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REGGE-SLOPE EXPANSION OF MESON BORN AMPLITUDES

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Abstract

The expansion, in the Regge slope parameter α' , of simplified meson Born amplitudes comprising the Euler B function and its multiparticle generalisation is considered in order to proceed towards a Lagrangian density L which goes beyond the zero-slope-limit of Scherk.

A discussion of the Laurent-Taylor expansion of the four-meson amplitude leads, at least formally, to a Lagrangian density in terms of a scalar field ϕ ; this density is, however, non-local since it contains an infinite number of derivatives. Certain approximation schemes, in terms of higher-mass poles of the full amplitude, are introduced.

Dimensionality arguments applied also to the multiparticle extension lead naturally to the expansion of L in the Regge slope parameter, α' . The full Lagrangian density is given, up to order α'^2 relative to the zero slope limit. At this order it is necessary to consider only the four- and five-point functions.

I. INTRODUCTION

The zero-slope limit of the scalar boson Born amplitudes in the dual resonance model was first investigated by Scherk¹. He has shown that the four point amplitude represented by an Euler Beta-function and the N-point amplitude represented by the generalisation of the beta-function reduce, in the zero-slope limit, to the conventional ϕ^3 Lagrangian theory. This demonstration and further work² along these lines is of considerable interest because they provide a linkage between the hadronic models on the one hand, and Lagrangian field theories on the other. Since the latter is considerably more fully explored and understood than the former, we may hope to learn a great deal from the connection.

The present work is aimed towards a further exploration of these ideas. We consider the expansion of a dual Born amplitude in the slope parameter α' and study the higher-order (non-linear) terms that would vanish in the limit considered by Scherk¹. We investigate whether one can set up (even formally) a Lagrangian that can reconstruct the duality properties of the four-meson amplitude, and whether such a field theory is local. Thus in Section II, we display the higher order terms in the four-meson amplitude represented by the Euler Beta-function and show how one can write a Lagrangian. We also discuss how various poles contribute to the higher order terms. In Section III, multiparticle extensions of the beta function are considered with special emphasis on the five-point function. The final Section IV is the summary and conclusions. Although we examine only the very simplest Born amplitudes - the beta-function and its generalisation to N-particle amplitudes, it will be obvious that similar considerations can be applied to the more realistic Born amplitudes now available.³

II. FOUR-MESON AMPLITUDE

Consider the elastic scattering amplitude T_4 of four identical spinless particles with four momenta $p_1 + p_2 \rightarrow (-p_3) + (-p_4)$. Following Scherk¹ we write

$$T_4 = (g^2/2)[B(-\alpha_s, -\alpha_t) + B(-\alpha_t, -\alpha_u) + B(-\alpha_u, -\alpha_s)], \quad (2.1)$$

where

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 dt \, t^{x-1}(1-t)^{y-1} \quad (2.2)$$

and with m as the mass of the external particle,

$$x = -\alpha_s = -\alpha(0) - \alpha's = \alpha'(m^2-s), \quad (2.3a)$$

$$y = -\alpha_t = -\alpha(0) - \alpha't = \alpha'(m^2-t). \quad (2.3b)$$

Now if we let $g, \alpha' \rightarrow 0$ keeping $\lambda = g/\sqrt{\alpha'}$ and m^2 fixed (Scherk limit), we obtain

$$B(x,y) = \frac{x+y}{xy} \frac{\Gamma(1+x)\Gamma(1+y)}{\Gamma(1+x+y)} \xrightarrow{x,y \rightarrow 0} \left(\frac{1}{x} + \frac{1}{y}\right), \quad (2.4)$$

and

$$T_4 \xrightarrow{\alpha' \rightarrow 0} \lambda^2 \left(\frac{1}{m^2-s} + \frac{1}{m^2-t} + \frac{1}{m^2-u} \right). \quad (2.5)$$

Physically, in the above limit, the masses of all the excited states go to infinity leaving only the lowest mass ground state. It is not surprising then that T_4 can be interpreted as the Born term arising from the Lagrangian

$$L_0 = L_{\text{free}} + \frac{1}{6} \lambda \phi^3, \quad (2.6)$$

$$L_{\text{free}} = \frac{1}{2} (\partial_\mu \phi)^2 - m^2 \phi^2, \quad (2.7)$$

which is the starting point of Scherk's work.

A. Higher Order Terms

To go beyond this result, we let g, α' be non zero and consider the expansion in terms of λ and α' . This can be achieved in several different ways. But the most straightforward way is to consider a Taylor Series expansion of the function

$$E(x,y) = \frac{\Gamma(1+x)\Gamma(1+y)}{\Gamma(1+x+y)} = \sum_{r,s=0}^{\infty} D_{rs} x^r y^s \quad (2.8)$$

in the region $|x| < 1, |y| < 1$. The expansion will converge in this domain whose projection on the plane $\text{Re } x - \text{Re } y$ is shown in Fig. 1.

