinds of model theories before further progress can be made.

An unsolved problem in perturbation theory concerns a conjecture which almost everyone makes implicitly. Consider an arbitrary transition amplitude. Do the Landau singularities associated with sufficiently complicated graphs lie very far from the low-energy part of the physical regions? This problem is a generalization of what is often called "majorization"; to state it in a more precise way is a non-trivial question in itself because what one really wants is some quantitative estimates. At the same time, one might like to try to restrict the singularities to some known domain, as in the Mandelstam representation.

III. Now let's go to the Formal-Constructive corner of my table. I include here first the problem of finding better approximation

methods. As you know, strong-interaction physics is still a qualitative science, and while many interesting qualitative results have been obtained, we shall ultimately have to crash the number barrier and make it quantitative.

Next I mention the "axiomatization conjecture"--the conjecture that one can find a few reasonable general properties of the S -matrix (such as unitarity), from which the remaining more specific information mentioned above can be derived. In view of the difficulties encountered in trying to base all results on a field-theory foundation, it is obvious that people should also try the alternate method of deducing relations among the various properties of the S -matrix. As in any axiomatization, the big problem is being sure that the axioms are independent and also consistent with each other. One stumbling block seems to be the question of how to define the idea of "maximal analyticity" without getting enmeshed in either circular or open-ended logical arguments.

The work of Gunson and Olive² suggests a way to postpone answering this question while one settles many important preliminary mathematical problems. They exploit the fact that the S -matrix for reactions involving many incoming particles possesses much interesting structure - poles, "anomalous" thresholds, etc. - even in the physical region, where direct use can be made of unitarity and of the cluster properties of the S -matrix. There are indications that one can give a reasonable definition of some sort of "minimal analyticity", involving analyticity just in the neighborhood of a physical region,

and obtain from such a notion a rather complete understanding of the singularities found in the physical region, including their locations and strengths³. If such a program can be carried through, we might go on to the more important question of finding out what additional assumptions have to be made in order to obtain crossing symmetry; this problem was also attacked directly by Gunson and Olive.

A problem related to axiomatization which has been almost entirely neglected is that of inversion (Dyson's problem⁴) - the task of constructing a field theory which is equivalent to a given S-matrix. This problem has been thoroughly studied in potential scattering. One question of particular interest is the following: Given an S-matrix with the analyticity properties inferred from, say, the perturbation expansion, does there exist an equivalent local Lagrangian field theory? Is it unique? Would the assumption that an equivalent local theory exists restrict the S-matrix in some way that has not yet been thought of? Another way to express these questions is to ask whether the S-matrix possesses an off-mass-shell extension. Some people (including myself) have been interested in these questions from the point of view of finding new and, hopefully, better ways to calculate, but so far the discussions have been too formal to settle the questions of existence and uniqueness.

IV. The empirical-descriptive aspect of the S-matrix includes its use in parametrizing and correlating experimental data. We note here the following invaluable concepts:

- 1) The Breit-Wigner resonance formula
- 2) The effective range formula
- 3) Polological formulas (small-angle scattering and production)
- 4) Regge families (of bound states and resonances)

It would not be appropriate to talk about these topics without discussing the actual phenomena to which they apply. It may be hoped that there are additional useful relations that are still waiting to be discovered. From a more general point of view, we should include here even such topics as the optical theorem and the partial-wave expansion. Finally, it should be remembered that the assumptions underlying S-matrix theory must be subject to experimental test. In fact, this is one of the historical purposes of dispersion relations.

V. It is not really possible to make a sharp distinction between the parametrization of experimental data and the construction of dynamical models. I shall draw a line in the following way: The expression of the S-matrix in terms of the masses and coupling constants of the particles I refer to as descriptive. The further problem of explaining why the masses and coupling constants have the values they do is put into the "constructive" category.

Roughly speaking, a dynamical model is a statement about the forces which act between certain particles, in terms of which one hopes to calculate the properties of bound states or resonances. The crossing symmetry of the S-matrix assures that the finite-range

forces are given by the exchange of particles, which might themselves be bound states or resonances. There is therefore a lot of feedback in the system - one speaks of "bootstrap dynamics". The logical structure is outlined in Figure 1.

