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SURFACE-RENEWAL MODELS FOR
HEAT TRANSFER BETWEEN
WALLS AND FLUIDIZED BEDS

by

R. D. Patel

MASTER

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by

R. D. Patel

Chemical Engineering Division

Based on a thesis submitted to the Faculty of
Purdue University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

November 1967

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NOMENCLATURE

A	= area of heater
a	= dimensionless age
C	= specific heat
c_1, c_2	= constants
D_p	= particle diameter
d	= fractional depth
$E\{ \}$	= expectation operator
f	= residence-time density function
F	= residence-time distribution function
$\mathcal{F}\{ \}$	= Fourier transform
h	= overall heat-transfer coefficient (Eq. 7)
h_c	= contact heat-transfer coefficient
k	= thermal conductivity
$K + 1$	= number of equally spaced observations
L	= dimensional characteristic length
M	= contact Nusselt number
n	= lowest number for which $Z_n \rightarrow 0$. [$n = o(K)$]
N_{Nu}	= Nusselt number (Eq. 8), or transport coefficient
$N(t)$	= number of elements in residence at time t ; random function
p	= Laplace-transform variable
p_N	= probability mass function
Q	= power to main heater
$Q()$	= heat input
q	= heat flux
R	= autocorrelation function
S	= spectral density function
Si	= sine integral
t	= dimensional temperature; time
T	= dimensionless temperature
$\text{Var}\{ \}$	= variance operator
v	= superficial gas velocity
$W(j\Delta t)$	= weight function in estimator for spectral density
x	= dimensionless distance
X_0	= estimator for $\bar{N}$ (Eq. 66)
y	= dimensional distance
Y_j	= estimator for $\psi(j\Delta t)$ (Eq. 67)
Z_j	= modified estimator for $\psi(j\Delta t)$ (Eq. 68)

Greek Letters

α	= thermal diffusivity; shape factor for gamma distribution
Δt	= time difference between observations
γ	= factor in Model II (Eq. 32)
θ	= dimensional time
λ	= factor in Model II (Eq. 32)
λ_n	= eigenvalues (Eq. 10)
μ	= particle packing factor, defined after Eq. 34
ρ	= density
σ^2	= variance
τ	= residence time (random variable)
Φ	= distribution function for age
ϕ	= age density function
ψ	= autocovariance function
ω	= frequency
ω_N	= Nyquist frequency (Eq. 72)

Subscripts

b	= bulk of bed
g	= gas
m	= packet
mf	= minimum fluidization
N	= Nyquist
p	= packet
s	= solid
w	= wall

Vinculum and Circumflex

$-$	= mean value over distribution
$\wedge$	= estimate

SURFACE-RENEWAL MODELS FOR HEAT TRANSFER BETWEEN WALLS AND FLUIDIZED BEDS

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R. D. Patel

ABSTRACT

This report reviews models previously presented for the heat transfer between a wall and a fluidized bed. Attention is focused on penetration-type models, since these appear to present a more accurate picture of the heat-transfer mechanism. These penetration models require a knowledge of the residence times of the fluidized particles at the wall.

Equipment was designed to measure heat-transfer coefficients and particle residence times at the identical wall surface under the same conditions. This was done by constructing a heat-transfer surface and a transparent surface which could be placed in a cavity in the wall to measure heat-transfer coefficients and residence times, respectively. The heat-transfer surface was an adiabatic heater. Residence times were measured by repeatedly photographing the wall of the bed with the transparent surface mounted in the cavity. Thus, heat-transfer coefficients could be related experimentally to the residence times of particles at the heat-transfer surface. Cellulose acetate, glass, copper, aluminum, and alumina were used in sizes ranging from 0.001 to 0.01 ft. Residence times were measured for only the glass and the cellulose acetate; the residence times for other particles were obtained from a dimensionless correlation between residence time and gas velocity using the glass and acetate data. Air was used as the fluidizing gas. A stirred, bubble-free bed was used to simplify residence-time measurements.