To evaluate the partial derivatives, we note that (putting $\phi(x,y) = \ln E(x,y)$)

$$\frac{\partial}{\partial x} (\ln E(x,y)) = \phi_x(x,y) = [\psi(1+x) - \psi(1+x+y)], \quad (2.9)$$

where $\psi(z)$ is the logarithmic derivative of the gamma function,

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}. \quad (2.10)$$

Likewise

$$\frac{\partial}{\partial y} (\ln E(x,y)) = \phi_y(x,y) = [\psi(1+y) - \psi(1+x+y)]. \quad (2.11)$$

Then

$$D_{rs} = D_{sr} = \left[\frac{1}{r!} \frac{1}{s!} \frac{\partial^r}{\partial x^r} \frac{\partial^s}{\partial y^s} E(x,y) \right]_{x=y=0}. \quad (2.12)$$

In terms of the logarithmic derivatives $\phi_x(x,y)$, $\phi_y(x,y)$ we find the following expression

$$D_{rs} = \frac{1}{r!s!} E(x,y) \left[\left(\phi_x + \frac{\partial}{\partial x} \right) \left(\phi_y + \frac{\partial}{\partial y} \right) 1 \right]_{x=y=0}. \quad (2.13)$$

This is convenient for direct evaluation of the Taylor coefficients, care being necessary to remark the non-commutativity of the operators. We can obtain an explicit formal expression for D_{rs} in the following way:

$$\frac{\partial^s}{\partial y^s} E(x,y) = E(x,y) \sum_{\substack{m=0 \\ \sum a_K = m \\ \sum ka_K = s}}^s (s; a_1, a_2, \dots, a_s) (\phi_y)^{a_1} (\phi_{yy})^{a_2} \dots (\phi_{sy})^{a_s}, \quad (2.14)$$

with the partition coefficients (e.g. Ref. 4)

$$(s; a_1, a_2, \dots, a_s) = \frac{s!}{(1!)^{a_1} a_1! (2!)^{a_2} a_2! \dots (s!)^{a_s} a_s!} \quad (2.15)$$

We note from (2.9) and (2.11) that

$$\begin{aligned} \left[\frac{\partial^r}{\partial x^r} \phi_x(x,y) \right]_{x=y=0} &= \left[\frac{\partial^r}{\partial y^r} \phi_y(x,y) \right]_{x=y=0} = 0, \\ \left[\frac{\partial^r}{\partial y^r} \phi_{(k+1)x}(x,y) \right]_{x=y=0} &= \left[\frac{\partial^r}{\partial y^r} \phi_{(k+1)y}(x,y) \right]_{x=y=0} = -\psi^{(r+k)}(1). \end{aligned} \quad (2.16)$$

Then

$$\left. \frac{\partial^r}{\partial x^r} E(x,y) \right|_{x=y=0} = 0 \quad (2.17)$$

and

$$\begin{aligned} & \frac{\partial^r}{\partial x^r} [(\phi_y)^{a_1} (\phi_{yy})^{a_2} \dots (\phi_{sy})^{a_s}] \\ &= \sum_{\substack{r_1, r_2, \dots, r_s \\ \sum_{k=1}^s r_k = r}} \frac{r!}{r_1! r_2! \dots r_s!} \left(\frac{\partial^{r_1}}{\partial x^{r_1}} (\phi_y)^{a_1} \right) \left(\frac{\partial^{r_2}}{\partial x^{r_2}} (\phi_{yy})^{a_2} \right) \dots \left(\frac{\partial^{r_s}}{\partial x^{r_s}} (\phi_{sy})^{a_s} \right). \end{aligned} \quad (2.18)$$

Hence the nonvanishing D_{rs} are given by

$$\begin{aligned} D_{rs} &= \frac{1}{r!} \frac{1}{s!} \sum_{\substack{m=0 \\ \sum a_K = m \\ \sum ka_K = s}}^s (s; a, a, \dots, a_s) \sum_{\substack{r_1, r_2, r_s \\ \sum r_K = r}} \frac{r!}{r_1! r_2! \dots r_s!} \\ & \quad \left[\left(\frac{\partial^{r_1}}{\partial x^{r_1}} (\phi_y)^{a_1} \right) \left(\frac{\partial^{r_2}}{\partial x^{r_2}} (\phi_{yy})^{a_2} \right) \dots \left(\frac{\partial^{r_s}}{\partial x^{r_s}} (\phi_{sy})^{a_s} \right) \right]_{x=y=0} \end{aligned} \quad (2.19)$$