Figure 1

If we now introduce the further assumption that all particles can be considered to be composite, we have a completely self-contained system of particles self-consistently "bootstrapping" each other. This assumption of compositeness is clearly related to the asymptotic behavior, but the precise nature of the relation is not understood.

There are several obvious questions which can be asked about bootstraps:

- 1) What is a good definition of a composite particle?
- 2) Why should all particles be composite?
- 3) How many self-consistent solutions exist?
- 4) Do any of the solutions agree with nature?

The definition one chooses for compositeness depends on what one puts into the box labelled "dynamics" in Figure 1. For example, in the N/D method of calculation, a composite particle comes from a zero of D, where one assumes D has certain asymptotic behavior.⁵ If one assumes that the transition amplitudes have extensions off of the mass shell, compositeness can be taken to imply certain asymptotic behavior far from the mass shell. In any case, at present we are reduced to exploiting analogies with the familiar properties of potential scattering in order to obtain a mathematical equation which expresses compositeness.

There seems to be no reason to doubt that many of the particles we see: the deuteron, the $J^P = \frac{3}{2}^+$ resonances, and even the baryons as well as the boson states, can be considered, at least to a very good approximation, to be composite particles. Moreover, as long as only the long range forces are taken into account, it doesn't really matter very much how composite particles are defined. The questions of principle obtrude when we envisage calculating with precision, and including all the particles in the bootstrap dynamics.

An appealing feature of the hypothesis of universal compositeness is that arbitrary parameters are eliminated and theorists are

given a definite program which will either succeed or fail. Can a more fundamental motivation be found? This would involve a proof that the equations used to define compositeness are necessary consequences of unitarity. On the other hand, perhaps a "compositeness condition" can be derived from the requirement that a local field theory exists which gives rise to finite results.

The possibility of uniqueness gives rise to most of the mystery about the bootstrap approach. It is an interesting exercise to try to describe this uniqueness question in field-theoretical language. There are several different views that one might take.

- 1) Dispersion relations are just a lot of identities and do not obviously restrict the theory. Moreover, any given local Lagrangian might lead to finite results if one just calculated in a clever way.
- 2) Properties of Lagrangians are unfortunately not trivially related to the properties of states. Particle states, for example, which are the elementary excitations of the vacuum state, have properties which are strongly affected by the structure of that stable ground state. (This corresponds to degenerate-vacuum theories.)
- 3) It is well-known to be hard to find a finite theory; perhaps there are not very many. Remember that we must also include the stringent condition that the vacuum state, the state of

zero total four-momentum, be actually the ground state, the state of lowest energy. A good way to learn the properties of a theory which successfully avoids all difficulties is just to start drawing conclusions under the assumption that these difficulties have been avoided.

- 4) There are many underlying Lagrangians that might work. At any event, a theory which confines itself to the strong-interactions, even with the inclusion of electromagnetic effects, can only be an approximate theory and leaves out many processes which might be extremely important at energies of, say, 10^2 BeV. However, since the structures of light particles tend to be strongly affected by the properties of virtual states which also have low energies, feedback effects certainly exist, even if they are not 100%. The low-lying states, accordingly, might be expected to show some special simple features which are independent of many aspects of the Lagrangian. This is a very familiar situation; it occurs in many-body theory, where it is a common feature of cooperative phenomena. In the highly excited states, on the other hand, further properties of the veiled Lagrangian would be uncovered.

From whichever point one views the bootstrap concept, a host of challenging and significant mathematical problems become prominently visible. We must be profoundly grateful for this focussing of attention. It should be noted, that while the four views given above are expressed in quite different words, whenever a practical

dynamical calculation is undertaken, the procedure that is followed does not depend on which view is held.

The most striking and characteristic features of the strong interactions are the symmetries which these interactions exhibit. What is the underlying reason for the existence of these symmetries? Can we understand the dynamical origins of the symmetry, or must we be content with an impositional theory? I mean by a dynamical theory of the symmetry, one in which the symmetry appears as an accidental degeneracy - a cunningly contrived accidental degeneracy. Even apart from the possibility of providing an answer, the framework constructed by Chew and his colleagues has been of inestimable value because it has made it inescapably clear that this is a most important question. In fact, we are very lucky in this, because with our present modest computational tools, it is hard to calculate relations among the properties of states with different spins and parities, but when we compare different states with the same spins and parities, which is what the symmetries are about, many approximations tend to cancel.

