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THE THEORY OF CONES

by

D. R. Morrison

MARCH 1965

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SANDIA CORPORATION MONOGRAPH

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D. R. Morrison

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THE THEORY OF CONES

1. Introduction and Basic Definitions

Definition 1.1. A system $(C, +, \times)$ is a cone if $(C, +)$ is a commutative semigroup, $(C, \times)$ is a group, and $\times$ distributes over $+$. $(C, +, \times)$ is a commutative cone if $(C, \times)$ is commutative.

The symbol $\times$ will often be omitted. The symbol 1 will be used to denote the unit element of $(C, \times)$. The symbols $J(c)$ and c^{-1} will be used, interchangeably, to denote the inverse of c in $(C, \times)$. The symbol 2 will denote $1 + 1$ and $1/2$ will denote 2^{-1} . It is not assumed in general, however, that $2 \neq 1$.

The operations $+$, $\times$, J will be called fractional operations, not to be confused with the rational operations, $+$, $-$, $\times$, J . Cones are closed under the fractional operations.

1.1 Examples

Some examples of cones follow:

The positive elements of an order sfield; in particular, the system F of positive rationals, hereafter called fractions, and the system P of positive reals.

The set $F[x]$ of fractional functions of x . (A fractional function of x has the form $P(x)(Q(x))^{-1}$, where P and Q are polynomials with coefficients in F .)

The intersection of any set of cones in which the fractional operations are compatibly defined. (See, however, the example in Section 8.)

The cartesian product of any set of cones.

The $n \times n$ upper (or lower) triangular matrices over an ordered sfield, with positive elements on the diagonal.

The cone L_1 of one element.

Any lattice ordered group, with $\times$ as the group operation, and $+$ as the least upper bound (or greatest lower bound) operation.

The set of all continuous functions on any topological space into any topological cone; in particular, into P .

Certain groups of order preserving automorphisms on ordered groups.

The sums of nonzero squares in a formally real field.

Numerous systems of subsets of a cone. In particular, systems of function classes, commonly referred to as orders. For example, the system MO of monomial orders $p\bar{x}^n$, where $p \in F$, $\bar{x}$ is an undefined class of functions, n is an integer and $+$ and $\times$ are defined as follows:

$$(p\bar{x}^n) \times (q\bar{x}^m) = pq\bar{x}^{m+n}$$

$$(p\bar{x}^n) + (q\bar{x}^m) = p\bar{x}^n \text{ if } n > m$$

$$(p\bar{x}^n) + (q\bar{x}^n) = (p + q)\bar{x}^n$$

A cone is prime if it has no proper subcones. F and L are the only prime cones.

The reader will find it helpful to think of cone theory as an analog of ring theory, with the roles of + and $\times$ partially interchanged. Rings are closed under the integral operations, +, -, $\times$; cones under the fractional operations, +, $\times$, J. Both theories begin with the extension of Ω , the set of natural numbers. Ring theory begins with the construction of the ring Z of integers as the group extension of $(\Omega, +)$; cone theory begins with the construction of the cone F as the group extension of $(\Omega, \times)$. Z is a prototype ring; F is a prototype cone.

Since cone theory generalizes the theory of lattice ordered groups, many of the basic definitions and theorems of cone theory are direct translations of basic definitions and theorems on lattice ordered groups (see Birkhoff¹). The following dictionary will be helpful in making the translations.

Cone	C	$\times$	+	$\oplus$	<	a^{-1}	1	$1 + C$	$1 + a$	$1 \oplus a$	upper	lower
L-group	G	+	$\cup$	$\cap$	$\leq$	$-a$	0	G^+	a^+	a^-	positive	negative

1.2 Duality Principle

Definition 1.2. If $a, b \in C$, we define the lower sum $a \oplus b$,

$$a \oplus b = (a^{-1} + b^{-1})^{-1} = a(a + b)^{-1}b = b(a + b)^{-1}a.$$

$(C, \oplus, \times)$ is a cone and is called the dual of the cone $(C, +, \times)$. Note that the transformation J

$$J(x) = x^{-1}$$

is an anti-automorphism between dual cones. Throughout all of cone theory, every definition and theorem has a dual, which may or may not be stated, formed by interchanging elements with their inverses, + with $\oplus$, and reversing the order of multiplication.

Some useful and easily verified identities which involve +, $\oplus$, $\times$ and J are the following.

$$(a \oplus b) + a \times (a + b)^{-1} \times a = a \quad (1)$$

$$(a \oplus b) \times (a + b) = (a + b) \times (a \oplus b) = ab \text{ if } a \times b = b \times a. \quad (2)$$

In particular,

$$(a \oplus 1) \times (a + 1) = (a + 1) \times (a \oplus 1) = a \quad (3)$$

$$(a + b)^{-1} = a^{-1} \oplus b^{-1} \quad (4)$$

$$(a \oplus b)^{-1} = a^{-1} + b^{-1} \quad (5)$$

Equations (2) and (3) are Birkhoff's¹ Corollary 1 and 2 to Theorem 4, Chapter XIV. Equation (3) factors an arbitrary element a of a cone C into an upper factor, $a + 1$, and a lower factor, $a \oplus 1$. Birkhoff calls this the Jordan decomposition. Equations (4) and (5) are convenient formulas for the inverse of a sum or lower sum, similar in form to the de Morgan laws.

Equation (1) is more conveniently expressed as the following.

Theorem 1.1. Compensation Theorem. For every a, b in a cone C , there is a $c \in C$ such that

$$(a \oplus b) + c = a.$$

Dually, there is a $d \in C$ such that

$$(a + b) \oplus d = a.$$

The reader who knows elementary electronics will recognize $\oplus$ as the operation by which resistances in parallel are combined, and the Compensation Theorem as one which states that any resistance in series can be compensated by another one in parallel and vice versa.

1.3 Monogenic Cones

A cone C is monogenic if there is an element $\bar{x}$ in C such that no proper subcone of C includes $\bar{x}$. The element, $\bar{x}$, is said to generate C . A simple example of a monogenic cone is $F[x]$, which consists of all quotients of polynomials in x , with positive integer coefficients. A generator of $F[x]$ is x . Every monogenic cone is a homomorphic image of $F[x]$; more specifically, if C is monogenic and $\bar{x}$ is a generator, then there is a unique cone homomorphism $h_{\bar{x}}$ of $F[x]$ onto C in which x is mapped onto $\bar{x}$. We shall denote such a monogenic cone as $h_{\bar{x}}(F[x])$ or as $C(\bar{x})$. $C(\bar{x})$ is the intersection of all subcones of C which include $\bar{x}$. Some immediate consequences are the following: Every monogenic cone is commutative, every cone is the union of its monogenic subcones, and therefore, every cone is the union of its commutative subcones.

2. Cones of Functions and Functions with Slope

Let D be any domain and let C be any cone. Denote by $D:C$ the set of all functions whose domain is D and whose range is in C . Define $+$, $\oplus$ and $\times$ in $D:C$, as usual, by

$$(f + g)(x) = f(x) + g(x)$$

$$(f \oplus g)(x) = f(x) \oplus g(x)$$

$$(f \times g)(x) = f(x) \times g(x)$$

for all $f, g \in D:C$, $x \in D$. Clearly $(D:C, +, \times)$ and $(D:C, \oplus, \times)$ are cones.

$D:C$ includes a subcone which we shall denote $D:C \rightarrow$, the cone of constant functions (functions with singleton range). $((D:C, \rightarrow), +, \times)$ is isomorphic with $(C, +, \times)$, and we shall, in general, identify its elements with those of C , using the same symbol, say c , to denote an element of C , and a function whose domain is D and whose value is c for every $d \in D$. Thus $f \in D:C \rightarrow$, or, more briefly, $f \rightarrow$ means that f is a constant function.

2.1 Slopes in Commutative Cones

In this section C is assumed to be a commutative cone and D is a subset of C .

Definition 2.1. A function $f \in D:C$ is with slope if the equation

$$f(x) + my = f(y) + mx$$

has at least one solution m for every pair x, y of elements of D .

The set of solutions is denoted $f'(x, y)$, and its elements are called slopes of f from x to y .

Definition 2.2. The set $D:C\uparrow$ is the set of functions with slope in $D:C$.

Thus $f \in D:C\uparrow$, or more briefly $f\uparrow$ means f is with slope. Thus, we may write the statement $f(x) + my = f(y) + mx$ in the form

$$m \in f'(x, y).$$

The set $f'(x, y)$ may consist of a single element, or may include many elements. The set $f'(x, x)$, for example, includes all of C .

Example 1. In a cone which is the cone of positive elements of an ordered field, $f\uparrow$ is equivalent to " f is monotone increasing." In this case, $f'(x, y)$, ($x \neq y$) has exactly one element, $m = \frac{f(x) - f(y)}{x - y}$.

Example 2. In a linearly ordered group, (where $x + y$ means $\max(x, y)$), $f\uparrow$ is equivalent to " f is monotone nondecreasing." In this case one element of $f'(x, y)$ is $\frac{f(x) + f(y)}{x + y}$. But if f is not one-to-one, and if $f(x) = f(y)$ for some $x \neq y$, then all elements $m \leq \frac{f(x) + f(y)}{x + y}$ are in $f'(x, y)$.

Theorem 2.1. $f\uparrow$ if and only if for every $x, y \in D$, there is an $m' \in C$ such that

$$f(x) \oplus m'x = f(y) \oplus m'x.$$

Proof. If $f\uparrow$, then there is an $m \in C$ such that

$$f(x) + my = f(y) + mx.$$

Taking inverses on both sides and using Equation (4) of paragraph 1.2

$$(f(x))^{-1} \oplus m^{-1}y^{-1} = (f(y))^{-1} \oplus m^{-1}x^{-1}.$$

Multiplying by $f(x)f(y)$

$$f(y) \oplus m^{-1}y^{-1}f(x)f(y) = f(x) \oplus m^{-1}x^{-1}f(x)f(y).$$

Thus $m^{-1}x^{-1}y^{-1}f(x)f(y)$ is the required m' .

The converse is the dual.

Thus "with slope" is self-dual; it has the same meaning in $(C, +, \times)$ and in $(C, \oplus, \times)$.

Definition 2.3. A system $(S, +, \times)$ is a semicone (= semiring) if $(S, +)$ is a commutative semigroup, $(S, \times)$ is a semigroup, and $\times$ distributes over $+$.

Lemma. $(D:C\uparrow, +, \times)$ is a semicone.

Proof. Let $f\uparrow$ and $g\uparrow$, and $x, y \in D$. Let $m \in f'(x, y)$, $n \in g'(x, y)$. Then $m + n \in (f + g)'(x, y)$.
Also

$$mg(y) + nf(x)$$

and

$$mg(x) + nf(y)$$

are elements of $(f \times g)'(x, y)$. Thus $f + g$ and $f \times g$ are in $(D:C\uparrow)$.

Theorem 2.2. $((D:C\uparrow) \cup (D:C\rightarrow), +, \times)$ is a semicone.

Proof. It has already been shown that $(D:C\uparrow)$ and $(D:C\rightarrow)$ are each closed under $+$ and $\times$. Let $f \in (D:C\uparrow)$ and $c \in (D:C\rightarrow)$, $x, y \in D$, $m \in f'(x, y)$. Then $m \in (f + c)'$ and $(m \times c) \in (f \times c)'$.

Notation. The identity function on the domain D is denoted I . Thus $I(x) = x$ for every $x \in D$.

Theorem 2.3. $I\uparrow$.

Proof. $I \in I'(x, y)$.

From Theorems 2.2 and 2.3 it follows that for all polynomials

$$P(x) = \sum_n p_n x^n, \quad p_n \in C,$$

where n ranges over some finite set of non-negative integers, which includes at least one positive integer, $P\uparrow$. The set of polynomials is the minimum subsemicone of $D:C$ containing I and $D:C\rightarrow$.

2.2 Functions with Inverse Slope

Definition 2.4. If $g \in D:C$ and $x, y \in D$, then any solution, m , of the equation

$$g(x) + mx = g(y) + my$$

is called an inverse slope of g between x and y . The set of all inverse slopes of g between x and y is called $-g'(x, y)$.

Definition 2.5. A function g is with inverse slope if for every pair x, y of elements in its domain, $-g'(x, y)$ is not empty.

Notation. The set of functions in $D:C$ with inverse slope is denoted $D:C\downarrow$. Thus $g \in D:C\downarrow$, or more briefly, $g\downarrow$ means g is with inverse slope.

Definition 2.6. Composition of Functions. If f and g are functions, then we define their composition fg by

$$fg(x) = f(g(x))$$

for those elements x in the domain of g (if any) such that $g(x)$ is in the domain of f .

In particular if g is any function whose range and domain are in C , then Jg and gJ are functions. If D is the domain of g , then $J(D)$ is the domain of gJ and D is the domain of Jg .

Theorem 2.4. The following conditions on a function g are equivalent

$$Jg \uparrow, g \downarrow, gJ \uparrow.$$

Proof. Suppose $Jg \uparrow$. Then for every $x, y \in C$, there is an $m \in C$ such that

$$Jg(x) + my = Jg(y) + mx. \quad (6)$$

Multiplying both sides by $g(x)g(y)$ yields

$$g(y) + mg(x)g(y)y = g(x) + mg(x)g(y)x. \quad (7)$$

Thus $mg(x)g(y) \in -g'(x, y)$. Thus $g \downarrow$.

Conversely, suppose $g \downarrow$. Then for every $x, y \in C$, there is an $n \in C$ such that

$$g(x) + nx = g(y) + ny. \quad (8)$$

Let $m = nJg(x)Jg(y)$ so that $n = mg(x)g(y)$.