where

$$\begin{aligned} \left[\frac{\partial^{r_n}}{\partial x^{r_n}} (\phi_{(p+1)y})^a \right]_{x=y=0} &= \left[\sum_{\substack{\ell_1, \ell_2, \dots, \ell_a \\ \sum \ell_i = r_n}} \frac{r_n!}{\ell_1! \ell_2! \dots \ell_a!} \left(\frac{\partial^{\ell_1}}{\partial x^{\ell_1}} \phi_{(p+1)y} \right) \left(\frac{\partial^{\ell_2}}{\partial x^{\ell_2}} \phi_{(p+1)y} \right) \right. \\ & \quad \left. \dots \left(\frac{\partial^{\ell_a}}{\partial x^{\ell_a}} \phi_{(p+1)y} \right) \right]_{x=y=0} \\ &= \begin{cases} 0 & \text{if any } \ell_i = 0 \\ \sum_{\substack{\ell_1, \ell_2, \dots, \ell_a \\ \sum \ell_i = r_n}} \frac{r_n!}{\ell_1! \ell_2! \dots \ell_a!} (-\psi^{(\ell_1+p)}(1)) (-\psi^{(\ell_2+p)}(1)) \\ & \quad \dots (-\psi^{(\ell_a+p)}(1)), \end{cases} \\ & \quad \text{if every } \ell_i \geq 1. \end{aligned} \quad (2.20)$$

It is seen that all the D_{rs} (related to the higher order coupling constants of the Lagrangian, discussed below) can be fully expressed in terms of derivatives,

up to order $(r+s-1)$ of the ψ function, taken with unit argument. The first few terms are as follows

$$\begin{aligned} E(x,y) = & 1 - xy \{ \psi^{(1)} + \frac{1}{2!} \psi^{(2)}(x+y) + \frac{1}{3!} \psi^{(3)}(x^2+y^2) \\ & - \frac{1}{2} (\psi^{(1)})^2 - \frac{1}{2!} \psi^{(3)} \}_{xy} + \frac{1}{4!} \psi^{(4)}(x^3+y^3) \\ & - \frac{1}{2} (\psi^{(1)} \psi^{(2)} - \frac{1}{3!} \psi^{(4)})_{xy}(x+y) \\ & + \frac{1}{5!} \psi^{(5)}(x^4+y^4) - \frac{1}{2} (\frac{1}{3} \psi^{(1)} \psi^{(3)} + \frac{1}{4} (\psi^{(2)})^2 - \frac{1}{4!} \psi^{(5)})_{xy}(x^2+y^2) \\ & - \frac{1}{3!} (- (\psi^{(1)})^3 + \frac{3}{2} (\psi^{(2)})^2 + \frac{3}{2} \psi^{(1)} \psi^{(3)} - \frac{1}{3!} \psi^{(5)})_{x^2 y^2} \} + O(\alpha'^7), \end{aligned} \quad (2.21)$$

where $\psi^{(m)} = \psi^{(m)}(1)$. Then

$$\begin{aligned} B(x,y) = & (\frac{1}{x} + \frac{1}{y}) - \psi^{(1)}(x+y) - \frac{1}{2!} \psi^{(2)}(x+y)^2 \\ & - \frac{1}{3!} \psi^{(3)}(x+y)^3 + \frac{1}{2} (\psi^{(1)})^2 + \frac{1}{3!} \psi^{(3)} \}_{xy}(x+y) + O(\alpha'^4), \end{aligned} \quad (2.22)$$

where we have noted that $x, y \sim \alpha'$.

When we sum over the permutations implied by (2.1), the contribution of the first nonsingular term to T_4 is given by

$$g^2 \alpha' [3m^2 - (s+t+u)] \psi^{(1)} = g^2 \alpha' [-3m^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^4 p_i \cdot p_j + \frac{3}{2} \sum_{i=1}^4 p_i^2] \psi^{(1)}. \quad (2.23)$$

To obtain this contribution from a Lagrangian, we have to add to (2.6) an additional piece⁵ L_2 where

$$L_2 = -\psi^{(1)} \lambda^2 \alpha'^2 [\frac{1}{4} (\partial_\mu \phi)^2 \phi^2 + \frac{1}{4} (\partial_\mu^2 \phi) \phi^3 + \frac{1}{8} m^2 \phi^4]. \quad (2.24)$$

It should be noted that we have deliberately not set $p_i^2 = m^2$. If the mass shell condition is used, (2.23) is simply $(\lambda^2 \alpha'^2 m^2 \psi^{(1)})$ which can be derived from any of the following three forms of the Lagrangian L_2

$$\psi^{(1)} \lambda^2 \alpha'^2 [\frac{1}{8} (\partial_\mu \phi)^2 \phi^2] , \quad \psi^{(1)} \lambda^2 \alpha'^2 [-\frac{1}{24} (\partial_\mu^2 \phi) \phi^3] \quad \text{or} \quad \frac{1}{24} \psi^{(1)} \lambda^2 \alpha'^2 m^2 \phi^4. \quad (2.25)$$

These alternate forms of the Lagrangian which give identical results as far as the four point function is concerned will not lead to the same result when we

consider the five-point function. We shall see this explicitly in the next section.