Two surface-renewal models are presented to describe heat transfer between the wall and the fluidized bed. Model I considers transient heat transfer to a "packet" or "clump" of particles at the wall, assuming a non-zero wall-to-packet thermal resistance. Model II considers the transient heating of a particle at the wall gaining heat by convection from the surrounding fluid and losing heat to the bulk of the bed behind. Both models gave predictions within an average 30% of the data for glass. For the other systems (for which measured residence-time data were lacking), the deviation was greater. Both models, being of film-penetration type, require a knowledge of the heat-transfer coefficient at minimum fluidization. The residence times were such that essentially a penetration mechanism was

in effect. Thus, pure penetration variants of the above models (which do not require a knowledge of minimum-fluidization heat-transfer coefficients) could be used with equal accuracy.

Methods are also presented to estimate mean residence times of particles at the surface, their age densities, and the average transport coefficients by observation of only the total tracer particle counts at the transporting surface. Two methods, employing the autocovariances and the spectral density of the counts, are outlined and tested on computer-simulated counts and actual counts, experimentally observed at the wall of the fluidized bed. Although there is some deviation in the predicted age densities from the correct values, the average residence times and the average transport coefficients agree with the true values.

INTRODUCTION

Over the past few decades, the fluidized-bed reactor has become an important tool in the chemical process industries. The very large surface area of the fluidized particles make this type of reactor suitable for solid-catalyzed reactions, the catalyst being the particles themselves. Of particular importance is the very high heat-transfer rate that may be achieved in such a reactor. Not only does this make such a bed eminently suitable for handling exothermic reactions, but it often results in the use of the bed as a heat exchanger.

The complex nature of the phenomena occurring in fluidized beds has forced an empirical approach when designing such a bed. Only recently have attempts been made to arrive at a mechanistic picture of the process. In particular, the important problem concerning the mechanism of heat transfer between the bed and the wall of the reactor has not been solved completely. This aspect of fluidized beds is reviewed and treated in this report.

I. BACKGROUND AND LITERATURE SURVEY

This section reviews some of the previous work on heat transfer in fluidized beds. Attention is focused on works that attempt to formulate a model for the phenomenon, rather than on those producing a correlation of experimental data. With this background in mind, the next section outlines the approach adopted in the present report.

Several excellent references have reviewed the literature on heat transfer between the bed and an internal or external surface.¹⁻⁴ In particular, Zabrodsky³ and Ainshtein and Gel'perin⁴ have compared the heat-transfer

correlations of various authors. These correlations often give contradictory results, and frequently apply only for the experimental system of the particular worker. To explain the phenomenon mechanistically, several authors have formulated models. Some pertinent ones are discussed below.

Several authors⁵⁻⁷ have stated that the primary resistance to heat transfer is located in a thin fluid film at the wall. This film is disturbed by the movement of the particles at the wall, with a consequent improvement in the heat-transfer rate. These models effectively ignore the high heat capacity of the particles compared to that of the gas as a contributing factor. Heat-transfer coefficients increase with particle heat capacity, as has been shown by Ziegler *et al.*⁸ and by Zabrodsky.³ This relationship cannot be explained by the film-type models.

Several authors^{8-11,14} have proposed penetration-type models. Mickley and Fairbanks⁹ envisage clumps of particles, or "packets" arriving at the heated surface, receiving heat by unsteady-state transfer, and returning to the bulk of the bed. The packets have the properties of the bed at minimum fluidization. The heat-transfer coefficient is proportional to the square root of the packet thermal conductivity, specific heat, and density, and also to a "stirring factor," which accounts for the frequency of renewal of the packets at the heated surface. The effect of particle diameter (which is important³) is assumed to be included in the stirring factor. The deficiencies in this model are that (a) the heat-transfer coefficient is directly proportional to the square root of particle heat capacity, which is unrealistic, as pointed out by Ziegler *et al.*,⁸ and (b) the form of the instantaneous heat-transfer coefficient, being inversely proportional to the square root of the age of the element, is such that at zero age the coefficient is infinite, whereas at infinite age (corresponding to minimum fluidization conditions) it is zero.

Baskakov¹¹ proposed a similar model with an added contact resistance to heat transfer located at the wall-facing surfaces of the particles in contact with the wall. Thus the heat-transfer coefficient for very small values of age does not become infinite, but is determined by the finite contact resistance at the wall. However, for infinitely large values of age, the coefficient goes to zero, which is clearly a deficiency.