Substituting this into (8) yields (7) which is equivalent to (6). This implies $m \in (Jg)'(x, y)$, hence $Jg \uparrow$.

Again, suppose $g \downarrow$ so that (8) holds. Elements of the domain of gJ have the form $J(x)$ where x is in D , the domain of g . Replacing g with the equivalent gJJ , and inserting the factors $yJ(y)$ and $xJ(x)$, (8) becomes

$$gJJ(x) + nxyJ(y) = gJJ(y) + nxyJ(x). \quad (9)$$

Thus $nxy \in (gJ)'(J(x), J(y))$, hence $gJ \uparrow$. Conversely, if $gJ \uparrow$, then for every $J(x), J(y)$ in $J(D)$, the domain of gJ , x and y in D , there is a $p \in C$ such that

$$gJJ(x) + pJ(y) = gJJ(y) + pJ(x) \quad (10)$$

Let $n = pJ(x)J(y)$ so that $p = nxy$. Then (10) becomes

$$\begin{aligned} gJJ(x) + nx &= gJJ(y) + ny \\ g(x) + nx &= g(y) + ny. \end{aligned}$$

Thus $n \in -g'(x, y)$, and $g \downarrow$.

Corollary 1: The following conditions on a function f are equivalent

$$f \uparrow, Jf \downarrow, fJ \downarrow.$$

Proof. Let $g = Jf$, then $f = Jg$. If $f \uparrow$ then $Jg \uparrow$ and by Theorem 2.3 $g \downarrow$. Thus $Jf \downarrow$. Similarly, if $Jf \downarrow$, then $g \downarrow$ and $Jg \uparrow$. But $Jg = f$, so $f \uparrow$. Thus $f \uparrow$ and $Jf \downarrow$ are equivalent.

Let $h = fJ$. Then $f = hJ$ and $f \uparrow$ implies $hJ \uparrow$ which implies $h \downarrow$, that is, $fJ \downarrow$. Similarly, $fJ \downarrow$ implies $h \downarrow$ which implies $hJ \uparrow$, that is $f \uparrow$. Thus $f \uparrow$ is equivalent to $fJ \downarrow$.

Definition 2.6. If f is a function, then f^* , the dual of f is

$$f^* = JfJ.$$

The domain of f^* is $J(D)$.

$$fJ = Jf^*$$

$$f^{**} = f$$

$$f^*g^* = (fg)^*.$$

Corollary 2: $f^* \uparrow$ if and only if $f \downarrow$, and $g \downarrow$ if and only if $g^* \uparrow$.

Proof. $f \uparrow$ is equivalent to $fJ \downarrow$ is equivalent to $Jf^* \downarrow$ is equivalent to $f^* \uparrow$. Similarly $g \downarrow$ is equivalent to $g^* \uparrow$.

Corollary 3: $(J(D):C \downarrow, +, \times)$ is a semicone. Also $((J(D):C \downarrow) \cup (J(D):C \rightarrow), +, \times)$ is a semicone.

Proof. The mapping $g \rightarrow gJ$ maps $D:C \downarrow$ isomorphically onto $J(D):C \uparrow$ and $(D:C \uparrow \cup D:C \rightarrow)$ isomorphically onto $(J(D):C \downarrow \cup D:C \rightarrow)$.

Corollary 4: The product $((D:C \uparrow) \times (D:C \downarrow), \times, +)$ is a cone; the closure under division of $(D:C \uparrow, \times, +)$.

Theorem 2.5. If $g \downarrow$ then g has at most one fixed point.

Proof. Let $g(x) = x$ and $g(y) = y$. Then there is an $m \in (-g'(x, y))$ such that

$$g(x) + mx = g(y) + my$$

$$x + mx = y + my$$

$$(1 + m)x = (1 + m)y.$$

Multiplying by $(1 + m)^{-1}$

$$x = y.$$

2.3 Functions with Definite (Inverse) Slope

Definition 2.7. A function $h(x)$ is with definite slope (abbreviated $h \uparrow \uparrow$) if $(J \times h) \uparrow$.

Equivalently $h \uparrow \uparrow$ if $h = I \times f$ for some function f such that $f \uparrow$. The set of functions in $D:C$ with definite slope is denoted $D:C \uparrow \uparrow$.

Theorem 2.6. $h \uparrow \uparrow$ implies h is one-to-one.

Proof. Suppose $h \uparrow \uparrow$ and

$$h(x) = a. \quad (11)$$

There is a function f such that $f \uparrow$ and $h(x) = (I \times f)(x) = xf(x)$. Thus (11) is equivalent to

$$xf(x) = a. \quad (12)$$

Let $g(x) = aIf(x)$. Then $g \uparrow$ and (12) is equivalent to

$$g(x) = x. \quad (13)$$

By Theorem 2.5, the solution of (13) (if any) is unique. Thus the solution of (11) (if any) is unique.

From Theorem 2.6, it follows that all polynomials in x , without a constant term, are one-to-one.

Corollary 1: 1 is the only root of 1 in C .

Proof. If $x^n = 1$ for some $x \in C$, n is a positive integer, and $1^n = 1$, it follows that $x = 1$ since $x^n \uparrow \uparrow$.

Corollary 2: 1 is the only finite subsemigroup of $(C, \times)$.

Proof. Let S be any finite subsemigroup of $(C, \times)$, and let $s \in S$. Then $s^n \in S$ for all positive integers n . Since S is finite, there exist positive integers m, n such that $s^{m+n} = s^n$. Multiplying by s^{-n} , $s^m = 1$, hence by Corollary 1, $s = 1$.

Corollary 3: L is the only finite cone.

Proof. Let C be any finite cone. Then $(C, \times)$ is a finite subsemigroup of itself, and by Corollary 2, $C = 1$. Thus C is the one-element cone, L .

Note: $h \uparrow$ does not imply that h is one-to-one. For example, if $h(x) = (x + c)$, (c constant) then $1 \in h'(x, y)$ for all x, y , so $h \uparrow$. But $a + c = b + c$ does not, in general, imply that $a = b$. If C is a lattice ordered group, for example, $a + c = b + c = c$ if $a \leq c$ and $b \leq c$. If c has additive cancellation, then $h \uparrow$ does imply that h is one-to-one.

2.4 Composition Rules

Theorem 2.7. (First Composition Theorem). If $f \uparrow$ and $h \uparrow \uparrow$, then $fh \uparrow$.

Proof. Since $h \uparrow \uparrow$, there is a function k such that $k \uparrow$ and $h(x) = xk(x)$ for all x . Since $f \uparrow$ and $k \uparrow$ for every pair x, y , of elements in the domain of fh , there exist elements $m, n, p \in C$ such that

$$f(h(x)) + mh(y) = f(h(y)) + mh(x) \quad (14)$$

$$k(x) + ny = k(y) + nx = p. \quad (15)$$

It follows that

$$fh(x) + myk(y) = fh(y) + mxk(x).$$

Adding $mnxy$,

$$fh(x) + myk(y) + mnxy = fh(y) + mxk(x) + mnxy$$

$$fh(x) + my(k(y) + nx) = fh(y) + mx(k(x) + ny).$$

(16)

From (15) and (16) it follows that

$$fh(x) + pmy = fh(y) + pmx.$$

Thus $pm \in (fh)'$, and $fh \uparrow$.

Remark. It is not known, by the author, whether the hypothesis of Theorem 2.7 can be weakened to $h \uparrow$ and still conclude that $fh \uparrow$. This is true if C has an additive cancellation law and in many other cases.

Corollary 1 (Second Composition Theorem): If $f \uparrow \uparrow$ and $h \uparrow \uparrow$ then $fh \uparrow \uparrow$.

Proof. Since $f(x) = x\ell(x)$ for some ℓ such that $\ell \uparrow$, and $k(x) = xk(x)$ for some k such that $k \uparrow$, it follows that

$$fh(x) = h(x)\ell h(x) = xk(x)\ell h(x).$$

Thus $k \uparrow$, $\ell h \uparrow$ by Theorem 2.6, so $(k \times \ell h) \uparrow$ by Theorem 2.1. Thus $fh = I \times (k \times \ell h)$. Thus $fh \uparrow \uparrow$ by definition of $\uparrow \uparrow$.

Corollary 2: If $h \uparrow \uparrow$, then $h^* \uparrow \uparrow$.

Proof. Since $h \uparrow \uparrow$, there is a function f such that $h(x) = xf(x)$ for all x in its domain, and $f \uparrow$. It follows that

$$h^*(x) = JhJ(x) = J(J(x)f(J(x))) = xJfJ(x) = xf^*(x).$$

Since $f^* \uparrow$ by Corollary 2 of Theorem 2.4, it follows that $h^* \uparrow \uparrow$.

Definition 2.8. A function g is with definite inverse slope (abbreviated $g \downarrow \downarrow$) if $gJ \uparrow \uparrow$.

Since $(gJ)^* = Jg$, by Corollary 2 and the definition, $g \downarrow \downarrow$ if and only if $Jg \uparrow \uparrow$.

Corollary 3 (Third Composition Theorem): The functions with definite slope or definite inverse slope in C are closed under composition. That is, if $f \uparrow \uparrow$ or $f \downarrow \downarrow$, and if $g \uparrow \uparrow$ or $g \downarrow \downarrow$, then $fg \uparrow \uparrow$ or $fg \downarrow \downarrow$.

Proof. If $f \uparrow \uparrow$ and $g \uparrow \uparrow$ then $fg \uparrow \uparrow$ by Corollary 1. If $f \uparrow \uparrow$ and $g \downarrow \downarrow$, then $gJ \uparrow \uparrow$, hence $fgJ \uparrow \uparrow$ by Corollary 1, hence $fg \downarrow \downarrow$. If $f \downarrow \downarrow$ and $g \uparrow \uparrow$, then $Jf \uparrow \uparrow$, hence $Jfg \uparrow \uparrow$ by Corollary 1, hence $fg \downarrow \downarrow$. If $f \downarrow \downarrow$ and $g \downarrow \downarrow$ then $Jf \uparrow \uparrow$ and $gJ \uparrow \uparrow$, hence $JfgJ \uparrow \uparrow$, hence $JfgJ \uparrow \uparrow$ by Corollary 1. Thus $(fg)^* \uparrow \uparrow$ so $fg \uparrow \uparrow$ by Corollary 2.

Corollary 4: If $f \uparrow \uparrow$ or $f \downarrow \downarrow$ then f is one-to-one.

Proof. If $f \uparrow \uparrow$ then f is one-to-one by Theorem 2.5. If $f \downarrow \downarrow$ then $Jf \uparrow \uparrow$ so Jf is one-to-one. Since $f = JJf$ and J is one-to-one, f is the composition of two one-to-one functions, so is one-to-one.

2.5 Inverse Functions

If a function f is one-to-one, then we define

$$f^{-1}(a) = x \ni f(x) = a$$

for every a in the range of f . By definition of composition $f^{-1}f = I$, the identity function on the domain of f .

Theorem 2.8. If $f \in D:C \uparrow$ and f is one-to-one, then $f^{-1} \uparrow$.

Proof. For every $x, y \in D$ there is an $m \in C$ such that

$$f(x) + my = f(y) + mx.$$

Since f is one-to-one, we may write $f^{-1}f(y)$ for y and $f^{-1}f(x)$ for x . Thus

$$mf^{-1}f(y) + f(x) = mf^{-1}f(x) + f(y).$$

Multiplying by m^{-1} ,

$$f^{-1}f(y) + m^{-1}f(x) = f^{-1}f(x) + m^{-1}f(y).$$

Thus $m^{-1} \in f^{-1}(f(x), f(y))$ and f^{-1} is with slope on its domain, the range of f .

Thus square roots, cube roots, and in general functions defined as solutions of equations of the form $xP(x) = a$, where x is a polynomial, are functions with slope on their domains.

3. Subsets of Groups--Generalized Cosets

In this section the notion of coset of an invariant subgroup is generalized. The material presented here has a substantial intersection with that of Lajos⁹ and of Chawla², although it neither includes nor follows in its entirety from either of these references. The results of this section will be applied in Section 4 to certain subsets of a cone.

3.1 The Power Set as a Semigroup

Let $(G, \times)$ be a group and let $P(G)$ be its power set (set of all subsets). Let 1 denote the unit element of G . Singleton sets will be denoted by the same symbol as their one element. Define $\times$ in $P(G)$ by complex multiplication

$$A \times B = \{ab \ni a \in A, b \in B\}$$

where $A, B \in P(G)$. Clearly $(P(G), \times)$ is a semigroup and 1 is its unit.

3.2 Unit Sets and Subgroups of $(P(G), \times)$

Definition 3.1. An idempotent element U of $(P(G), \times)$ will be called a unit set of G .

Thus U is a unit set of G if $U \subset G$ and

$$U \times U \subset U \quad ((U, \times) \text{ is a semigroup}),$$

and

$$U \times U \supset U \quad ((U, \times) \text{ has the divisor property});$$

every element of U is the product of two elements of U .

Definition 3.2. If $X, Y, A \in P(G)$ and if $XY = YX = A$, then X and Y are divisors of A and A is a multiple of X and of Y .

Notation. The symbol $X|A$ will mean X is a divisor of A , or A is a multiple of X . $M(X)$ will denote the set of all multiples of X , and $D(A)$ the set of all divisors of A .

Theorem 3.1. If U is a unit set of G , then $(M(U), \times)$ is a semigroup and U is its unit. Conversely if $(H, \times)$ is any subsemigroup with unit of $(P(G), \times)$ and if U is its unit, then $(H, \times)$ is a subsemigroup of $(M(U), \times)$.

Theorem 3.2. $(D(U), \times)$ is a semigroup.