It is clear that we can, proceeding in this manner, write a Lagrangian density which reproduces any or all of the terms in the series expansion of $B(x,y)$. In principle, therefore, we can reconstruct the dual amplitude in two steps, as follows:

- (i) Go to a kinematical region where $|\alpha_s|, |\alpha_t|, |\alpha_u| < 1$. Use the full Lagrangian (containing arbitrarily many derivatives acting on ϕ^4) to deduce the S-matrix in a Born-approximation.
- (ii) This will reproduce, accurately, $B(-\alpha_s, -\alpha_t)$ which may then be analytically continued to all α_s, α_t from its defining functional form.

B. Higher Mass Resonance Poles and Higher Order Terms

As remarked at the beginning of this section, in the limit, $g, \alpha' \rightarrow 0$, all the higher mass states in $B(x,y)$ go to infinity. It is instructive to consider their contributions when $g, \alpha' \neq 0$.

For this purpose consider the expansion of $E(x,y)$ in terms of t-channel poles:

$$E(x,y) = \sum_{N=1}^{\infty} (-1)^{N-1} \frac{\Gamma(1+x)}{(N-1)! \Gamma(x-N+1)} \frac{1}{y+N}; \quad (2.26)$$

Now

$$\frac{1}{y+N} = \sum_{p=0}^{\infty} (-1)^p \frac{1}{N} \left(\frac{y}{N}\right)^p \quad (2.27)$$

Hence

$$E(x,y) = 1 - \sum_{N=1}^{\infty} \sum_{p=1}^{\infty} (-1)^p \frac{\Gamma(N-x)}{\Gamma(-x)} \frac{1}{N! N^p} y^p. \quad (2.28)$$

Therefore,

$$\left. \frac{\partial E(x,y)}{\partial x} \right|_{x=0} = \sum_{p=1}^{\infty} (-1)^p y^p \sum_{N=1}^{\infty} \frac{1}{N^{p+1}} = \sum_{p=1}^{\infty} (-1)^p y^p \zeta(p+1), \quad (2.29)$$

where $\zeta(z)$ is the Riemann zeta function.

From (2.8),

$$\left. \frac{\partial E}{\partial x} \right|_{x=0} = \psi(1) - \psi(1+y) = - \sum_{p=1}^{\infty} \frac{\psi^{(p)}}{p!} y^p = \sum_{p=1}^{\infty} D_{1p} y^p \quad (2.30)$$

From the important identity

$$(-1)^p \zeta(p+1) = -\frac{\psi^{(p)}}{p!}, \quad (2.31)$$

we have

$$D_{1p} = -\frac{\psi^{(p)}}{p!} = (-1)^p \zeta(p+1) = (-1)^p \sum_{N=1}^{\infty} \frac{1}{N^{p+1}}, \quad (2.32)$$

which shows how individual poles contribute to D_{1p} which is the coefficient of $(x+y)^p$ in the expansion (2.22) of $B(x,y)$. Thus the contribution of the N^{th} pole (i.e. $y = -N$ or $\alpha(t) = N$) to $\psi^{(p)}/p!$ falls off as $\frac{1}{N^{p+1}}$. This suggests an approximation scheme in which

$$\frac{\psi^{(p)}}{p!} \approx (-1)^{p+1} \quad (2.33)$$

i.e. we retain only the $N = 1$ pole. In this case

$$E(x,y) \approx 1 + x \sum_{p=1}^{\infty} (-1)^p y^p = 1 - x + \frac{x}{1+y} \quad (2.34)$$

and hence

$$B(x,y) \approx \left(\frac{1}{x} + \frac{1}{y} \right) - \frac{(x+y)}{(1+y)}. \quad (2.35)$$

From the point of view of writing an equivalent Lagrangian, eqs. (2.34) and (2.35) are unsuitable as they are manifestly not cyclic symmetric. If we use the approximation (2.33) in (2.19), we are led to the more interesting form

$$E(x,y) \approx 1 - \frac{xy}{(1+x)(1+y)} \quad (2.36)$$

and

$$B(x,y) \approx \left(\frac{1}{x} + \frac{1}{y} \right) - \frac{x+y}{(1+x)(1+y)}. \quad (2.37)$$

It is clear that even in this approximation the Lagrangian corresponding to (2.37) and its permutations written in terms of ϕ contains an infinite number of terms⁶ involving derivatives of ϕ . Thus it is only in the strict $g, \alpha' \rightarrow 0$ limit, one has a simple renormalizable local Lagrangian, although another possible approximation scheme, this time in α' , will be discussed in the next section.

To conclude this section, we note that, instead of expanding around the poles $\alpha_s = \alpha_t = 0$, we can alternatively expand around any arbitrary pole. Again we can separate the non-pole terms and write expansions similar to (2.20) and (2.21). This does not shed any new light on the structure of the Lagrangian.