Ziegler *et al.*⁸ have proposed a model similar to that of Mickley and Fairbanks. Instead of considering the packets, Ziegler *et al.* consider the transient heating of a cold sphere surrounded by a stagnant gas at the hot-wall temperature. They then obtain a time-averaged heat-transfer coefficient by assuming a residence-time distribution of gamma form. There is an error in their final expression, since they find the mean coefficient by integrating the instantaneous coefficient with respect to the residence-time density, rather than the age density. The gamma form is not suitable for

representing the age density, since in general it is not monotone-decreasing. As shown by Koppel, Patel, and Holmes,¹² the age density is a monotone-decreasing function. However, their result is exact for a special case of the gamma form (the exponential density function), and the correction for other gamma forms is small.¹³ Ziegler's model correctly predicts a maximum heat-transfer coefficient, corresponding to zero age, which is finite. However, for infinite age, the predicted coefficient is zero. Their model does predict the relation between the heat-transfer coefficient and the particle thermal properties in good agreement with their experimental results.

Botterill et al.^{14,15} have also considered the transient heating of spheres at a hot wall. They calculated the variation in the heat flux with age of a sphere by numerical solution of the energy equation for the gas-sphere system. The average heat flux was compared to experimental data obtained in a scraped-surface fluidized bed. (A set of stirring blades swept particles away from the hot wall at a fixed frequency. An artificial residence time was thus imposed on the particles.) There was good agreement with the theoretical predictions. Botterill et al.¹⁴ have also demonstrated the important role played by the gas as a heat-transfer medium. By applying a vacuum to their stirred-bed system, they found a sharp drop in the transport coefficient. Botterill et al.¹⁶⁻¹⁸ have improved this model and presented further experimental data from the scraped-surface beds. In particular, they have postulated¹⁷ that certain of their data (with copper particles) are at variance with the theoretical predictions because of the presence of a thin film of gas between the particles and the wall. The discrepancy is more marked for the copper, since the gas film becomes the controlling resistance. For materials of poorer conductivity, the effect is not as pronounced. The data agree closely with the theoretical model when the presence of a gas film is included in the model. These authors further state¹⁷ that "even with short residence times, particle thermal properties are only of secondary importance and the controlling factor is the gas conductivity" --presumably because of the presence of the gas film. The assumption of this gas film is open to criticism. Visual observations during this work show that the particles appear to be in contact with the wall. Closer examination is necessary before a definitive statement can be made. The particles may intermittently touch the wall and bounce away a minute distance, thus effectively setting up a film of gas between them and the wall. Apparently Botterill et al. postulate the gas film to explain their data and do not have any physical basis for it.

More recently, Couderc et al.^{19,20} have reported work similar to that of Botterill. They point out the importance of the particle residence time at the wall, along with the distance separating the surface particles from the wall, in predicting the wall-to-bed coefficient.

Although the residence times of particles are of fundamental importance to these penetration-type models, few attempts have been made to measure them experimentally or to predict them theoretically. Mickley, Fairbanks, and Hawthorn²¹ determined the renewal frequency of packets of particles at the surface of a low-heat-capacity heater in a fluidized bed by measuring the temperature fluctuations in the heater. They then used these renewal frequencies in combination with their heat-transfer model⁹ to arrive at good predictions of the time-average heat-transfer coefficient. Ruckenstein²² has used instability theory to predict the renewal frequency as defined by Mickley *et al.* He shows that the renewal frequency is a weak function of the velocity of fluidization, as was also found experimentally by Mickley *et al.*²¹ Apart from these indirect attempts to obtain particle residence times, no reference was found in the literature wherein the particle residence times were measured directly.

Other models for heat-transfer have been formulated. Among these may be mentioned those of Zabrodsky,²³ Ruckenstein,²⁴ and Wasmund and Smith.²⁵ Wasmund and Smith's model applies only to particulates fluidized beds which generally arise when a liquid fluidizing medium is employed.