Proof. If $A, B \in D(U)$, then there exist sets X, Y such that $AX = XA = YB = BY = U$. Thus $ABYX = YXAB = U$, so $AB \in D(U)$.

Definition 3.3. If $A, B \in P(G)$ and if $A|B$ and $B|A$, then A and B are cosets of each other.

Definition 3.4. If U is a unit set of G then the set of all cosets of U is denoted G/U .

Thus $G/U = M(U) \cap D(U)$.

Theorem 3.3. Subgroups of $(P(G), \times)$. If U is a unit set of G , then $(G/U, \times)$ is a group and U is its unit. Conversely, if $(H, \times)$ is any subgroup of $(P(G), \times)$ and if U is its unit, then $(H, \times)$ is a subgroup of $(G/U, \times)$.

Proof. The group properties of $(G/U, \times)$ are obvious with the possible exception of the inverse. If $A \in G/U$, then A is a coset of U . Since A is a multiple of U , $A \in M(U)$, hence $AU = UA = A$.

But also, A is a divisor of U , so that there is an $X \in P(G)$ (not necessarily in G/U) such that

$$AX = XA = U.$$

It follows that

$$A(UXU) = (UXU)A = U$$

and UXU is also in G/U . Thus UXU is the inverse of A in G/U .

3.3-A Theorem on Divisors in $P(G)$

Theorem 3.4. If $A, B \in P(G)$ then there is an $X \in P(G)$ such that

$$AX \supset B \supset XA \quad (17)$$

if and only if for every $b \in B$ there is an $a \in A$ such that

$$bA \subset aB \quad (18)$$

Furthermore, if (18) is satisfied, then the set

$$\bar{X} = \bigcap_{a \in A} Ba^{-1} \quad (19)$$

is such an X , and includes every such X .

Proof. Suppose (18) holds. Then for every $a' \in A$, $ba' \in aB$, $a^{-1}b \in Ba'^{-1}$, hence

$$a^{-1}b \in \bigcap_{a' \in A} Ba'^{-1} = \bar{X}$$

$$a^{-1}b \in \bar{X}$$

$$b \in a\bar{X} \subset A\bar{X}$$

$$b \in A\bar{X}.$$

Since this is true for every $b \in B$,

$$B \subset A\bar{X}.$$

Thus $\bar{X}$ satisfies the first half of (17).

But also, for every $a \in A$

$$Ba^{-1} \supset \bar{X}$$

$$B \supset \bar{X}a$$

and since this is true for every $a \in A$,

$$B \supset \bar{X}A$$

and $\bar{X}$ satisfies both parts of (17).

Now, conversely, let X be any set which satisfies the last part of (17),

$$B \supset XA.$$

Then for every $a \in A$,

$$B \supset Xa$$

$$Ba^{-1} \supset X$$

and since this is true for every $a \in A$,

$$X \subset \bigcap_{a \in A} Ba^{-1} = \overline{X}$$

so $X \subset \overline{X}$.

Now, suppose, conversely, that (17) holds. Let $b \in B$. Then $b \in AX$ and $b = ax$ for some $a \in A$, $x \in X$. It follows that

$$bA = axA \subset aXA \subset aB,$$

$$bA \subset aB$$

and (18) holds.

Note. Equation (18) will be called the left inclusion condition on A and B . Its dual, $Ab \supset Ba$, will be called the right inclusion condition on A and B . The set $\overline{X}$ will be called the maximal solution of $B \supset XA$ and the dual set,

$$\bigcap_{a \in A} a^{-1}B,$$

the maximal solution of $B \supset AX$.

3.4 Normal Unit Sets and Induced Quasi-Order Relations

In this section the results of Section 3.2 will be elaborated in the special case of unit sets in the center of $(P(G), \times)$. Such unit sets induce quasi-order relations in G .

Definition 3.5. A subset A of G is normal if $gA = Ag$ for every element $g \in G$. Equivalently, A is normal if $AB = BA$ for every subset B of G .

Thus the normal sets comprise the center of the semigroup $(P(G), \times)$.

If E is a normal unit set, then E induces a quasi-order relation $\rightarrow$ (precedes) defined by

$$a \rightarrow b \text{ if and only if } b \in aE$$

Equivalently, since E is normal

$$a \rightarrow b \text{ if and only if } b \in Ea.$$

The relation $\rightarrow$ has the following properties

- (1) Homogeneity: $a \rightarrow b$ implies $ca \rightarrow cb$ and $ac \rightarrow bc$. (From the definitions.)
- (2) Transitivity: $a \rightarrow b$ and $b \rightarrow c$ implies $a \rightarrow c$. (Because $(E, \times)$ is a semigroup.)
- (3) Interpolation: $a \rightarrow c$ implies there is a $b \in G$ such that $a \rightarrow b$ and $b \rightarrow c$. (Because $(E, \times)$ has the divisor property.)

Conversely, if $\rightarrow$ is any relation in G with properties (1), (2) and (3), then the set $E = \{e \in G \mid 1 \rightarrow e\}$ is a normal unit set and induces $\rightarrow$. Thus there is a one-to-one correspondence between the normal unit sets in G and the homogeneous, transitive interpolation quasi-order relations.

Definition 3.6. If A is a normal subset and g is an element of G , then gA is a principal coset of A .

Note that a principal coset is a coset. For

$$g^{-1}(gA) = A = (gA)g^{-1}$$

so gA is a divisor of A , and

$$(g)(A) = gA = (A)(g)$$

so A is a divisor of gA .

Theorem 3.5. If E is a normal unit set in G , then the mapping h_E ,

$$h_E(g) = gE$$

is a homomorphism on G into G/E .

Theorem 3.5 generalizes the classical "natural homomorphism" on a group G onto its quotient group modulo an invariant subgroup. Note that h_E is into G/E , and onto the set of principal cosets. We shall show later that if $1 \in E$ then every coset of E is principal, so that h_E is onto G/E . There are cases, however, in which $1 \notin E$ and some cosets are not principal, so that h_E is properly into G/E .

Definition 3.7. A set A satisfies a left Cauchy condition relative to a normal unit set E if for every $e \in E$ there is a $g \in G$ such that

$$gE \supset A \supset egE.$$

The right Cauchy condition is defined dually, with ge replacing eg . Since E is normal it makes no difference if E appears to the left or right of g and e or between them. The Cauchy conditions assert that A is "trapped" between two principal cosets of E , and that the cosets may be taken in some sense, as close together as the elements of E are to 1.

Lemma. If A is a multiple of E , then the left Cauchy condition on A relative to E is equivalent to the left inclusion condition on A and E .

Proof. Suppose that the left Cauchy condition on A relative to E is satisfied. Let $e \in E$. Since E has the divisor property, $e = e_1 e_2$ for some $e_1, e_2 \in E$. By the left Cauchy condition, there is a $g \in G$ such that

$$gE \supset A \supset e_2 gE.$$

It follows that

$$eg = e_1 e_2 g \in E e_2 g = e_2 gE \subset A,$$

$$eg \in A.$$

Let $a = eg$. Then

$$aE = egE \supset eA,$$

$$aE \supset eA.$$

Thus the left inclusion condition on A and E is satisfied.

Conversely, suppose the left inclusion condition on A and E is satisfied. Let $e \in E$. Then there is an $a \in A$ such that

$$aE \supset eA.$$

Let $g = e^{-1}a$. Then

$$gE = e^{-1}aE \supset e^{-1}eA = A = AE \supset aE = egE,$$

$$gE \supset A \supset egE.$$

Thus A satisfies the left Cauchy condition relative to E .

Theorem 3.6. A multiple A of a normal unit set E is a coset of E if and only if A satisfies the left and right Cauchy conditions relative to E .

Proof. If A is a coset of E , then there is a set X such that

$$AX = E = XA.$$

Hence in particular,

$$AX \supset E \supset XA.$$

By Theorem 3.4, the left inclusion condition on A and E follows and from this, by the previous lemma, the left Cauchy condition on A relative to E . Dually, the right Cauchy condition on A relative to E follows.

Conversely, if the left and right Cauchy conditions hold, then by the previous lemma, the two inclusion conditions hold. Thus by Theorem 3.4 each of the inclusion relations

$$AX \supset E \supset XA$$

and

$$YA \supset E \supset AY$$

has a solution. Moreover, the maximal solutions are

$$\bar{X} = \bigcap_{a \in A} Ea^{-1}$$

and

$$\bar{Y} = \bigcap_{a \in A} a^{-1}E,$$

respectively. Since E is normal, it follows that $\bar{X} = \bar{Y}$, hence

$$A\bar{X} = E = \bar{X}A$$

and A is a coset of E .

Definition 3.8. A normal unit set E is reflexive if $1 \in E$, symmetric if $J(E) = E$.

Reflexive unit sets induce reflexive relations; symmetric unit sets induce symmetric relations, i. e., equivalence relations.

A nonempty symmetric normal unit set is necessarily a reflexive unit set. In a symmetric unit set the divisor property is assured by the presence of 1; every element is the product of itself and 1.

Corollary: If E is reflexive, then every coset of E is principal.

Proof. By the Cauchy conditions with $e = 1$, if A is a coset of E , there is a $g \in G$ such that

$$gE \supset A \supset g1E = gE$$

$$A = gE.$$

It follows also that if $1 \in E$, then the natural homomorphism h_E is on G onto G/E .

An Example of Nonprincipal Cosets. Let $(F, \times)$ be the multiplicative group of fractions (positive rationals). Let $1 + F$ be the semigroup of fractions > 1 . Then the multiples of $1 + F$ are the additive ideals of F , that is, sets which include with any f , all $g > f$. The Cauchy condition (right or left, since $(F, \times)$ is commutative) on a multiple A of $1 + F$, is that for every $e \in 1 + F$ there is an $f \in F$ such that $f(1 + F) \supset A \supset ef(1 + F)$. Thus f is a lower bound on A . Thus the cosets of $1 + F$ are the additive ideals with a positive lower bound, i.e., Dedekind cuts. Thus $F/1 + F$ is the set of positive reals. The rational cuts are principal, the irrational cuts, nonprincipal.

3.5 Operations on Normal Unit Sets

If E is a normal unit set, then there are several other sets defined in terms of E which are necessarily normal unit sets. Some of them are

$$J(E) = \{e^{-1} \ni e \in E\}$$

$$M(E) = \{m \in C \ni mE \subset E\}$$

$$K(E) = M(E) \cap J(M(E)).$$

Clearly $M(E)$ includes E and 1, and is therefore reflexive. If E is reflexive, then $M(E) = E$. Thus in any case $M(M(E)) = M(E)$. Thus M is a kind of closure operator. M is not, however, monotone. In the direct product $(F \times F, \times)$, $(1 + F) \times (1 + F)$ and $(1 + F) \times (1 + F) \cup 1$ are two normal unit sets, properly ordered by inclusion. But $M((1 + F) \times (1 + F)) = ((1 + F) \cup 1) \times ((1 + F) \cup 1)$ and $M((1 + F) \times (1 + F) \cup 1) = (1 + F) \times (1 + F) \cup 1$. Thus the M of the larger unit set is properly included in the M of the smaller.

The natural homomorphisms h_E , $h_{J(E)}$, $h_{M(E)}$ and $h_{K(E)}$ onto the groups G/E , $G/J(E)$, $G/M(E)$ and $G/K(E)$ all have the same kernel $K(E)$. It follows that all are onto isomorphic cones. The first two, however, may be into cones that are not isomorphic to the others, if they include nonprincipal cosets. Thus they are into extensions, or in a sense, completions of $G/K(E)$.

4. Cones of Subsets and Convex Subsets

We now return to the study of a cone $(C, +, \times)$, and apply the results of Section 3 to the construction of cones in $(P(C), +, \times)$. The operations $+$ and $\oplus$ are defined in $P(C)$, the power set (set of all subsets) of C , as usual

$$A + B = \{a + b, \exists a \in A, b \in B\}$$

$$A \oplus B = \{a \oplus b, \exists a \in A, b \in B\}$$

for all $A, B \in P(C)$.

Clearly $(P(C), +)$ and $(P(C), \oplus)$ are commutative semigroups and $(P(C), \times)$ is a semigroup.

The distributive laws, however, do not hold throughout $(P(C), +, \times)$. While it is true for all $A, B, X \in P(C)$, that

$$(A + B)X \subset AX + BX$$

the inclusion is an equality only in special cases.

4.1 Left Convex Subsets of C

Definition 4.1. A subset X of a cone C is left convex in C if

$$(A + B)X = AX + BX$$

for every $A, B \in P(C)$.

Definition 4.2. The set of left convex subsets of C will be denoted $\mathcal{L}(C)$.

Theorem 4.1. $\mathcal{L}(C)$, is closed under $\times$.

Proof. If $X, Y \in \mathcal{L}(C)$, then for every $A, B \in P(C)$.

$$\begin{aligned} (A + B)(XY) &= ((A + B)X)Y && \text{by associative law of } (P(C), \times) \\ &= (AX + BX)Y && \text{because } X \in \mathcal{L}(C) \\ &= (AX)Y + (BX)Y && \text{because } Y \in \mathcal{L}(C) \\ &= A(XY) + B(XY) && \text{by associative law of } (P(C), \times). \end{aligned}$$

Thus $(A + B)(XY) = A(XY) + B(XY)$ and $XY \in \mathcal{L}(C)$.