III. MULTIPARTICLE EXTENSION

In the zero-slope limit, g and $\alpha' \rightarrow 0$ the simple $\lambda\phi^3$ Lagrangian of Eq. (2.6), found from the four-meson amplitude faithfully reproduces the corresponding limit of the multiparticle amplitudes provided that one normalized the multiparticle T-matrix by

$$T_N = \sum_{\substack{\text{inequivalent} \\ \text{permutations}}} F_N(p_1 p_2 \dots p_N) \quad (3.1)$$

$$F_N(p_1 p_2 \dots p_N) = \frac{g^{N-2}}{2^{N-3}} (\alpha')^{\frac{1}{2}(N-4)} B_N(p_1 p_2 \dots p_N). \quad (3.2)$$

The powers of α' in the definition of F_N is essential from dimensionality considerations, when we use covariant normalizations for the external wave functions. The numerical factor 2^{3-N} is such that the Feynman-Born diagrams all appear with the correct coefficient at each N value.

The question arises: Once we know the higher terms in the Regge slope expansion at the four-meson level, derivable from ϕ^4 interaction terms as discussed in the previous section, is this again sufficient to reproduce accurately the Regge slope expansions of the multiparticle dual amplitudes.

A. Types of Terms which may Occur.

On the grounds of dimensionality alone, it is straightforward to discuss the possible interaction terms which might be necessary to reproduce the multi-particle amplitudes. Taking as the fundamental dimension that of mass (M) we note that the scalar field has dimension $[\phi] = M$, the zero-slope coupling has dimension $[\lambda] = [g \alpha'^{-\frac{1}{2}}] = M$ and for the Lagrangian density itself $[L] = M^4$.

The possible terms in the Lagrangian density are listed in Table 1. It is convenient to refer to the simplest (Scherk) term $\lambda \phi^3$ as the zero-slope limit, to the terms $\lambda^{N-2} (\alpha')^{N-2} \phi^N$ ($N \geq 4$) as contact terms⁷ and to all other terms (necessarily of higher order in α') as derivative terms. These derivative terms must contain either derivatives or powers of m^2 , or both, acting on ϕ^N .

The pattern of terms that are allowed by dimensionality will suggest an approximation where we truncate the Lagrangian L , at a definite order in α' relative to the zero-slope limit.

B. The Contact Term for $N = 5$.

We now consider the $B_5(p_1 p_2 p_3 p_4 p_5)$ Born amplitude for a fixed permutation of the external legs, and attempt to write for it a Laurent expansion in the five complex variables $x_i = -\alpha_{i,i+1} = -\alpha(0) - \alpha'(p_i + p_{i+1})^2$.

It is particularly convenient to separate out explicitly the known single and double ground-state poles of B_5 in order to extract a function $E_5(x_1 x_2 x_3 x_4 x_5)$ which is analytic in $|x_i| < 1$. The analytic function E_5 plays a role analogous to that of the function $E(x,y)$ at the four-meson level. It seems to be impossible to separate such a function multiplicatively from B_5 , but it can be separated additively. To be specific we write

$$\begin{aligned}
 B_5(p_1 p_2 p_3 p_4 p_5) = & \frac{1}{x_1} B_4(x_3 x_4) + \frac{1}{x_2} B_4(x_4 x_5) + \frac{1}{x_3} B_4(x_5 x_1) \\
 & + \frac{1}{x_4} B_4(x_1 x_2) + \frac{1}{x_5} B_4(x_2 x_3) \\
 & - \frac{1}{x_1 x_3} - \frac{1}{x_2 x_4} - \frac{1}{x_3 x_5} - \frac{1}{x_4 x_6} - \frac{1}{x_5 x_1} \\
 & + E_5(x_1 x_2 x_3 x_4 x_5)
 \end{aligned} \tag{3.3}$$

which defines E_5 . This definition, together with the internal bootstrap property of B_5 , ensures that $E_5(x_1 x_2 x_3 x_4 x_5)$ is analytic inside $|x_i| < 1$, $i = 1, 2, 3, 4, 5$.

The question of whether a $\lambda^3 \alpha'^2 \phi^5$ contact term must be present in L is now seen to be equivalent to asking whether the analytic function E_5 has a non-vanishing zeroth-order term in its five-fold Taylor expansion around $x_i = 0$.