Studies with a number of simple models have shown that crossing symmetry can induce structural symmetries, but further work to clarify the scope of the phenomenon is needed. I should like to call attention to a number of interesting mathematical problems that arise in studies of dynamic symmetry induction. To be specific, let me write down some equations that one can get in any kind of symmetrized ladder approximation:

$$g_{abc} = \sum_{\alpha\beta\gamma} D_{abc}^{\alpha\beta\gamma} g_{a\beta\gamma} g_{\alpha b\gamma} g_{\alpha\beta c} \quad (2)$$

$$1 = \sum_{\alpha\beta\gamma bc} W_{bc\beta\gamma}^{a\alpha} g_{abc} g_{a\beta\gamma} g_{\alpha b\gamma} g_{\alpha\beta c}. \quad (3)$$

Here g_{abc} is an over-all coupling constant for the vertex $a \rightarrow b + c$.

The D 's and W 's are certain highly-symmetrical functions of the spins and masses of the particles. They are gotten from the solutions of non-linear integral equations; their precise form depends on the method of calculation one happens to use, and I do not wish to dwell on them except to remind you of the problem of proving existence and uniqueness of solutions. Let us take them as given functions (which must, of course, not be all independent).

These equations define what I should like to call "bootstrap algebras". Can we determine the properties of all bootstrap algebras? A case of particular interest occurs when the particles are found in multiplets, so there are equalities among some of the D 's and W 's. It is known that to obtain interesting restrictions on bootstrap algebras one needs to impose some further conditions. For example, one can require some inequalities on the eigen values of the "potential" matrices

$$V_{bc, \beta\gamma} = \sum_{\alpha} D_{abc}^{\alpha\beta\gamma} g_{\alpha b\gamma} g_{\alpha\beta c}. \quad (4)$$

Such inequalities bring in, in a rather weak way, additional dynamical information. The relations among bootstrap algebras with different mass values can be studied by making further assumptions about the mass dependence of the D 's and W 's. For example, how much can one deduce about broken-symmetry bootstrap algebras if one requires these functions to be monotone?

Our motivation in posing such questions is that we should like to find out to what extent dynamic symmetry induction is independent of details of the forces between particles, and also to be able to focus our attention upon the details which really are important. These questions also have a more abstract interest. When one thinks of Lie algebras as being special cases of bootstrap algebras which hold under certain additional restrictions, one obtains new insight into their structure. This insight can deepen our understanding of the role which group theory has to play in physics.

I should like to close by mentioning one of the embarrassments of current bootstrap calculations. It is easy to demonstrate the existence of strong attractive forces in scalar states. It is hard to understand why these forces are not so strong that the catastrophe of bound states with $m^2 \leq 0$ doesn't occur. For example, it is generally believed that the Pommeranchuk trajectory, containing the f_0 with $J = 2$, crosses $m^2 = 0$ when $J = 1$. It is hard to understand why $J = 0$ does not correspond to a state with $m^2 < 0$. Gell-Mann has suggested a way out which involves certain spin-coupling effects⁶. If this is the right answer, however, models of the physical vector mesons which include components with only two pseudoscalar mesons are quite suspect. A second question involves the existence of $m^2 = 0$ scalar mesons associated with the broken symmetries⁷. This question has been discussed in connection with "degenerate-vacuum" theories, but it might be a more general difficulty.

I hope that this brief survey at least suggests the enormous variety of work which is going on in S-matrix theory. I have

discussed in most detail the "constructive" side of S-matrix theory because I think that many interesting questions lie there, and also because new approaches would be helpful. No person would be able to follow all of the work which is being done, so I have had to give rather sketchy discussions of many points. Another reporter would give a different emphasis and express an interest in quite a different set of questions. There is only one point on which there could be no disagreement, which is that this field is full of deep and challenging mathematical problems.

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