Theorem 4.2. A set $X \in P(C)$ is left convex if and only if for every $a, b \in C$,

$$(a + b)X = aX + bX. \tag{20}$$

Proof. Clearly (20) is necessary, since a and b are elements (singleton sets) in $P(C)$. Suppose (20) holds. Let $A, B \in P(C)$, and let

$$\begin{aligned} u &\in AX + BX \\ u &= ax_1 + bx_2 \end{aligned}$$

for some $x_1, x_2 \in X$, $a \in A$, $b \in B$. It follows that

$$\begin{aligned} u &\in (aX + bX) \\ &= (a + b)X && \text{by (20)} \\ &\subset (A + B)X. \end{aligned}$$

Thus

$$AX + BX \subset (A + B)X.$$

Clearly

$$(A + B)X \subset AX + BX.$$

Thus

$$(A + B)X = AX + BX,$$

and X is left convex.

Corollary: Every singleton set is left convex.

Proof. This follows from Theorem 4.2 and the left distributive law.

Theorem 4.3. $\mathcal{L}(C)$ is closed under $+$.

Proof. Suppose X and Y are left convex. Then for every $a, b \in C$,

$$\begin{aligned} (a + b)(X + Y) &= (a + b)X + (a + b)Y && \text{because } a + b \in \mathcal{L}(C) \\ &= (aX + bX) + (aY + bY) && \text{because } X, Y \in \mathcal{L}(C) \\ &= (aX + aY) + (bX + bY) && \text{because } (P(C), +) \text{ is a commutative semigroup} \\ &= a(X + Y) + b(X + Y) && \text{because } a, b \in \mathcal{L}(C). \end{aligned}$$

Thus $(a + b)(X + Y) = a(X + Y) + b(X + Y)$. Hence by Theorem 4.2, $(X + Y) \in \mathcal{L}(C)$.

Theorem 4.4. A set X is left convex if and only if, for every $a, b \in C$, $x_1, x_2 \in X$, there is an $x \in X$ such that

$$ax_1 + bx_2 = (a + b)x \tag{21}$$

Proof. Suppose X is left convex. Then

$$\begin{aligned} ax_1 + bx_2 &\in aX + bX \\ &= (a + b)X && \text{by Theorem 4.2} \end{aligned}$$

Thus

$$ax_1 + bx_2 = (a + b)x$$

for some $x \in X$.

Conversely, suppose that (21) holds.

Let

$$u \in aX + bX.$$

Then

$$\begin{aligned} u &= ax_1 + bx_2 && \text{for some } x_1, x_2 \in X \\ &= (a + b)x && \text{for some } x \in X, \text{ by (21)} \\ &\in (a + b)X. \end{aligned}$$

Thus

$$aX + bX \subset (a + b)X.$$

Clearly $aX + bX \supset (a + b)X$, hence $aX + bX = (a + b)X$ and X is left convex.

Note. Equation (21) will be called a left interpolation equation and x will be called a left interpolate of x_1 and x_2 . The coefficients a and b will be called interpolation coefficients. Multiplying (21) on the left by $(a + b)^{-1}$ yields

$$(a + b)^{-1} a x_1 + (a + b)^{-1} b x_2 = x$$

which is a normalized left interpolation equation, characterized by the fact that the sum of the interpolation coefficients is 1. Left convex sets are characterized by the fact that they include with any two elements, all their left interpolates.

It follows that

Theorem 4.5. The intersection of any collection of left convex subsets of C is a left convex subset of C .

Theorem 4.6. If X is left convex, so is $J(X)$.

Proof. Let $x_1, x_2 \in X$ so that $x_1^{-1}, x_2^{-1} \in J(X)$. Let $a, b \in C$, and suppose that

$$ax_1^{-1} + bx_2^{-1} = (a + b)x^{-1}.$$

Multiplying by x and inserting factors $x_1^{-1} x_1$ and $x_2^{-1} x_2$ yields

$$(ax_1^{-1} + bx_2^{-1})x = ax_1^{-1} x_1 + bx_2^{-1} x_2,$$

which is a left interpolate equation with coefficients ax_1^{-1} and bx_2^{-1} . It shows that x is an interpolate of x_1 and x_2 hence is in X . Thus $x^{-1} \in J(X)$. Thus $J(X)$ is left convex by Theorem 4.4.

Corollary: $\mathcal{L}(C)$ is closed under $\oplus$.

Proof. $A \oplus B = J(J(A) + J(B))$. The corollary follows from Theorems 4.3 and 4.6.

Theorem 4.7. Powers of c in a Left Convex Set. Let $c \in C$, and let p, q, r be integers, $p < q < r$. Let X be a left convex subset of C . If $c^p, c^r \in X$ then $c^q \in X$.

Proof. Case 1. $p = 0, q = 1, C$ is commutative. In this case, the function $f(x) = x^{r-1}$ is with slope, so there is an $m \in f'(x, 1)$, and

$$x^{r-1} + m = 1 + mx.$$

$$x^r + mx = x + mx^2.$$

$$1x^r + mx1 = (1 + mx)x.$$

This is a left interpolate equation with 1 and mx as interpolation coefficients, and x as interpolate. Since $x^r, 1 \in X$, it follows that $x \in X$.

Case 2. $q = p + 1$, C commutative, generalizing Case 1 to arbitrary p . If $x^p, x^r \in X$, then $1, x^{r-p} \in (x^{-p}X)$, which is also left convex. By Case 1, $x \in (x^{-p}X)$, so $x^q = x^{p+1} \in X$.

Case 3. C commutative, generalizing Case 2 to arbitrary q , by induction on $q - p$. The case $q - p = 1$ is already established as Case 2. Suppose the theorem is true when $q - p = k$. And suppose $q - p = k + 1$. Then $x^p, x^r \in X$ implies $x^{p+k} \in X$, by induction hypothesis. Then $x^{p+k}, x^r \in X$, and this implies $x^{p+k+1} \in X$ by Case 2. Thus $x^q = x^{p+k+1} \in X$ and the induction is established.

Case 4. C noncommutative. The element c generates a monogenic, hence commutative subcone $C[c]$. Furthermore $X \cap C[c]$ is left convex in $C[c]$. If $c^p, c^r \in X$, then they are in $C[c] \cap X$, and by Case 3, it follows that $c^q \in C[c] \cap X$, hence $c^q \in X$.

Theorem 4.8. Partial sums in a Left Convex Set. If $x, y, z \in C$, and if x and $x + y + z$ are in a left convex subset X of C , then $x + y \in X$.

Proof. Case 1. $y = 1$. In this case

$$1(x + 1 + z) + zx = (1 + z)(x + 1).$$

This is a left interpolation equation with coefficients 1 and z , and it shows that $x + 1$ is an interpolate of x and $x + 1 + z$. Since x and $x + 1 + z$ are in X , it follows that $x + 1 \in X$.

Case 2. $y \neq 1$. In this case, x and $x + y + z$ are in X . It follows that $y^{-1}x$ and $y^{-1} + 1 + y^{-1}z$ are in $y^{-1}X$, which is also left convex. It follows from Case 1, that $y^{-1}x + 1 \in y^{-1}X$, hence $x + y \in X$.

Corollary 1. For every $a, b, c \in C$, a is a left interpolate of $a + b$ and $a \oplus c$.

Proof. By the Compensation Theorem 1.1, $a = a \oplus c + x$ for some $x \in C$. The corollary then follows from Theorem 4.8.

Corollary 2. If $a, b, c, d, e \in C$ and if $a = b + c = d \oplus e$ then a is a left interpolate of b and d .

Proof. By the Compensation Theorem, there exist $x, y \in C$ such that $b = a \oplus x$ and $d = a + y$. Corollary 2 then follows from Corollary 1.

4.2 The Semicone of Convex Subsets

A set is convex if it is both left convex and right convex. For normal sets, convex, left convex, and right convex are equivalent adjectives. From the duals of the theorems of Section 4.1, it follows that the set $\mathcal{R}(C)$ of right convex subsets of C is closed under $+$, $\oplus$, $\times$, J , and arbitrary intersection. It follows also that the set

$$\mathcal{K}(C) = \mathcal{L}(C) \cap \mathcal{R}(C)$$

of convex subsets of C also has these properties, and in addition, the left and right distributive laws. Thus

Theorem 4.9. $(\mathcal{K}(C), +, \times)$ and $(\mathcal{K}(C), \oplus, \times)$ are semicones.

4.3 Cones of Convex Subsets

If a subset of $\mathcal{K}(C)$ is to be a cone, it must first of all be a multiplicative group, and the results of Section 3 tell us which subsets of $P(C)$ (which includes $\mathcal{K}(C)$) are groups. Specifically, the subgroups of $(P(C), \times)$ are the groups of cosets of unit sets and their subgroups, in particular, groups of cosets of normal unit sets.

Definition 4.3. A convex normal unit set D in $(C, \times)$ is a direction in C . A coset of D is a D-sector.

Theorem 4.10. A normal unit set D in a cone C is convex if and only if the relation $\rightarrow$ induced by D is additive.

Proof. For $a, b, c, d \in C$

$$a \rightarrow b \quad \text{if and only if } b \in aD$$

$$c \rightarrow d \quad \text{if and only if } d \in cD.$$

Thus, if D is convex

$$b + d \in (aD + cD)$$

$$b + d \in (a + c)D, \quad \text{because } D \text{ is convex}$$

$$a + c \rightarrow b + d$$

and $\rightarrow$ is additive.

Conversely, if $\rightarrow$ is additive, and if $d_1, d_2 \in D$, and $a, b \in C$, and $a + b = 1$, it follows that

$$1 \rightarrow d_1$$

$$1 \rightarrow d_2$$

$$a \rightarrow ad_1$$

$$b \rightarrow bd_2$$

$$a + b \rightarrow ad_1 + bd_2 \quad \text{because } \rightarrow \text{ is additive}$$

$$1 \rightarrow ad_1 + bd_2 \quad \text{because } a + b = 1$$

$$ad_1 + bd_2 \in D.$$

Thus D is convex.

Theorem 4.11. If D is a direction in C then all D -sectors are convex.

Proof. Let A be a D -sector, hence a coset of D . By Theorem 3.4 the maximal solution of the inclusion

$$D \supset XA$$

is the set

$$\bar{X} = \bigcap_{a \in A} Da^{-1}.$$

Since $\bar{X}$ is the intersection of a collection of convex sets, it follows that $\bar{X}$ is convex. Thus $D\bar{X}D$ is also convex, and $D\bar{X}DA = A D\bar{X}D = D$. Thus $D\bar{X}D$ is a coset of D , and is, in fact, the inverse

of A in C/D . We have proved that the inverse in $(C/D, \times)$ of an arbitrary element of C/D is convex. But every element of C/D is the inverse in $(D/D, \times)$ of some element of C/D , hence every element of C/D is convex.

Lemma. If A and B are cosets of a direction D and if S and T are their inverses, respectively, in $(C/D, \times)$ then

$$(A + B)(S \oplus T) = D.$$

Proof. We have $AS = SA = BT = TB = D$. Let $u \in (A + B)(S \oplus T)$. Then

$$u = (a + b)(s \oplus t)$$

for some $a \in A$, $b \in B$, $s \in S$, $t \in T$. It follows that

$$u = a(s \oplus t) + b(s \oplus t)$$

$$u = as(s + t)^{-1}t + bt(s + t)^{-1}s$$

This is a normalized right interpolation equation with coefficients $(s + t)^{-1}t$ and $(s + t)^{-1}s$, whose sum is 1. Thus u is a right interpolate of as and bt , both in D . Since D is convex, $u \in D$. Thus

$$(A + B)(S \oplus T) \subset D.$$

Conversely, let $d \in D$. Then $d \in AS$ and $d \in BT$. Thus

$$d = as = bt$$

for some $a \in A$, $b \in B$, $s \in S$, $t \in T$.

It follows that

$$d = d(s + t)^{-1}(t + s)$$

$$= d(s + t)^{-1}t + d(s + t)^{-1}s$$

$$= as(s + t)^{-1}t + bt(s + t)^{-1}s$$

$$= a(s \oplus t) + b(s \oplus t)$$

$$= (a + b)(s \oplus t) \in (A + B)(S \oplus T)$$

Thus

$$D = (A + B)(S \oplus T)$$

Theorem 4.12. If D is a convex direction, then $(C/D, +, \times)$ and $(C/D, \oplus, \times)$ are cones.

Proof. By the results of Section 3, $(C/D, \times)$ is a group. If A , B are elements of C/D , and S , T are their inverses in C/D , by the previous lemma and its dual,

$$(A + B)(S \oplus T) = D = (S \oplus T)(A + B)$$

so that $(A + B)$ and $(S \oplus T)$ are divisors of D . Clearly $(A + B)D = AD + BD = A + B$ and similarly $(A + B) = D(A + B)$. Thus $A + B$ is also a multiple, hence a coset of D . Thus C/D is closed under $+$. Similarly, C/D is closed under $\oplus$. The distributive laws in C/D follow from the convexity of its elements in C .

Theorem 4.13. The natural mapping h_D of C into C/D is a cone homomorphism.

Proof. According to Theorem 3.5, h_D is a homomorphism of $(C, \times)$. Also

$$\begin{aligned} h_D(A) + h_D(B) &= AD + BD \\ &= (A + B)D \quad \text{because } D \text{ is convex} \\ &= h_D(A + B) \end{aligned}$$

Thus h_D is a cone homomorphism.

Theorem 4.14. The operations J , M , and K on unit sets map directions into directions.

Proof. J maps directions into directions, by Theorem 4.6. M was defined

$$M(D) = \{m \in C \ni mD \subset D\}.$$

It was shown in Section 3 that $M(D)$ is a normal unit set. If $m_1, m_2 \in M(D)$ and if $a, b \in C$, $a + b = 1$, it follows that

$$\begin{aligned} (am_1 + bm_2)D &= am_1D + bm_2D, & \text{because } D \text{ is convex} \\ &\subset aD + bD, & \text{because } m_1, m_2 \in M(D) \\ &= (a + b)D, & \text{because } D \text{ is convex} \\ &= D, & \text{because } a + b = 1. \end{aligned}$$

Thus $(am_1 + bm_2) \in M(D)$ and $M(D)$ is convex. Also

$$K(D) = M(D) \cap J(M(D))$$

is the intersection of two reflexive directions, and so is a reflexive direction.