Let us therefore examine carefully the double-integral representation⁸

$$\begin{aligned}
 B_5(p_1 p_2 p_3 p_4 p_5) = & \int_0^1 dt_1 \int_0^1 dt_2 t_1^{x_1-1} \left(\frac{1-t_1}{1-t_1 t_2} \right)^{x_2-1} \left(\frac{1-t_2}{1-t_1 t_2} \right)^{x_3-1} t_2^{x_4-1} \\
 & (1-t_1 t_2)^{x_5-2}
 \end{aligned} \tag{3.4}$$

and divide up the domain of integration as indicated in Figure 2(a). The boundaries are defined by

$$1. \quad 0 \leq t_1, t_2 \leq \epsilon \tag{3.5}$$

$$2. \quad 0 \leq t_2 \leq \epsilon; \quad \epsilon \leq t_1 \leq (1-\epsilon) \tag{3.6}$$

$$3. \quad 0 \leq t_1 \leq \epsilon; \quad \epsilon \leq t_2 \leq (1-\epsilon) \tag{3.7}$$

$$4. \quad 0 \leq (1-t_1), t_2 \leq \epsilon \tag{3.8}$$

$$5. \quad 0 \leq t_1, (1-t_2) \leq \epsilon \tag{3.9}$$

$$6. \quad \epsilon \leq t_2 \leq (1-\epsilon); \quad (1-\epsilon)(1-\epsilon t_2)^{-1} \leq t_1 \leq 1 \tag{3.10}$$

$$7. \quad \epsilon \leq t_1 \leq (1-\epsilon); \quad (1-\epsilon)(1-\epsilon t_1)^{-1} \leq t_2 \leq 1 \tag{3.11}$$

$$8.9.10. \quad (1-\epsilon) \leq t_1 t_2 \leq 1 \tag{3.12}$$

where $\epsilon \geq 0$ is a small positive number. The final region contains three distinct singularities which may be further separated by defining

$$t_1 = 1 - u_1 u_2 \quad (3.13)$$

$$t_2 = \frac{1-u_1}{1-u_1 u_2} \quad (3.14)$$

and defining the sub-regions (Fig. 2(b))

$$8. \quad 0 \leq u_1, u_2 \leq \epsilon \quad (3.15)$$

$$9. \quad 0 \leq u_1 \leq \epsilon; \quad \epsilon \leq u_2 \leq (1-\epsilon) \quad (3.16)$$

$$10. \quad 0 \leq u_1, (1-u_2) \leq \epsilon. \quad (3.17)$$

It is now straightforward to check that these ten contributions are given by

$$\begin{aligned} & \left(\frac{1}{x_1} + \epsilon_1\right) \left(\frac{1}{x_4} + \epsilon_4\right) + \left(\frac{1}{x_2} + \epsilon_2\right) \left(\frac{1}{x_5} + \epsilon_5\right) \\ & + \left(\frac{1}{x_3} + \epsilon_3\right) \left(\frac{1}{x_5} + \epsilon_5\right) + \left(\frac{1}{x_4} + \epsilon_4\right) \left(\frac{1}{x_1} + \epsilon_1\right) \\ & + \left(\frac{1}{x_5} + \epsilon_5\right) \left(\frac{1}{x_2} + \epsilon_2\right) + \left(\frac{1}{x_1} + \epsilon_1\right) \tilde{B}_4(x_3 x_4) \\ & + \left(\frac{1}{x_2} + \epsilon_2\right) \tilde{B}_4(x_4 x_5) + \left(\frac{1}{x_3} + \epsilon_3\right) \tilde{B}_4(x_5 x_1) \\ & + \left(\frac{1}{x_4} + \epsilon_4\right) \tilde{B}_4(x_1 x_2) + \left(\frac{1}{x_5} + \epsilon_5\right) \tilde{B}_4(x_2 x_3) \end{aligned} \quad (3.18)$$

where we have made the definitions:

$$\epsilon_i = \int_0^\epsilon dt \, t^{x_i-1} - \frac{1}{x_i} = \frac{\epsilon^{x_i} - 1}{x_i} \quad (3.19)$$

for $i = 1, 2, 3, 4, 5$ and

$$\tilde{B}_4(x, y) = \int_\epsilon^{1-\epsilon} dt \, t^{x-1} (1-t)^{y-1} \quad (3.20)$$

$$= B_4(x, y) - \frac{1}{x} - \frac{1}{y} - \epsilon_x - \epsilon_y. \quad (3.21)$$

Combining terms we may therefore re-write expression (3.18) as

$$\begin{aligned}
& \frac{1}{x_1} B_4(x_3 x_4) + \frac{1}{x_2} B_4(x_4 x_5) + \frac{1}{x_3} B_4(x_5 x_1) + \frac{1}{x_4} B_4(x_1 x_2) + \frac{1}{x_5} B_4(x_2 x_3) \\
& - \frac{1}{x_1 x_3} - \frac{1}{x_2 x_4} - \frac{1}{x_3 x_5} - \frac{1}{x_4 x_1} - \frac{1}{x_5 x_2} \\
& + \epsilon_1 (B_4(x_3 x_4) - \frac{1}{x_3} - \frac{1}{x_4}) + \epsilon_2 (B_4(x_4 x_5) - \frac{1}{x_4} - \frac{1}{x_5}) \\
& + \epsilon_3 (B_4(x_5 x_1) - \frac{1}{x_5} - \frac{1}{x_1}) + \epsilon_4 (B_4(x_1 x_2) - \frac{1}{x_1} - \frac{1}{x_2}) \\
& + \epsilon_5 (B_4(x_2 x_3) - \frac{1}{x_2} - \frac{1}{x_3}) \\
& - \epsilon_1 \epsilon_3 - \epsilon_2 \epsilon_4 - \epsilon_3 \epsilon_5 - \epsilon_4 \epsilon_1 - \epsilon_5 \epsilon_2 .
\end{aligned} \tag{3.22}$$

