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FIELDS PRODUCED BY CYLINDRICAL CURRENT ARRAYS

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Introduction

The fact that very high fields can be produced by superconducting coils gives added interest to theoretical methods of designing and computing two-dimensional fields of current arrays.

The purpose of this report is, (1), to formulate a general "current sheet theorem" for treating planar fields due to cylindrical current arrays as functions of a complex variable, and, (2), to illustrate the theorem for the case of an arbitrary distribution of currents flowing along the elements of a right circular cylinder.

Cylindrical Current Sheets

A cylinder is, by definition, the surface traced by a straight line moving parallel to a fixed straight line and intersecting a fixed curve. Each position of the moving straight line is an element of the cylinder. We call a cylinder in which current flows only along the elements (not transverse to them) a "cylindrical current sheet." In other words, a cylindrical current sheet is composed of current filament elements. The resulting magnetic field will be two-dimensional, or planar, since it varies only with coordinates transverse to the elements but not with distance parallel to the elements.

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Complex Variable Formulation

Using the complex formulation previously defined for two-dimensional fields,^{1,2} the complex potential due to a filamentary current I abamperes perpendicular to the $Z = X + iY$ plane at $Z = z$ is an analytic function of Z :

$$W(Z) = 2I \log (Z - z) + \text{constant} . \quad (1)$$

We use capital Z for field points and lower case z for source (current) points. The complex logarithm is multiple-valued; therefore the value of $W(Z)$ increases by $2I \times 2\pi i = 4\pi i I$ when Z moves in any closed path once around the current I in a clockwise direction.

The same mathematical fact can be expressed in terms of the complex field strength due to the current filament, namely

$$H(Z) = \frac{dW}{dZ} = H_Y + iH_X . \quad (2)$$

By the Cauchy Integral Formula the integral of $H(Z)$ taken clockwise around any path enclosing $Z = z$ is

$$\oint H(Z) dZ = \oint \frac{2I}{Z - z} dZ = 4\pi i I . \quad (3)$$

Here $2I$ appears as the "residue" of $H(Z)$ at $Z = z$. The relation (3) contains the physical fact that the line integral of magnetic field strength along a closed path around the current I is $4\pi I$.

Current Sheet Theorem

We can apply the residue theorem to the assemblage of current filaments comprising any section of a cylindrical current sheet. Let the filaments of which the sheet is composed be perpendicular to the Z plane along the curve

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1. R.A. Beth, "Two-Dimensional Fields as Functions of a Complex Variable," BNL Accelerator Dept. Internal Report AADD-66, February 1965.
 2. R.A. Beth, "Complex Representation and Computation of Two-Dimensional Magnetic Fields," J. Appl. Phys. (in press).

Fig. 1. Intersection of cylindrical current sheet with the complex $Z = X + iY$ plane.

$Z = z$ in Fig. 1, and let ΔI be the total current flowing upward in the interval from z to $z + \Delta z$. Then

$$\oint H(Z) dZ = 4\pi i \Delta I$$

for any closed path enclosing Δz . If we deform the path toward $Z = z$ from both sides and use the subscripts R and L to denote functions to the right and left of the curve, we obtain the "current sheet theorem":

$$H_R(z) - H_L(z) = 4\pi i \frac{dI}{dz}. \quad (4)$$

where $H_R(z)$ and $H_L(z)$ are the limits of the analytic functions $H_R(Z)$ and $H_L(Z)$ as Z approaches z from the right and left, respectively, and dI/dz is written symbolically for the limit of $\Delta I/\Delta z$ as $\Delta z \rightarrow 0$ at any z .

To evaluate dI/dz , which is, in general, complex, it is convenient to express both $I = I(p)$ and $z = z(p)$ in terms of some real parameter p which increases monotonically along the curve. In general, I is real and z is complex. If both functions are differentiable,

$$\frac{dI}{dz} \equiv \frac{dI}{dp} / \frac{dz}{dp} . \quad (5)$$

Note that $I(p) - I(p_0)$ is the cumulative total current flowing in the sheet interval from $z(p_0)$ to $z(p)$, and that $|dI/dz|$ is the linear current density (abamperes/centimeter) in the sheet at z .

Another useful form of the current sheet theorem is obtained by integrating Eq. (4):

$$W_R(z) - W_L(z) = 4\pi i I + \text{constant} , \quad (6)$$

where the potentials $W_R(z)$ and $W_L(z)$ are again limits of the analytic functions $W_R(Z)$ and $W_L(Z)$ as $Z \rightarrow z$ from right and left respectively, and we interpret z and I as differentiable functions of a real parameter which increases along the curve as before.

Right Circular Cylinder

The current sheet theorem enables us to evaluate the field inside and outside a right circular cylinder carrying current along its elements.