4.4 Some Examples of Convex Directions and the Cones and Quasi-Order Relations They Induce

In the cone F of fractions, the convex directions are

1. F
2. ϕ , the empty set,
3. 1 ,
4. $1 + F = \{f \in F \ni f > 1\}$
5. $1 \oplus F = \{f \in F \ni f < 1\}$
6. $M(1 + F) = \{f \in F \ni f \geq 1\}$
7. $M(1 \oplus F) = \{f \in F \ni f \leq 1\}$

The relations they induce are the all relation, the null relation, $=$, $>$, $<$, $\geq$, and $\leq$, respectively.

The cones F/D are isomorphic to L , L , F , P , P , F and F , respectively. The elements are, respectively, F , ϕ , the singleton sets, the open upper Dedekind cuts, the open lower Dedekind cuts, the closed upper Dedekind cuts with fractional minimum and the closed lower Dedekind cuts with fractional maximum.

Let C be a cone of functions on an arbitrary domain S into P , the cone of positive reals. Let $\{f(n), n \in N\}$ be any net (see Kelley,⁸ Chapter 2) so that $\lim_n f(n)$, $\overline{\lim}_n f(n)$, and $\frac{\lim}{n} f(n)$ have meaning.

Then the sets

1. $\{f \in C \ni \lim_n f(n) = 1\}$
2. $\{f \in C \ni \lim_n f(n) > 1\}$
3. $\{f \in C \ni \lim_n f(n) < 1\}$
4. $\{f \in C \ni \lim_n f(n) = 0\}$
5. $\{f \in C \ni \lim_n f(n) = \infty\}$
6. $\{f \in C \ni f(n) \text{ is eventually } > 1\}$

and the corresponding sets defined by $\underline{\lim}$ and $\overline{\lim}$ are directions in C . The relations they induce are equivalence and domination relations in the limit, respectively. The cones C/D they define consist of classes of functions exhibiting classes of limiting behavior.

Suppose the domain S of the functions in C is well ordered by $<$. Then the sets

$$f^+ = \{s \in S \ni f(s) > 1\}$$

and

$$f^- = \{s \in S \ni f(s) < 1\}$$

and

$$f^1 = \{s \in S \ni f(s) = 1\}$$

are disjoint and their union is S . Each of these sets f^* has a first element $w(f^*)$ if f^* is not empty. The set

$$D = \{f \in C \ni f^+ \text{ is not empty and either } f^- \text{ is empty or } w(f^+) < w(f^-)\}$$

is a direction. D induces the lexicographic order on C .

4.5 The Upper and Lower Directions, and the Relations They Induce

If C is any cone then $1 + C$ is a convex direction which we shall call the upper direction. Its elements will be called upper elements of C . The inverse direction is $1 \oplus C$ which we shall call the lower direction of C , and whose elements will be called lower elements. Their product, according to Equation (3) of Section 2.1 is C . Their intersection may be empty or the singleton set 1.

Theorem 4.15. If $c \in (1 + C) \cap (1 \oplus C)$ then $c = 1$.

Proof. Suppose $c \in 1 + C$ and also $c \in 1 \oplus C$, so that $c^{-1} \in 1 + C$. Then $c = 1 + x$, $c^{-1} = 1 + y$ for some $x, y \in C$. It follows that

$$(1 + x)(1 + y) = 1 + x + y + xy = 1.$$

Since 1 and $1 + x + y + xy$ are both in the convex set 1, it follows from Theorem 4.8 that the partial sum $1 + x$ is also in 1. Thus $1 + x = 1$, $c = 1$.

It follows from Theorem 4.15 that all cones fall into two disjoint categories,

1. Cones with dominance, or D-cones, in which $(1 + C) \cap (1 \oplus C) = 1$.
2. Cones without dominance, or $\sim$ D-cones (not D-cones), in which $(1 + C) \cap (1 \oplus C)$ is empty.

We shall say that an element $a \in C$ dominates an element $b \in C$ if $a + b = a$. If $a + b = a$ then $1 + a^{-1}b = 1$ so $1 \in 1 + C$. Since $1 \oplus C = J(1 + C)$ it follows also that $1 \in 1 \oplus C$, and $(C, \oplus, \times)$ is a cone with dominance. In a D-cone, every element dominates some element and is dominated by some element. In a $\sim$ D-cone, no element dominates another element.

The relations induced by $1 + C$ and $1 \oplus C$ will be denoted by $>$ and $<$, respectively, pronounced "greater than" and "less than." The following conditions on a pair a, b of elements of C are thus equivalent:

$$a < b, a \in b(1 + C), a = b + bc,$$

for some $c \in C$, $a = b + x$ for some $x \in C$, $b > a$, $b \in a(1 \oplus C)$, $b = a \oplus ac$ for some $c \in C$, $b = a \oplus x$ for $x \in C$.

If C is a D-cone then $>$ and $<$ are reflexive and it is more appropriate in this case to use the symbols $\geq$ and $\leq$ for the partial order relations induced by the upper and lower directions.*

4.6 The Sum of a Cone and a Normal Subcone. Convex Closure of a Normal Subcone.

Theorem 4.16. Let C be a cone and let S be a normal subcone of C . Then $C + S$ is a direction in C , closed under $+$ and $\oplus$. Its inverse, $C \oplus S$ is also a direction in C . Their intersection, $(C + S) \cap (C \oplus S)$ is a convex subcone of C , and is the convex closure of S in C .

Proof. Clearly $C + S$ is a subsemicone of C ; that is, $C + S$ is closed under $+$ and $\times$. Also, $C + S$ is closed under interpolation. For suppose that $c_1 + s_1, c_2 + s_2 \in C + S$, $c_1 \in C, s_1 \in S$. Then there exist $s_3, s_4 \in S$ such that

$$s_1 = (s_1 \oplus s_2) + s_3, s_2 = s_1 \oplus s_2 + s_4,$$

(by the Compensation Theorem 1.1); hence if $a, b \in C$, $a + b = 1$,

$$\begin{aligned} & a(c_1 + s_1) + b(c_2 + s_2) \\ &= a((c_1 + s_3) + (s_1 \oplus s_2)) + b((c_2 + s_4) + (s_1 \oplus s_2)) \\ &= a(c_1 + s_3) + b(c_2 + s_4) + (s_1 \oplus s_2) \\ &\in C + S. \end{aligned}$$

Thus $C + S$ is convex.

Similarly by the Compensation Theorem, there exist elements c_3 and c_4 in C such that

$$\begin{aligned} (c_1 + s_1) \oplus c_3 &= s_1 \\ (c_2 + s_2) \oplus c_4 &= s_2. \end{aligned}$$

Taking lower sums,

$$(c_1 + s_1) \oplus (c_2 + s_2) \oplus c_3 \oplus c_4 = s_1 \oplus s_2$$

* $(C, <)$ is a poset and J is an involution of C .⁷

Applying the Compensation Theorem again, there is a $c_5 \in C$ such that

$$\begin{aligned}(c_1 + s_1) \oplus (c_2 + s_2) &= ((c_1 + s_1) \oplus (c_2 + s_2) \oplus c_3 \oplus c_4) + c_5 \\ &= (s_1 \oplus s_2) + c_5.\end{aligned}$$

Thus

$$(c_1 + s_1) \oplus (c_2 + s_2) \in C + S$$

and $C + S$ is closed under $\oplus$.

Dually, $J(C + S) = C \oplus S$ is closed under $+$, $\oplus$, $\times$, and interpolation. Thus their intersection, $(C + S) \cap (C \oplus S)$ is closed under $+$, $\oplus$, $\times$, and interpolation and also under J . Thus $(C + S) \cap (C \oplus S)$ is a convex subcone of C . Since $C + S$ includes $S + S = S$, and dually, $C \oplus S$ includes S , their intersection includes S . Let u be any element of the intersection. Then

$$u = c_1 + s_1 = c_2 \oplus s_2$$

for some $c_1, c_2 \in C, s_1, s_2 \in S$.

By Corollary 2 of Theorem 4.8 on Partial Sums it follows that u is an interpolate of s_1 and s_2 . Thus u is in the convex closure of S . Thus $(C + S) \cap (C \oplus S)$ is the convex closure of S .

The convex cone $(C + S) \cap (C \oplus S)$ will be called the cone of S-bounded elements of C. Each of its elements is an interpolate of elements of S .

4.7 Powers of c in a Direction

Theorem 4.7 states that the set N of integers n such that c^n is in a convex set X is a convex set of integers. If D is a direction, then N is also closed under addition because D is closed under $\times$. It follows that if N includes a positive integer n , then N includes 1, and therefore N includes every positive integer n .

Theorem 4.17. Let D be a direction, $c \in D$, and let n be a positive integer such that $c^n \in D$. Furthermore, let n be the least such integer. Then $n = 1$.

Proof. Suppose $n = 2$. Then $c^2 \in D$. By the divisor property of D , $c^2 = ab$, $a, b \in D$. Since D is normal, $c^{-1}ac \in D$. Thus

$$(c + a)c = c^2 + ac$$

$$(c + a)c = ab + ac$$

$$(c + a)c = ab + c(c^{-1}ac).$$

This is a left interpolation equation with coefficients c and a and makes ca a left interpolate of b and $c^{-1}ac$, both in D . Thus $c \in D$, and 2 is not minimal, a contradiction.

Clearly n is not an even number, for if $n = 2k$, $c^{2k} \in D$ implies, by the preceding argument, that $c^k \in D$, and n is not minimal.

If n is an odd integer, $2k + 1$, $k \geq 1$, then $c^{2k+1} \in D$ implies $c^{4k+2} \in D$, which, by Theorem 4.7 implies $c^{2k+2} \in D$. Thus by the argument above, $c^{k+1} \in D$. Thus $2k + 1$ is not minimal. Having exhausted all other possibilities, $n = 1$.

5. Classification of Cones and Cone Homomorphisms

5.1 L-Cones

It was noted in Theorem 4.7 and its corollaries that $(C, \times)$ has no finite subsemigroups but 1, and therefore the only finite cone is L, the cone of one element. If a cone C includes a finite subcone isomorphic to L, we shall say that C is an L-cone. Clearly L-cones are characterized by the identity $1 + 1 = 1$. It follows that $c + c = c$ for every $c \in C$. Thus L-cones are D-cones, since every element in an L-cone dominates itself. It follows also that in an L-cone the relations $\leq$ and $\geq$ induced by the upper and lower directions are lattice order relations. For if $a, b \in C$, then $a + b \geq a$, $a + b \geq b$ and furthermore, if $c \geq a$ and $c \geq b$, it follows that $c + c \geq a + b$. But $c + c = c$, so $c \geq a + b$. Thus $a + b$ is the least upper bound of a and b . Dually, $a \oplus b$ is the greatest lower bound of a and b . Thus

Theorem 5.1. If $(C, +, \times)$ is an L-cone, then $(C, \times)$ is a lattice ordered group and $+$ is the least upper bound operation in C. Conversely if $(C, \times)$ is a lattice ordered group and $+$ is the least upper bound operation, then $(C, +, \times)$ is an L-cone.

5.2 F-Cones

If $(C, +, \times)$ is not an L-cone, then $1 + 1 \neq 1$ in C. It follows that the subsemigroup S, of $(C, +)$ generated by 1 is not the singleton set 1. But S is also a semigroup under $\times$, and by the corollary to Theorem 4.8 it follows that S is infinite. It follows that $(S, +, \times)$ satisfies Peano's postulates and is, therefore, isomorphic to the semicone Ω of natural numbers. It follows immediately, that $S \times J(S)$ is a cone isomorphic with F, the cone of fractions. Thus C includes a cone isomorphic with F.

Definition 5.1. A cone is an F-cone if it includes a subcone isomorphic with F.

Theorem 5.2. Every cone is an L-cone or an F-cone and no cone is both.

5.3 F-Bounded Cones and Cancellation Cones.

Theorem 4.16 furnishes us with another method of classifying F-cones. According to Theorem 4.16, if C is an F-cone, and therefore includes F as a subcone, then C includes a normal subcone $FB(C) = (C + F) \cap (C \oplus F)$ whose elements b are F-bounded, that is, satisfy a relation

$$p < b < q$$

for some $p, q \in F$. $FB(C)$ is the convex closure of F in C. If $FB(C) = C$, then we shall say that C is an F-bounded cone or an FB-cone. An F-cone which is not an FB-cone, will be called an $F \sim B$ (F not B) cone. F and P are examples of FB cones; the monogenic cone $F[x]$ of fractional functions of x , and the cone of monomial orders are examples of $F \sim B$ -cones.

A cone $(C, +, \times)$ will be called a cancellation cone or C-cone if $(C, +)$ is a cancellation semigroup. If C is not a C-cone it will be called a $\sim C$ -cone. DC, $D \sim C$, $C \sim D$ and $\sim C \sim D$ have the obvious meanings.

Theorem 5.3. L is the only DC-cone.

Proof. Let C be a DC-cone. Since C is a D-cone there is a $c \in C$ such that $1 + c = c$. Let $a \in C$. Then $a + ac = ac$, hence $a + c + ac = ac + c = ac + c + 1$. Since C is a C-cone it follows that $a = 1$. Thus all elements of C are equal to 1; $C = L$.

Theorem 5.4. Every FB-cone is a C-cone (due to M. Schlessinger).