II. OBJECTIVE OF PRESENT WORK

The objective of the present work was to develop a model to describe heat transfer from wall surfaces in a fluidized bed and to demonstrate the model's accuracy experimentally.

Attention was devoted to penetration-type models since, as was pointed out earlier, they appear to present a physically realistic picture of the process. One objective was to remove some of the earlier models' deficiencies, which were pointed out in the preceding section (for example, predicting zero coefficients at minimum fluidization). A primary objective was the measurement of particle residence times at the heat-transfer surface. To achieve this objective, attention was devoted to simple methods of obtaining such residence-time data. The relationship found experimentally between heat-transfer coefficients and particle residence times (at the identical surface) was to be compared with that predicted theoretically by the model.

III. APPARATUS

This section describes the apparatus used to pursue the experimental objectives of the current work. The equipment was designed to measure both heat-transfer coefficients and particle residence times at the identical wall surface of the fluidized bed, and under the same fluidization conditions. (This wall surface will henceforth be referred to as the test surface.)

A. General

Figure 1 is a photograph of the salient parts of the equipment. The fluidized-bed column was a 4-in.-nominal-diam Schedule 40 stainless steel pipe 18 in. high. A $2\frac{1}{2}$ -in.-wide, $\frac{3}{8}$ -in.-thick, flat Lucite plate was mounted in a slot cut along the entire length of the pipe. This provided a flat surface, which facilitated the taking of residence-time data. The flat Lucite plate was itself broken 5 in. above the bottom of the pipe by a rectangular cavity 4 in. high and $2\frac{1}{2}$ in. wide. This cavity accommodated either the heating

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Fig. 1. General View of Experimental Equipment

surface or the transparent surface used in the residence-time studies. These will be described in detail below. A 2-in.-high by $1\frac{1}{2}$ -in.-wide area, centrally located at this cavity, was chosen as the test surface. The pipe was supported vertically on a Unistrut frame by two of its flange bolts.

The fluidizing gas (compressed air from the building supply) was passed through a pressure regulator and then humidified. Humidification reduced the buildup of static charges on the particles within the column. The humidifier was a bed packed with Raschig rings and partly filled with water. Part of the air was allowed to bubble through the water. Adjusting the air flow through the humidifier eliminated most electrostatic effects. The gas then passed through an accurate rotameter and control valve and was distributed into the column through a $1/8$ -in.-thick sintered bronze plate mounted in the pipe flange. A pressure gauge measured the outlet pressure at the rotameter. Air flowed out of the column into the room.

A water manometer measured the pressure drop across the bed (necessary in the determination of the minimum fluidization velocity). Three static-pressure taps were mounted symmetrically at the base of the column. Pieces of 400-mesh screen placed in the taps prevented the fluidized particles from entering the pressure lines. The pressure at the top of the bed was assumed atmospheric, one end of the manometer being open to the air.

Early in the research, gas bubbles at the Lucite surface presented a considerable problem. Residence-time data could not be obtained with any accuracy when bubbles were present at the test surface. To eliminate the bubbles, we installed a stirrer in the bed to break up the bubbles before they rose above the level of the test surface. The stirrer proper was a 2-in.-diam, three-blade paddle wheel, located approximately 4 in. above the distributor plate. It was not known initially whether fluctuations in the stirring rate would cause variations in the observed coefficients or residence times. Therefore, the stirrer was driven by a synchronous-type motor through a "Zeromax," variable-speed mechanical drive. With this setup, constant stirring speeds could be obtained from a few rpm up to 1000 rpm. A strobe-type tachometer measured the stirring speed. It was found later that 10% variations in the stirring speed did not affect the heat-transfer coefficient significantly. Theoretical implications of stirring are discussed in Section V.B.1 of this report.

B. Heat Transfer

An adiabatic heater was constructed to measure heat-transfer coefficients. The construction of the heater guaranteed negligible heat losses to the surroundings. Figure 2 is an exploded view of the heater; Fig. 3 shows the assembled device ready for insertion into the cavity in the column.