Let the cylinder of radius r cut the Z plane in the circle

$$Z = z = z(\theta) = r e^{i\theta} , \quad r = \text{constant} \quad (7)$$

in Fig. 2. With θ as curve parameter the current in the cylinder is given by

$$I = I(\theta) .$$

If we assume that the algebraic total current in the cylinder is zero,

$$I(\theta) = I(\theta + 2\pi)$$

then $I(\theta)$ is a periodic function of θ and may be expressed in a Fourier series

Fig. 2. Circular cylinder current sheet.

$$I(\theta) = \sum_{n=1}^{\infty} \left[A_n \sin n\theta + B_n \cos n\theta \right] \quad (8a)$$

or

$$I(\theta) = \sum_{n=1}^{\infty} C_n \sin (n\theta + \theta_n) \quad (8b)$$

where

$$C_n e^{i\theta_n} = A_n + iB_n \quad (8c)$$

gives the connection between the constants C_n , θ_n and the constants A_n , B_n , all of which are assumed to be real.

Using Eq. (7) we have

$$\begin{aligned} \sin (n\theta + \theta_n) &= \frac{1}{2i} \left[e^{i(n\theta + \theta_n)} - e^{-i(n\theta + \theta_n)} \right] \\ &= \frac{1}{2i} \left[e^{i\theta_n} \left(\frac{z}{r} \right)^n - e^{-i\theta_n} \left(\frac{r}{z} \right)^n \right] . \end{aligned}$$

Therefore the form (6) of the current sheet theorem gives, for the current function (8b)

$$W_R(z) - W_L(z) = 2\pi \sum_{n=1}^{\infty} C_n \left[e^{i\theta_n} \left(\frac{z}{r} \right)^n - e^{-i\theta_n} \left(\frac{r}{z} \right)^n \right] + \text{constant} . \quad (9)$$

In order to make $W(Z)$ finite for both $|Z| \rightarrow 0$ and $|Z| \rightarrow \infty$, we must have

$$W_R(Z) = - 2\pi \sum_{n=1}^{\infty} C_n e^{-i\theta_n} \left(\frac{r}{Z} \right)^n + \text{constant} \quad (10a)$$

and

$$W_L(Z) = - 2\pi \sum_{n=1}^{\infty} C_n e^{i\theta_n} \left(\frac{Z}{r} \right)^n + \text{constant} \quad (10b)$$

to satisfy (9) for $Z \rightarrow z$, as required. By differentiating each with respect to Z we find the field strengths in the form (2) for the regions outside and inside the cylinder, respectively:

$$H_R(Z) = 2\pi \sum_{n=1}^{\infty} n C_n r^n e^{-i\theta_n} Z^{-(n+1)} \quad (11a)$$

and

$$H_L(Z) = - 2\pi \sum_{n=1}^{\infty} n C_n r^{-n} e^{i\theta_n} Z^{n-1} . \quad (11b)$$

If the net total current in the cylinder is not zero as assumed above, but I_c , we may assume I_c to be distributed uniformly around the cylinder. The contribution to the inside potential and field is nil, while the term to be added to the outside potential is

$$\Delta W_R(Z) = 2 I_c \log Z$$

and to the outside field strength is

$$\Delta H_R(Z) = 2 I_c / Z .$$

The current sheet theorem thus solves the field problem for a right circular cylinder current sheet in general: the solution enables us to find the field for a given current distribution (8) or the current distribution to produce a given field (11).

Example

A simple application of the general solution given in Equations (10) and (11) is to find the current distribution to produce an ideal constant gradient field:

$$H_Y = B + GX , \quad H_X = GY \quad \text{gauss}$$

inside the cylinder. In complex form we have to identify

$$H_L(Z) = H_Y + iH_X = B + G(X + iY) = B + GZ$$

with Eq. (11b). Thus

$$\begin{aligned} - 2\pi n C_n r^{-n} e^{i\theta_n} &= B \quad \text{for } n = 1 \\ &= G \quad \text{for } n = 2 \\ &= 0 \quad \text{for } n = 3, 4, \dots \end{aligned}$$

or $C_1 = -Br/2\pi$, $C_2 = -Gr^2/4\pi$, $C_n = 0$ for $n = 3, 4, \dots$ and $\theta_n = 0$ for $n = 1, 2, \dots$. Then (8b) gives the current function

$$I(\theta) = -\frac{Br}{2\pi} \sin \theta - \frac{Gr^2}{4\pi} \sin 2\theta .$$

The required current density is, therefore

$$\frac{dI}{d\theta} = -\frac{Br}{2\pi} \cos \theta - \frac{Gr^2}{2\pi} \cos 2\theta \quad \text{abamperes/radian}$$

which implies a linear current density in amperes/cm of

$$f(\theta) = \frac{10}{r} \frac{dI}{d\theta} = -\frac{10}{2\pi} B \cos \theta - \frac{10}{2\pi} Gr \cos 2\theta$$

on the surface of a cylinder of r centimeters in radius.

Thus, to produce an AGS type of field with $B = 10,000$ gauss and $G/B = 0.0425 \text{ cm}^{-1}$, the required linear current density $f(\theta)$ on the surface of a circular cylinder of 9.0 cm radius is given by

$$- f(\theta) = 15915 \cos \theta + 6088 \cos 2\theta \text{ amperes/cm}$$

with the maximum density on the high field side of about 22000 amperes/cm.

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