Proof. Let $a, b, c \in C$, an FB-cone, and suppose $a + b = a + c$. Since C is an FB-cone, there is a fraction $p \in F$ such that $a^{-1}b > p$. Furthermore, there is a positive integer n , such that $p > 2^{-n}$, hence $a^{-1}b > 2^{-n}$. Thus $a < 2^n b$.

Let

$$2^n b = a + x.$$

Then

$$a + x + b = a + x + c,$$

$$2^n b + b = 2^n b + c.$$

Now let k be any positive integer such that

$$2^k b + b = 2^k b + c.$$

Then

$$2^k b + b + b = 2^k b + b + c = 2^k b + c + c,$$

$$2^k b + 2b = 2^k b + 2c.$$

Multiplying by $1/2$

$$2^{k-1} b + b = 2^{k-1} b + c.$$

Hence by reverse induction

$$2b = b + c.$$

Similary

$$2c = b + c,$$

$$2b = 2c,$$

Multiplying by $1/2$,

$$b = c.$$

Thus C is a C-cone.

Summarizing, cones fall into six disjoint categories, with examples as follows.

Category	Finite	Infinite				
	L-Cones		F-Cones			
	D-Cones			$\sim$ D-Cones		
	LCD-Cone	$\sim$ C-Cones			FC-Cones	
		Includes all lattice ordered groups except L		$F \sim C \sim D$	FC $\sim$ B - cones	FB - cones
Examples	L (only example)	(F, max, $\times$) (P, max, $\times$)	MO (monomial orders)	(F, +, $\times$) $\times$ (Z, min, +)	F[x]	F P

5.4 Cone Homomorphisms

If $(C, \times, +)$ is a cone and h maps C homomorphically onto another cone, then in particular, h is a group homomorphism of $(C, \times)$. It follows that the kernel, K , of h is a normal subgroup of $(C, \times)$. From the additivity of h , it follows also, that K is convex. For every left interpolate of elements of K has the form $ak + b\ell$ where $k, \ell \in K$, $a, b \in C$ and $a + b = 1$. It follows that

$$\begin{aligned} h(ak + b\ell) &= h(a)h(k) + h(b)h(\ell) \\ &= h(a) + h(b) \quad \text{because } k, \ell \in K \\ &= h(a + b) \\ &= h(1) \quad \text{because } a + b = 1. \end{aligned}$$

Thus $ak + b\ell \in K$, and K is convex. Thus K is a symmetric direction.

Conversely, every symmetric direction K is the kernel of a cone homomorphism, in particular of the natural cone homomorphism h_K of C onto C/K .

Thus the symmetric direction, or convex normal subgroup, plays the same role in cone theory as the normal subgroup in group theory, or the ideal in ring theory. The analogy breaks down in one respect, however, the symmetric direction is in general, not a subcone. The special case in which it is a subcone will be treated in the next theorem.

Because symmetric directions in C are kernels of cone homomorphisms on C , we shall denote them henceforth, simply as kernels in C .

Theorem 5.5. Let C be an F-cone and K a kernel in C . Then the following conditions upon K are equivalent.

- (1) K is a cone
- (2) $2 \in K$
- (3) $K \supset F$
- (4) $K \supset FB(C)$
- (5) C/K is an L-cone.

Proof. Clearly (1) implies (2). Let $a, b \in K$ and suppose (2) holds. Then $1/2(a + b)$ is an interpolate of elements of K , hence, since K is convex, is in K . Since also, $2 \in K$ and K is closed under $\times$, it follows that $a + b = 2 \times 1/2 \times (a + b)$ is in K . Thus K is closed under addition, and is a cone. Thus (2) implies (1).

If (1) holds, then K is either an F-cone or an L-cone. If K is an L-cone, then C is an L-cone, contrary to hypothesis. Thus $K \supset F$ and (1) implies (3). Clearly (3) implies (2). Thus (3) is equivalent to (1) and to (2).

Since $FB(C) \supset F$, (4) implies (3). Since $FB(C)$ is the convex closure of F in C , and K is convex in C , (3) implies (4).

An L-cone is characterized by the identity $\bar{1} + \bar{1} = \bar{1}$ where $\bar{1}$ is the identity element of the L-cone. Thus (2) and (5) are equivalent.

Corollary. Let C be any F -cone and let h be any homomorphism on C into an L -cone, $h(C)$. Then there exists a homomorphism $\bar{h}$ on $C/FB(C)$ into $h(C)$ such that $h = h_{FB(C)} \bar{h}$. Schematically, the diagram

is analytic.

Proof. This follows from (4) of Theorem 5.5.

This corollary splits cone theory into three parts

- (1) The theory of FB -cones
- (2) The theory of L -cones
- (3) The extension problem: given an FB -cone B and an L -cone Q , to characterize the class of F -cones C , such that $FB(C)$ is isomorphic to B and $C/FB(C)$ is isomorphic to Q .

Part (1) is treated extensively in a forthcoming paper¹⁰ by S. van Domelen. Part (2) is treated extensively in References 1 and 6 as the theory of lattice ordered groups.

5.5 Cuts and the Cut Kernel

A $(1 + C)$ -sector, or coset of $1 + C$ is a subset of C which includes with any c , all elements $> c$ and some but not all elements $< c$. Such a set is known classically as a cut (e.g. Dedekind cut) of C . By Theorem 4.12 the set of all cuts of C is a cone, $C/1 + C$. There is a natural homomorphism h_{1+C} on C into the cone of cuts, $C/1 + C$. The kernel of h_{1+C} is $K(1 + C)$. We shall call this kernel the cut kernel of C . An element k of C is in the cut kernel if and only if for every $c \in C$

$$1 \oplus c < k < 1 + c,$$

that is, k is less than every element greater than 1 and greater than every element less than 1. Thus $1 \in K(1 + C)$ and if $<$ linearly orders C , then 1 is the only element of $K(1 + C)$. There are interesting examples, however, in which $<$ does not linearly order C and the cut kernel includes other elements.

Let F_n be the cone of $n \times n$ upper triangular matrices over the rational field, with positive coefficients on the diagonal. The relation $<$ is then defined in F_n strictly in terms of the relations between diagonal elements. The cone $F_n/1 + F_n$ is isomorphic with the cartesian product of n copies of P . Thus the cut kernel $K(1 + F_n)$ consists of those matrices in F_n whose diagonal elements are all 1.

Definition 5.2. A subgroup $(G, \times)$ of a cone $(C, +, \times)$ is uniformly bounded above, and b is a uniform upper bound for G if $g < b$ for every $g \in G$.

Theorem 5.6. If C is an FB -cone and if $(G, \times)$ is a subgroup of $(C, \times)$ uniformly bounded, then b is a uniform upper bound for $(G, \times)$ if and only if $b \in 1 + C$.

Proof. Since $1 \in G$, if b is a uniform upper bound for G , then in particular, $b > 1$, $b \in 1 + C$. Conversely, let u be any uniform upper bound for G . Then $u \in 1 + C$, $u = 1 + x$ for some $x \in C$. Since C is an FB-cone, there is an $f \in F$ such that $x < f$, so $1 + f$ is also a uniform upper bound for G . It follows that $1 + f + (1/2f)^2$ is also a uniform upper bound for G . Thus for every $g \in G$, $g^2 \in G$, and $g^2 < 1 + f + (1/2f)^2 = (1 + 1/2f)^2$. Thus $((1 + 1/2f)g^{-1})^2 \in 1 + C$ and by Theorem 4.17 it follows that $(1 + 1/2f)g^{-1} \in 1 + C$, $g < 1 + 1/2f$. By induction $g < 1 + 2^{-n}f$ for every positive integer n . Now let $b = 1 + y$ be any element of $1 + C$. Since C is an FB-cone, there is a positive integer n such that $2^{-n}f < y$, so $g < 1 + 2^{-n}f < 1 + y = b$, $g < b$ for every $g \in G$, $b \in 1 + F$.

Corollary. The cut kernel, $K(1 + C)$, is the union of all uniformly bounded subgroups of $(C, \times)$.

Proof. Since $K(1 + C)$ is itself uniformly bounded, it is in the specified union. Conversely, if g is in some uniformly bounded subgroup of $(C, \times)$ then by Theorem 5.6 $g < 1 + c$ for every $c \in C$. Also $g^{-1} < 1 + c^{-1}$ for every $c \in C$, and taking inverses, $g > 1 \oplus c$. Thus

$$1 \oplus c < g < 1 + c$$

for every $c \in C$, and $g \in K(1 + C)$.

6. Cones of Automorphisms

Conrad³ has studied the group A of order-preserving automorphisms of a linearly ordered, additively written, abelian group $(G, +, <)$. In particular, he has shown that under certain conditions on $(G, +, <)$, A can be ordered. If we define $<$ in terms of the min function (or max), then $(G, +, \min)$ is a cone and the order preserving automorphisms become, simply, the cone automorphisms of $(G, +, \min)$. We shall consider the conditions under which A is a cone. The conditions imposed are very similar to those found by Conrad to be sufficient for the orderability of A .

Conrad notes that the order-preserving automorphisms of G can be represented by a set Δ of triangular matrices. If G is the group of n -dimensional real vectors, ordered lexicographically then Δ is the set of $n \times n$ triangular matrices with positive elements on the diagonal. We have noted in Section 1, that this system is a cone. It is this fact which will be generalized here.

Fuchs⁶ and others have made use of a natural valuation, w , on a linearly ordered group, defined as follows. For $g \in G$,

$$w(g) = \text{the minimum convex subgroup of } G \text{ which includes } g.$$

A subgroup of G is convex if it includes with f and g , all h such that $f < g < h$. The valuation w has the following properties

- (1) The range of w is linearly ordered by inclusion.
- (2) $w(g) = 0$ if and only if $g = 0$.
- (3) $w(ng) = w(g)$ for every integer $n \neq 0$.
- (4) $w(a + b) \subset w(a) \cup w(b)$.

If a and b have the same sign or are both zero, then (4) becomes an equality.

Another useful result stated by Fuchs⁶ as a lemma to Hahn's embedding theorem is the following.

Lemma B. If $x, y \in G$, and $y \neq 0$, and if $w(x) = w(y)$, then there is a real number r such that $w(y - rx) \subsetneq w(y)$.

6.1 Order-Preserving Endomorphisms of a Linearly Ordered Real Vector Space

Let $(V, +, <)$ be a linearly ordered real vector space, and let Γ be the range of the natural valuation $w(v)$ as v ranges over V . Then Γ is a set of convex subspaces of V , linearly ordered by inclusion.

Definition 6.1. An order endomorphism, c , of V is a mapping of V onto itself such that

1. $c(v_1 + v_2) = c(v_1) + c(v_2)$,
2. $c(av) = ac(v)$,
3. $v_1 > v_2 \rightarrow c(v_1) > c(v_2)$.

If also,

4. There is an inverse mapping c^{-1} such that $c(c^{-1}(v)) = v$ and $c^{-1}(c(v)) = v$ for all $v \in V$, then c is an order automorphism of V .

It is clear that the order automorphisms of V form a group. It is the purpose of the next few lemmas to show that if Γ is well ordered then the order automorphisms of V form a cone (compare with Conrad's Lemmas 1 and 2 and corollary).

Lemma 1. If the Γ is well ordered by inclusion, then every order automorphism of V preserves value.

Proof. Suppose the contrary. Then there is an order automorphism c and a vector v such that

$$w(c(v)) \subsetneq w(v).$$

(If the reverse inequality holds, replace c by c^{-1} and v by $c(v)$.) That is, there is a vector u such that $u \in w(c(v))$, but $u \notin w(v)$.

It follows that

$$c(u) \in w(c^2(v)),$$

but

$$c(u) \notin w(c(v)).$$

Thus also

$$w(c^2(v)) \subsetneq w(c(v)).$$

Similarly

$$w(v) \subsetneq w(c^{-1}(v)).$$

By induction, it follows for all integers n , both positive and negative, that

$$w(c^{n+1}(v)) \subsetneq w(c^n(v)).$$

But Γ is well ordered, so there is an n for which $w(c^n(v))$ is minimal. This is a contradiction, since $w(c^{n-1}(v)) \subsetneq w(c^n(v))$. This proves the lemma.

Lemma 2. The value preserving, order preserving endomorphisms of V are closed under $+$.

Proof. Let a and b be order preserving, value preserving endomorphisms of V , and let u be a vector in V .

Then

$$\begin{aligned} w((a+b)(u)) &= w(a(u) + b(u)), \\ &= \max(w(a(u)), w(b(u))), \\ &= \max(w(u), w(u)), \\ &= w(u). \end{aligned}$$

The first equality holds because $a(u)$ and $b(u)$ have the same sign or are both zero.

Lemma 3. If Γ is well ordered, then every order endomorphism of V which preserves values is an order automorphism of V .

Proof. Let c be an order endomorphism which preserves values. We must show the existence of an inverse mapping, c^{-1} . This is equivalent to showing that c is one-to-one and onto.

c is one-to-one because it preserves order. For suppose $u > v$. Then $c(u) > c(v)$. Similarly, if $u < v$, then $c(u) < c(v)$. Thus only if $u = v$ does $c(u) = c(v)$.