In the limit $\epsilon \rightarrow 0+$ it is obvious that

$$\epsilon_1 \rightarrow \ln \epsilon \quad (\rightarrow \infty). \tag{3.23}$$

Hence these regions of integration give a contribution to $B_5(x_1 x_2 x_3 x_4 x_5)$

that is

$$-5(\ln \epsilon)^2. \tag{3.24}$$

There is no ϵ -independent x_i -independent term from these regions. The divergence (3.24) must be cancelled by a corresponding term in the non-singular region. We may easily calculate the latter, in the limit $x_i \rightarrow 0$, from its expression

$$\begin{aligned}
& \left(\int_{\epsilon}^{1-\epsilon} dt_1 \int_{\epsilon}^{1-\epsilon} dt_2 + \int_{1-\epsilon}^1 dt_1 \int_{1-\epsilon}^{\frac{1-\epsilon}{\epsilon_1}} dt_2 \right. \\
& \left. + 2 \int_{\epsilon}^{1-\epsilon} dt_1 \int_{1-\epsilon}^{\frac{1-\epsilon}{1-\epsilon t_1}} dt_2 \right) t_1^{-1} (1-t_1)^{-1} t_2^{-1} (1-t_2)^{-1} \\
& = ((\ln \epsilon)^2) + (\psi'(1)) + ((\ln \epsilon)^2 + 2\psi'(1))
\end{aligned} \tag{3.25}$$

$$= 5(\ln \epsilon)^2 + 3\psi'(1). \tag{3.26}$$

Hence the divergence cancels as expected, and we are left with the final result

for the contact term, which is

$$E_5(x_1 x_2 x_3 x_4 x_5) \xrightarrow{x_i \rightarrow 0} 3\psi'(1). \quad (3.27)$$

C. The Lagrangian Density to Order $(\alpha')^2$

We now wish to obtain the Lagrangian density which reproduces correctly the terms up to order $(\alpha')^2$ in B_5 . For this purpose, we find that it is necessary to add to L_2 (Eqn. 2.24) a contact term (ϕ^5) so that the complete contribution to L up to order $(\alpha')^2$ is given by

$$L_2 = -\psi^{(1)} \lambda^2 \alpha'^2 \left[\frac{1}{4} (\partial \mu \phi)^2 + \frac{1}{4} (\partial^2 \mu \phi) \phi^3 + \frac{1}{8} m^2 \phi^4 \right] + \frac{9}{5!} \lambda^3 \alpha'^2 \psi^{(1)} \phi^5. \quad (3.28)$$

It is straightforward to show that the first term in (3.28) when combined with the $\frac{\lambda}{6} \phi^3$ part of the Lagrangian gives the following contribution to the five point function:

$$\psi^{(1)} \lambda^3 \alpha'^2 \left[\sum_{\substack{i,j \\ i \neq j}} \frac{m^2}{m^2 - (p_i + p_j)^2} - 10 \right]. \quad (3.29)$$

This contribution agrees precisely with the single pole contributions arising from the terms $\sum \frac{1}{x_i} B(x_j, x_k)$ (Eqn. 3.3) summed over all inequivalent permutations.

More explicitly

$$\frac{1}{x_1} B(x_3, x_4) = \frac{1}{x_1} \left(\frac{1}{x_3} + \frac{1}{x_4} \right) - \psi^{(1)} \frac{(x_3 + x_4)}{x_1} + \dots \quad (3.30)$$

The first set of terms are correctly reproduced by $\frac{\lambda}{6} \phi^3$ calculated in the second order. The second set of terms coming from all inequivalent permutations lead to (3.29).

The ϕ^5 term in (3.28) is necessary to reproduce the contact terms (3.27).

This now explains why we wrote the ϕ^4 derivative term in precisely the form (2.24) in the previous section, rather than any of the alternative forms noted in expression (2.25). The reason is that one requires that this part

of L_2 reproduce correctly the contributions, $\sum \frac{1}{x_i} B(x_j, x_k)$ of the single poles in B_5 . This latter condition is, of course, the bootstrap condition that the internal ground state poles consistently factorise into Born amplitudes for external ground state particles.

Note that if we did not impose the bootstrap requirement in this way, we could choose the (simple) form $\frac{1}{24} \psi^{(1)} \lambda^2 \alpha' m^2 \phi^4$ from (2.25) whereupon the corresponding L'_2 is given by⁹

$$L'_2 = \frac{1}{4!} \psi^{(1)} \lambda^2 \alpha'^2 m^2 \phi^4 - \frac{1}{5!} \psi^{(1)} \lambda^3 \alpha'^2 \phi^5. \quad (3.31)$$

Although this form (3.31) happens to be simpler, and hence more attractive, it is not expected to have a simple generalisation to higher multiparticle functions.