Next we shall show that c is onto because it preserves values. Suppose c is not onto. Then the range $c(V)$ is not all of V . Let $U = V \setminus c(V)$, the complement of $c(V)$ in V . Then U is not empty. Let $w(U) = \{w(u) \mid u \in U\}$. Then $w(U)$ is a nonempty set of values. Since Γ is well ordered, $w(U)$ has a least element, $w(u)$ for some $u \in U$. Now $c(u)$ has the same value as u , so by Lemma B there is a scalar a such that

$$c(u) = a u + r$$

where $w(r) < w(u)$. Thus $w(r) \notin \text{deg}(U)$, since $w(u)$ is the smallest element of $w(U)$. Thus $r \notin U$, $r \in c(V)$. That is, $r = c(v)$ for some $v \in V$. Thus

$$c(u) = a u + c(v)$$

$$u = \frac{c(u) - c(v)}{a} = c\left(\frac{u - v}{a}\right) \in c(V)$$

Thus $u \in c(V)$. This is a contradiction since $u \in U = V \setminus c(V)$. Thus c is onto. Thus c^{-1} exists and c is an order automorphism.

Theorem 6.1. The order automorphisms of a real, totally ordered vector space, with well ordered values, form a cone.

Proof. It is clear that they form a multiplicative group. Further, the sum of two such automorphisms is an order endomorphism which preserves values. Thus it is also an automorphism. Thus the order automorphisms form a cone.

7. Monogenic Cones

Throughout this section B will denote $FB(F[x])$, the cone of F -bounded fractional functions of x . A monogenic cone is the image of the cone $F[x]$ of fractional functions of x under some homomorphism h . Thus every monogenic cone is isomorphic to $F[x]/K$ for some kernel K of $F[x]$. Furthermore, $F[x]/K$ is an L -cone if and only if K includes B .

7.1 The Monogenic L -Cones

Let $p(x)$ be any polynomial in x with coefficients in F . Let $m(p)$ be the smallest exponent of x occurring in $p(x)$. If $q(x)$ is another polynomial with coefficients in F , then

$$m(p + q) = \min(m(p), m(q)),$$

$$m(p \times q) = m(p) + m(q).$$

Let

$$r(x) = \frac{p(x)}{q(x)},$$

be any fractional function of x , and define

$$m(r) = m(p) - m(q).$$

Although p and q are not uniquely determined by r , and hence $m(p)$ and $m(q)$ are not uniquely determined by r , it is nevertheless true, that $m(r)$ is uniquely determined by r . For if

$$r(x) = \frac{p_1(x)}{q_1(x)} = \frac{p_2(x)}{q_2(x)}$$

where p_1, p_2, q_1, q_2 are polynomials, it follows that $p_1(x)q_2(x) = p_2(x)q_1(x)$, hence

$$m(p_1 \times q_2) = m(p_2 \times q_1),$$

$$m(p_1) + m(q_2) = m(p_2) + m(q_1),$$

$$m(p_1) - m(q_1) = m(p_2) - m(q_2) = m(r).$$

Thus the value of $m(r)$ is independent of the representation of r .

It is clear that if r_1 and r_2 are two fractional functions of x , then

$$m(r_1 + r_2) = \min(m(r_1), m(r_2)),$$

$$m(r_1 \times r_2) = m(r_1) + m(r_2).$$

Thus

Lemma 1. The function m maps $(F[x], +, \times)$ homomorphically onto $(Z, \min, +)$, where Z is the set of integers.

Thus $(Z, \min, +)$ is a monogenic L -cone and 1 is a generator of it.

Similarly, define $M(p)$ to be the largest exponent of x occurring in $p(x)$, and define

$$M(r) = M(p) - M(q),$$

where

$$r(x) = \frac{p(x)}{q(x)}.$$

Then

$$M(r_1 + r_2) = \max(M(r_1), M(r_2)),$$

$$M(r_1 \times r_2) = M(r_1) + M(r_2),$$

so that

Lemma 2. The function M maps $(F[x], +, \times)$ homomorphically onto $(Z, \max, +)$.

It follows that $(Z, \max, +)$ is also a monogenic cone and 1 is a generator of it. In fact, $(Z, \max, +)$ and $(Z, \min, +)$ are dual and isomorphic L-cones. Now define $mM(r) = (m(r), M(r))$.

Lemma 3. The function mM maps $(F[x], +, \times)$ homomorphically onto the cartesian product cone $(Z, \min, +) \times (Z, \max, +)$.

Thus $(Z, \min, +) \times (Z, \max, +)$ is also a monogenic L-cone, and $(1, 1)$ is a generator of it. The kernel of mM is the intersection of the kernels of M and m and consists of those fractional functions $r(x)$ such that

$$M(r) = m(r) = 0$$

Note that in general, $M(r) \geq m(r)$. Note also that M and m translate $\oplus$ into min and max, respectively.

Definition 7.1. If $p(x)$ is a polynomial in x with coefficients in F , then $\text{gap}(p)$ means the length of the longest block of consecutive integers between $m(p)$ and $M(p)$ which do not occur as exponents in terms of $p(x)$. If $\text{gap}(p) = 0$ then $p(x)$ is said to be gapless.

Lemma 4. If $p(x)$ is not gapless then $\text{gap}(p(x)(x+1)) = (\text{gap}(p)) - 1$.

Corollary. The polynomial $p(x)(x+1)^n$ is gapless if $n \geq \text{gap}(p)$.

Corollary. Every fractional function of x is the quotient of two gapless polynomials in x .

Proof. Any fractional function can be written

$$\frac{p(x)}{q(x)} = \frac{p(x)(x+1)^n}{q(x)(x+1)^n}$$

for any positive integer n . If

$$n \geq \max(\text{gap}(p), \text{gap}(q))$$

then the expression on the right is the quotient of two gapless polynomials.

Lemma 5. A fractional function $r(x)$ is in B if and only if

$$m(r) = M(r) = 0.$$

Proof. If $r(x) \in B$ then there is another fractional function $s(x)$ such that

$$r(x) + s(x) = f, f \in F.$$

Applying the homomorphism m ,

$$\begin{aligned} \min(m(r), m(s)) &= m(r + s), \\ &= m(f) = 0. \end{aligned}$$

Thus

$$m(r) \geq 0$$

Dually, there is a fractional function $t(x)$ such that

$$r(x) \oplus t(x) = g, g \in F.$$

Thus

$$\max(m(r), m(t)) = m(g) = 0,$$

hence

$$m(r) \leq 0.$$

Thus $m(r) = 0$. Similarly, $M(r) = 0$.

Conversely, suppose $m(r) = M(r) = 0$.

Let

$$r(x) = \frac{p(x)}{q(x)}$$

where p, q are gapless polynomials, with coefficients in F . Since $m(r) = M(r) = 0$ it follows that we can define

$$\begin{aligned} m &= m(p) = m(q) \\ M &= M(p) = M(q). \end{aligned}$$

Thus m and M are, respectively, the smallest and largest exponent of x appearing in p . They are also the smallest and largest exponents, respectively in q .

Since p and q are gapless,

$$p(x) = \sum_{i=m}^M p_i x^i$$

$$q(x) = \sum_{i=m}^M q_i x^i$$

where $p_i, q_i \in F$.

The quotients p_i/q_i are all in F and since they form a finite subset of F , they have a common lower bound ℓ , and a common upper bound u in F . Thus for all i , $m \leq i \leq M$,

$$\ell < \frac{p_i}{q_i} < u$$

Thus there are elements $s_i \in F$ such that

$$uq_i = p_i + s_i$$

Let

$$s(x) = \left(\sum_{i=m}^M s_i x^i \right) / q(x)$$

Then

$$r(x) + s(x) = u,$$

$$r(x) < u.$$

Similarly

$$l < r(x).$$

Thus

$$r(x) \in B.$$

This proves the theorem.

Thus

Theorem 7.1. $F[x]/B$ is isomorphic to the cartesian product $(Z, \min, +) \times (Z, \max, +)$.

Proof. The function mM of Lemma 3 maps $F[x]$ homomorphically onto $(Z, \min, +) \times (Z, \max, +)$, and by Lemma 4, B is its kernel.

From Theorem 7.1 and the corollary of Theorem 5.4 it follows that $(Z, \min, +) \times (Z, \max, +)$ and its homomorphs $(Z, \min, +)$ and $(Z, \max, +)$ and L are, up to isomorphism, the only monogenic L -cones. The homomorphisms which map $F[x]$ onto them are, respectively, mM (of Lemma 3), M (of Lemma 2), m (of Lemma 1) and the trivial homomorphism z which maps every element onto the one element of L . Their kernels, B , $F^0[x] = \{r \ni M(r) = 0\}$, $F^0[x^{-1}] = \{r \ni m(r) = 0\}$ and $F[x]$, respectively, are, therefore, the only kernels of $F[x]$ which include any element but 1 of F . $(Z, \min, +)$ and $(Z, \max, +)$ are dual cones, and isomorphic. Thus their cartesian product is isomorphic with $(Z, \min, +) \times (Z, \min, +)$, for which $(1, -1)$ is a generator.

7.2 Monogenic FB-Cones

The image $mM(K)$ of a kernel K of $F[x]$ will tell us something about the monogenic cone $F[x]/K$.

Since K is a convex subgroup of $(F[x], \times)$, and mM maps $(F[x], \times, +)$ homomorphically onto $(Z, \min, +) \times (Z, \max, +)$, it follows that $mM(K)$ is a convex subgroup of $(Z, +) \times (Z, +)$. But $(Z, +) \times (Z, +)$ has only four convex subgroups, $Z \times Z$, $Z \times 0$, $0 \times Z$ and 0×0 .

Theorem 7.2. $F[x]/K$ is a monogenic FB-cone if and only if $mM(K) = Z \times Z$.

Proof. Let $\bar{x}$ be the image of x under the natural homomorphism on $F[x]$ onto $F[x]/K$. If $F[x]/K$ is an FB-cone, then there exists an element $\bar{y} \in F[x]/K$ and $f \in F$ such that

$$\bar{x} = \bar{y} + fK.$$

Thus there exist $k_1, k_2, k_3 \in K$ such that

$$xk_1 = yk_2 + fk_3$$

Applying the homomorphism m on $(F[x], +, \times)$ into $(Z, \min, +)$

$$m(xk_1) = \min(m(yk_2), m(fk_3)),$$

$$m(xk_1) \leq m(fk_3),$$

$$m(x) + m(k_1) \leq m(f) + m(k_3).$$

Since $m(f) = 0$, and $m(x) = 1$

$$1 = m(x) \leq m(k_3) - m(k_1) = m(k_3 k_1^{-1})$$

Since $m(k_3 k_1^{-1}) \in m(K)$, there is an element greater than or equal to 1 in $m(K)$. Thus $m(K) \neq 0$.

Dually, $M(K) \neq 0$. Thus $mM(K) = Z \times Z$.

Now suppose, conversely, that $mM(K) = Z \times Z$. Let $r(x)$ be any fractional function of x . Since $mM(k)$ takes all values in $Z \times Z$ as k ranges over K , there is, in particular, a $k \in K$ such that

$$m(k) < m(r) \leq M(r) < M(k).$$

Now let $d(x)$ be a common multiple of the denominators of k and r so that kd and rd are both polynomials. Also, let d be a multiple of $(1+x)^n$ where n is not less than the gaps of the numerators of k and r so that kd and rd are gapless.

Multiplication by d increases m by $m(d)$ and M by $M(d)$ so the above inequality is preserved,

$$m(kd) < m(rd) \leq M(rd) < M(kd)$$

Since kd is gapless

$$kd = \sum_{i=m(kd)}^{M(kd)} k_i x^i \quad k_i \in F$$

and since rd is gapless

$$rd = \sum_{i=m(rd)}^{M(rd)} r_i x^i \quad r_i \in F$$

Now r_i/q_i , $m(rd) \leq i \leq M(rd)$ are a finite number of elements of F , and they have a common lower bound, ℓ in F and a common upper bound u in F . Thus we can define s_i, t_i , $m(kd) \leq i \leq M(kd)$ so that

$$\left. \begin{aligned} s_i &= uk_i \\ t_i &= \ell k_i \end{aligned} \right\} \quad \begin{aligned} &\text{for } m(kd) \leq i < m(rd) \\ &\text{and } M(rd) < i \leq M(kd). \end{aligned}$$

$$\left. \begin{aligned} s_i + r_i &= uk_i \\ r_i &= \ell q_i + t_i \end{aligned} \right\} \quad \text{for } m(rd) \leq i \leq M(rd)$$

and define

$$t(x) = \sum_{i=m(kd)}^{M(kd)} t_i x^i$$

and

$$s(x) = \sum_{i=m(kd)}^{M(kd)} s_i x^i$$

Thus

$$s(x) + r(x) d(x) = u k(x) d(x),$$

and

$$r(x) d(x) = \ell k(x) d(x) + t(x).$$

Dividing by $d(x)$

$$\frac{s(x)}{d(x)} + r(x) = u k(x),$$

$$r(x) = \ell k(x) + \frac{t(x)}{d(x)}.$$

Applying the natural homomorphism h_k on $F[x]$ into $F[x]/K$,

$$h_k\left(\frac{s(x)}{d(x)}\right) + h_k(r(x)) = u,$$

$$h_k(r(x)) = \ell + h_k\left(\frac{t(x)}{d(x)}\right).$$

Thus u and ℓ are upper and lower bounds, respectively, of $h_k(r(x))$ in F . Thus every element in $F[x]/K$ has upper and lower bounds in F . Thus $F[x]/K$ is an FB-cone. This proves the theorem.

7.3 Monogenic FD-Cones

$F[x]/K$ is clearly a D-cone if and only if K includes at least one upper element of $F[x]$. It is an F-cone, on the other hand, if and only if K includes no upper element of F . Since K is convex, this implies that K must include no element of $1 + F + F[x]$. Thus K must include an element of the form $1 + s$ where s has no lower bound in F . One element meeting these requirements is $1 + x$. In the next few paragraphs the minimum kernel $K(1 + x)$ of $F[x]$ containing $1 + x$ will be described.

Lemma. $K(1 + x)$ is the set of fractional functions of the form

$$\frac{1 + x p(x)}{1 + x q(x)}$$

where p and q are polynomials.

Proof. For every positive integer n , $(1 + x)^n \in K(1 + x)$. Let

$$(1) \quad p(x) = \sum_i p_i x^i$$

where i ranges over a finite set of positive integers and $p_i \in F$. The binomial coefficient $\binom{n}{i}$ is an increasing and unbounded function of n , for fixed $i \geq 1$. It follows that for some n , $\binom{n}{i+1} > p_i$ for all values of i occurring in the summation of (1). Thus

$$1 < 1 + xp(x) = 1 + \sum_i p_i x^{i+1} < 1 + \sum_{i=0}^{n-1} \binom{n}{i+1} x^{i+1} = (1+x)^n$$

$$1 < 1 + xp(x) < (1+x)^n$$

By Theorem 4.8 (Partial Sums in a Convex Set) it follows from the convexity of $K(1+x)$, that $1 + xp(x) \in K(1+x)$. Similarly $1 + xq(x) \in K(1+x)$ and since $K(1+x)$ is a group,

$$\frac{1 + xp(x)}{1 + xq(x)} \in K(1+x).$$

Conversely, the set of functions of this form is a multiplicative group. We shall show that it is convex in $F[x]$.

Suppose $a_1(x)$, $a_2(x)$ and $s(x)$ are three fractional functions of x , and suppose that

$$(1) \quad a_1(x) \frac{1 + xp_1(x)}{1 + xq_1(x)} + a_2(x) \frac{1 + xp_2(x)}{1 + xq_2(x)} = (a_1(x) + a_2(x)) s(x).$$

We may suppose that $a_1(x)$ and $a_2(x)$ are polynomials, for if they are not, we may multiply (1) by a common multiple of their denominators. We may further normalize by dividing by the lowest order term of $a_1(x) + a_2(x)$, so that the low order term becomes 1. Solving (1) for $s(x)$

$$s(x) = \frac{a_1(x)(1 + xp_1(x))(1 + xq_2(x)) + a_2(x)(1 + xp_2(x))(1 + xq_1(x))}{(a_1(x) + a_2(x))(1 + xq_1(x))(1 + xq_2(x))}$$

Since the low order term of $a_1(x) + a_2(x)$ is 1, the numerator and denominator of $s(x)$ have low order terms of 1, and $s(x)$ has the form

$$\frac{1 + xp(x)}{1 + xq(x)}$$

Thus the set of such elements of $F[x]$ is a convex group and is the smallest such convex group including $1+x$. This proves the lemma.

Functions of this kind will be called monotropic functions of x because they approach 1 as x approaches zero.

Clearly no element of F but 1 is monotropic and no integral power of x but x^0 is monotropic.

Theorem 7.3. $F[x]/K(1+x)$ is a monogenic FD-cone. It is generated by an element $\bar{x}$ which satisfies the equation

$$1 + \bar{x} = 1$$

It consists of elements of the form $f\bar{x}^n$ where $f \in F$, $n \in \mathbb{Z}$ and $+$, $\times$ and J are defined by

$$f\bar{x}^n + g\bar{x}^m = \begin{cases} f\bar{x}^n & \text{if } n < m \\ (f+g)\bar{x}^n & \text{if } n = m \\ g\bar{x}^m & \text{if } n > m \end{cases}$$

$$(f\bar{x}^n) \times (g\bar{x}^m) = fg\bar{x}^{m+n}$$

$$J(f\bar{x}^n) = f^{-1} \bar{x}^{-n}$$

Proof. $\bar{x}$ is the image of x under the natural homomorphism. The identity $1 + \bar{x} = 1$ follows from the fact that $1 + x \in K(1 + x)$. The other identities follow by induction. The elements $f\bar{x}^n$ are all distinct because no element of F but 1 and no power of x but x^0 is in $K(1 + x)$. Clearly $F[x]/K(1 + x)$ is a monogenic FD-cone.

The cone $F[x]/K(1 + x^{-1})$ is constructed similarly, and the image $\bar{x}$ of x in it satisfies the identity

$$1 + \bar{x} = \bar{x}.$$

From this follows the relations

$$f\bar{x}^n + g\bar{x}^m = \begin{cases} f\bar{x}^n & \text{if } n > m \\ (f+g)\bar{x}^n & \text{if } n = m \\ g\bar{x}^m & \text{if } n < m \end{cases}$$

$$(f\bar{x}^n)(g\bar{x}^m) = fg\bar{x}^{m+n}$$

Cones of this kind are called cones of monomial orders.

Other monogenic FD-cones can be constructed by dividing out other kernels which intersect $1 + F[x]$ but not $1 + F + F[x]$. Each of them must include elements of the form z , where $1 + z = 1$, and therefore a subcone isomorphic to $1 + F[x]/K(1 + x)$, by the isomorphism which maps $\bar{x}$ onto z .

7.4 Monogenic FC ~ B Subcones of P

If p is any transcendental element of P , the cone of positive reals, then there is an isomorphism of $F[x]$ onto a monogenic subcone $F[p]$ of P , in which x is mapped onto p . All such cones $F[p]$ are isomorphic to each other and to $F[x]$. They are, therefore, like $F[x]$, FC ~ B monogenic cones. They provide examples; therefore of F ~ B subcones of an FB-cone. This seeming paradox arises out of the dependence of the meaning of "convex," "<" and "F-bounded" on the cone in which it is applied. Thus, for example in the cone, P , $\pi < 4$, because there is an element x in P such that $\pi + x = 4$. In $F[\pi]$, on the other hand π has no upper bound in F , because there is no x in $F[\pi]$ such that $\pi + x \in F$.

The subcones $F[p]$ of P , where p is an algebraic element of p will be discussed in Section 7.5.

7.5 Monogenic Cones of Square Matrices

We have seen examples of cones of $n \times n$ matrices. Each matrix which is in a cone of matrices is the generator of a monogenic cone of matrices.

All matrices referred to in this section and the next are $n \times n$ matrices ($n \in \Omega$), with rational coefficients. Whether a matrix A is in a cone or not depends on m_A , the minimum function of A .

Theorem 7.4. A matrix A is the generator of a monogenic cone of matrices if and only if the minimum function, $m_A(x)$ of A is relatively prime to every polynomial $f(x)$ with coefficients in F .

Proof. Suppose $m_A(x)$ is relatively prime to every $f(x)$ with coefficients in F . Then for every such f , $f(A)$ is nonsingular. Thus for every pair f, g of such polynomials, $f(A)g(A)^{-1}$ exists. The mapping $f(x)g(x)^{-1} \rightarrow f(A)g(A)^{-1}$ is a cone homomorphism, mapping x onto A . Thus $F[A]$ is a monogenic cone and A is a generator of it.

Now suppose conversely, that A lies in a cone, C . Let $f(x)$ be any polynomial with coefficients in F . Then $f(A)$ also lies in C .

Now suppose that $f(x)$ and $m_A(x)$ have a common factor $d(x)$. Then there exist polynomials $p(x), q(x)$ such that $f(x) = p(x)d(x)$, $m_A(x) = d(x)q(x)$. Thus $f(A) = p(A)d(A)$, $m_A(A) = d(A)q(A) = 0$. It follows that $f(A)q(A) = p(A)m_A(A) = 0$, and since $f(A)$ is in C , $f(A)^{-1}$ exists, so $q(A) = 0$. Since $m_A(A)$ is the minimum function of A it follows that $q(x)$ is not of lower degree than $m_A(x)$, so $d(x)$ is not of higher degree than 0. Thus the only common factors of $f(x)$ and $m_A(x)$ are those of degree 0, and $f(x)$ and $m_A(x)$ are relatively prime. This proves the theorem.

Theorem 7.5. If A is a matrix such that $F[A]$ is a cone, then $F[A]$ is an FB-cone.

Proof. The minimum function of A can be written

$$m_A(x) = p(x) - q(x)$$

where p and q both have coefficients in F and no exponent of x occurs in both. Thus $m(p) \neq m(q)$ and $M(p) \neq M(q)$. Since

$$m_A(A) = p(A) - q(A) = 0$$

it follows that $p(A) = q(A)$, hence

$$\frac{p(A)}{q(A)} = I$$

Thus the homomorphism h_A which maps x onto A and $F[x]$ onto $F[A]$ maps $p(x)/q(x)$ into I . Thus $p(x)/q(x) \in K_A$ where K_A is the kernel of h_A . Since $m(p) \neq m(q)$ it follows that $m(p/q) \neq 0$. Thus $m(K_A) \neq 0$. Similarly $M(K_A) \neq 0$. Thus by Theorem 7.2, $F[A]$ is an FB-cone.

Corollary. If p is an algebraic element of P , then $F[p]$ is an FB-subcone of P .

Proof. If $m(x)$ is the minimum function of p , with rational coefficients, and A is its companion matrix, then $F[p]$ and $F[A]$ are isomorphic. The corollary thus follows from Theorem 7.5.

7.6 Monogenic Cones of Matrices with Nontrivial Cut Kernel

The matrix cone

$$F \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right]$$

consists of all matrices of the form

$$\begin{pmatrix} f & x \\ 0 & f \end{pmatrix} \quad f > 0.$$

Its cut kernel consists of all matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Lemma 1. If z is an element of an FC-cone C , and if $z + z^{-1} = 2$, then $z^n + z^{-n} = 2$ for every $n \in \Omega$.

Proof. By induction on n . Suppose as induction hypothesis, that $z^k + z^{-k} = 2$ for $k < N$.

We have

$$z + z^{-1} = 1 + 1$$

Taking N th powers,

$$(z + z^{-1})^N = (1 + 1)^N$$

Expanding by the binomial theorem

$$\sum_{i=0}^N \binom{N}{i} z^{2i-N} = \sum_{i=0}^N \binom{N}{i}$$

Reversing the order on the left side

$$\sum_{i=0}^N \binom{N}{i} z^{N-2i} = \sum_{i=0}^N \binom{N}{i}$$

Adding

$$\sum_{i=0}^N \binom{N}{i} (z^{N-2i} + z^{2i-N}) = \sum_{i=0}^N \binom{N}{i} 2.$$

By induction hypothesis, corresponding terms on the two sides, except the first and last, are equal.

Since C is an FC-cone, we may cancel these. Thus

$$2(z^N + z^{-N}) = 4$$

$$z^N + z^{-N} = 2,$$

establishing the induction

Lemma 2. Under the conditions of Lemma 1, z is in the cut kernel of C .

Proof. By the corollary of Theorem 5.4 it suffices to show that z lies in a uniformly bounded multiplicative group. Since

$$z^n + z^{-n} = 2$$

for every $n \in \Omega$, it follows that z^n and z^{-n} are less than 2 for every $n \in \Omega$. Thus z is in the cut kernel. This proves the lemma.

Theorem 7.6. If A is a matrix and $F[A]$ is a monogenic matrix cone, then $F[A]$ has a nontrivial cut kernel, if the minimum function $m_A(x)$ of A is not square-free.

Proof. Suppose $m_A(x)$ is not square-free. Then there is a polynomial $p(x) - q(x)$, where p, q have coefficients in F , such that $m_A(x)$ divides $(p(x) - q(x))^2$ but not $p(x) - q(x)$. Thus $(p(A) - q(A))^2 = 0$ but $p(A) - q(A) \neq 0$. Thus $p^2(A) + q(A)^2 = 2p(A)q(A)$, but $p(A) \neq q(A)$. It follows that

$$\frac{p(A)}{q(A)} + \frac{q(A)}{p(A)} = 2$$

so $\frac{p(A)}{q(A)}$ is in the cut kernel, but is not equal to 1. Thus the cut kernel includes at least one element different from 1. It follows from Corollary 2 of Theorem 2.6 that the cut kernel is infinite.

8. A Pathological Cone

The cone described below has demolished several of the author's favorite conjectures, in particular, that the commutator of a cone lies in the cut kernel. The reader may find it useful in testing other conjectures.

Let M_Z be the class of infinite matrices over an ordered field, $A = (a_{i,j})$ with rows and columns indexed by the set Z of integers, and subject to the restriction that there is an integer $n(A)$ such that $a_{i,j} = 0$ for $i - j > n(A)$ and $a_{i,j} > 0$ for $i - j = n(A)$. Thus the positions for which $i - j = n(A)$ comprise the lowest nonzero diagonal of A , and it is occupied by positive elements. Infinite matrices are notoriously treacherous.^{11,12} Their products may not be defined at all, and even if defined, may not be associative. M_Z however is a cone. One element of M_Z is $a = (\delta_{i+1,j})$, whose elements are 1 immediately above the main diagonal, zero elsewhere. Another is $b = (2^i \delta_{i,j})$ whose elements are powers of 2 along the main diagonal. Their commutator, $a b a^{-1} b^{-1}$ is $2I$, clearly not in the cut kernel. ($I = (\delta_{i,j})$ is the identity matrix of M_Z .)

M_Z also exhibits the possibility of two cones in which $+$ and $\times$ are defined compatibly but J is not. The set M_Z^T , consisting of transposes of elements of M_Z is obviously a cone, and has a nonempty intersection with M_Z . The intersection is not a cone, however, for it is not closed under J . The matrix $I + a$, for example has one inverse in M_Z and another in M_Z^T , but none in the intersection.

M_Z can be thought of as the cone of order automorphisms of the vector space V_Z , of vectors $v = (\dots v_i, v_{i+1} \dots)$ whose coordinates are indexed by Z , the integers, both positive and negative, and subject to the restriction that there is a first nonzero coordinate, ordered lexicographically. Note that V_Z does not have well ordered degrees.

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