We should add that calculations along the above lines can be easily extended to six and higher-point functions. The computations are, however, very lengthy and no simple answer has emerged as yet.

IV. CONCLUDING REMARKS

We have investigated the expansion of the four point function $B(-\alpha_s, -\alpha_t)$ in the Regge slope parameter α' with the hope of finding a Lagrangian density which goes beyond the zero-slope limit of Scherk¹. Such a Lagrangian density can be written down formally by a Laurent-Taylor series expansion of $B(-\alpha_s, -\alpha_t)$ around the ground state pole $\alpha_s = \alpha_t = 0$. Such a formal Lagrangian written in terms of the scalar field ϕ is an infinite Lagrangian containing an infinite number of derivatives. This essential feature persists even in the drastic approximation of retaining only the next excited state beyond the ground state. Thus the Lagrangian is inherently non local and only in the strict zero slope limit it reduces to the simple local ϕ^3 Lagrangian. The nonlocality is not too surprising when one thinks of the arbitrary high mass and high spin states included in the dual models. It is also expected if one believes, for example,

in the idea that the underlying internal hadronic structure is that of an extended relativistic massless string, embedded in space-time¹⁰.

We have also studied the higher order terms in the five-point function (order α'^2). We have shown that, under certain requirements, the Lagrangian necessary to describe all the terms to this order needs a ϕ^5 contact term. Study of higher point functions with the possibility of finding some simple generalization is currently under progress.

In concluding, we would like to add one rather speculative remark concerning the renormalizability of the Lagrangian truncated at order α'^2 ; the Lagrangian $L = L_0 + L_1 + L'_2$ leads to a non-renormalizable theory according to the conventional rules because of the presence of the ϕ^5 contact term. We recall, however, that the dual model itself can lead to finite results if momentum-dependent counter terms are subtracted¹¹. This raises the very interesting possibility of a renormalization scheme for the Lagrangian L by using techniques similar to those used in the renormalization of dual models. This needs further investigation.

References and Footnotes

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3. P. H. Frampton, Physical Review D7, May 15 1973 and Syracuse University preprints SU-4205-12 and SU-4205-14; A. P. Balachandran and H. Rupertsberger, Syracuse University preprint SU-4205-18.
4. M. Abramowitz and J. A. Stegun, Handbook of Mathematical Functions, Dover (1965) page 823, (Faa di Bruno's formula).
5. The subscript in L_k refers to the order in α' relative to the zero-slope limit. With this convention $L_1 = 0$ and we begin with L_2 .
6. This is essential to obtain only the double poles without single poles at $x, y = -1$.
7. We have seen that the ϕ^4 contact term is absent for $N = 4$. This is actually an "accidental" property of the Euler β function, that can be related for example to its straight-line zeros; the contact term will in general be present, as discussed below, for $N > 4$ and even at $N = 4$ in more realistic Born amplitudes.
8. For rather an extensive bibliography on dual models in general and their multiparticle extensions in particular see, for example, the ends of the various sections of P. H. Frampton, Lecture Notes on Dual Resonance Models (Bielefeld University and Syracuse University preprints 1972-73).
9. It is perhaps worth remarking that one can even find a linear combination of the alternate forms in (2.25) that obviates the need for any ϕ^5 interaction at the B_5 level, at the expense of a complicated (ad hoc) ϕ^4 term; such a choice, however, is of course not consistent with our idea of maintaining bootstrap consistency order by order.

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Simplest Born Amplitude to which Term Contributes	Zero-Slope Limit	NON-LINEAR CORRECTION TERMS		
		$O(\alpha')$	$O(\alpha'^2)$	$O(\alpha'^3)$
B_4	$\lambda \phi^3$	$[\lambda^2 \alpha' \phi^4]$	$\lambda^2 \alpha'^2$ term	$\lambda^2 \alpha'^3$ term
B_5			$\lambda^3 \alpha'^2 \phi^5$	$\lambda^3 \alpha'^3$ term
B_6				$\lambda^4 \alpha'^3 \phi^6$

TABLE 1.

Terms in the Lagrangian density.

We refer to the $\lambda \phi^3$ term as the zero-slope limit, the terms $\lambda^{N-2} (\alpha')^{N-3} \phi^N$ ($N \geq 4$) as genuine contact terms, and all others, which must involve derivatives or powers of μ^2 or both, collectively as derivative terms.

FIGURE 1

(a)

(b)

FIGURE 2

Figure Captions

Fig. 1. Region of analyticity in $\text{Re } x - \text{Re } y$ plane.

Fig. 2. Subdivisions of integration region for B_5 .