Fig. 2
Adiabatic Heater; Exploded View

108-9819A

The main heater plate (shown in Fig. 2) was made of copper and was 2 in. high, $1\frac{1}{2}$ in. wide, and $\frac{3}{16}$ in. thick. Three thermocouple junctions were mounted on it to measure its temperature. The main heating element provided a 15-W electrical-resistance heat source. The main heater plate was enclosed by a copper guard-heater enclosure. This was a hollow rectangular box, open on one face, in which the main heater was positioned. Guard-heating elements were installed on the five outer faces of the guard-heater enclosure. (Only one guard heater is shown in Fig. 2.)

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The main heater and its element were insulated from the heater enclosure by means of insulating foam cast in place. Thermocouple junctions were installed on all five inside faces of the guard-heater enclosure to monitor their temperatures, and the enclosure was surrounded by insulation (as shown in Fig. 2). The complete heater assembly was then mounted in a Lucite support (as shown in Fig. 3) designed to fit snugly into the cavity in the fluidized-bed column. It was held there by two U-bolts and sealed with a foam-rubber gasket. Figure 1 shows the heater assembly mounted in the column. When the heater was in operation, power to the guard heaters was adjusted (by Variacs) so that the main plate and the guard-heater enclosure were at the same temperature. This ensured that no heat flowed between the two, and that all power supplied to the main heater went into the bed as heat. In practice, there was usually a small difference in temperature, but the heat loss was negligible because of the insulation between the main and guard sections.

Fig. 3. Adiabatic Heater; Assembled

The dc power to the main heating element was measured by a calibrated voltmeter-ammeter combination.

Two sheathed thermocouples were installed in the bed, 3 in. and 5/8 in. away from the test surface. An additional unsheathed thermocouple monitored the temperature of the inlet air to the column. All thermocouples were of iron-constantan, type J. Temperatures were recorded on a 12-point temperature recorder. The final steady-state potentials from the thermocouples were measured by a portable, precision potentiometer with a cold junction in an ice bath.

C. Residence Time

Residence times of particles at the test surface were obtained by a tracer technique. A small quantity of white particles was included in a bed of black particles. The test surface was photographed repeatedly to provide a film record of the entrances and exits of the tracer particles. (Further details are deferred to Section IV.)

A transparent Lucite plate was fitted into the cavity in the wall of the column when residence time was being measured. (The experimental setup for this measurement is shown in Fig. 4.) Lines were scribed on the plate to identify the test surface.

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Fig. 4. Experimental Setup for Residence-time Measurements

The camera used (a 35-mm Arriflex motion-picture camera) was fitted with a 40-mm Makroilar lens (focusing down to 4 in.), a variable shutter, and a constant-speed synchronous-drive motor. The latter ensured that the framing rate was a constant 24 frames/sec. This was checked with a Stroboscope. A constant framing speed was essential to the analysis of the film record. The Makroilar lens provided convenient focusing of the test area at small distances. The variable shutter enabled a shutter speed to be chosen to stop the motion of the particles on the film. A speed of $1/192$ sec was used throughout the work. Unfortunately, this proved to be too slow to stop the particle motion at higher velocities of fluidization. These film records could be interpreted manually, but trouble arose when attempts were made to scan them by machine. (Details are presented in Section IV.) The camera was provided with a film magazine of 400-ft capacity.

The test area was illuminated by photoflood lamps. Three were used--one of 500 W in the center, and two of 150 W each on either side. This gave fairly uniform lighting as measured by a light meter.

A high-contrast negative was necessary to perform machine scanning of the film. Ideally, this would mean a clear field representing the black particles, with sharp black dots representing the white tracers. Accordingly, Ansco FPC 335 black and white instrumentation film (made by General Aniline and Film Corp.) was employed throughout. This is a high-speed, high-contrast film with high resolving power and extended red sensitivity. It is ideally suited for scientific recording of transient phenomena with low inherent subject contrast. It is nominally rated at 160 A.S.A., but with special processing in the high silver-deposition range, it gave a superior image to Kodak Tri-X or Shellburst. A high-contrast negative with sharp black dots in an almost clear field was obtained. Since the film was made on a very thin Plestar base, 500 ft could be accommodated in the 400-ft magazine of the camera.

D. Fluidized System

1. Fluidizing Medium

The fluidizing medium used in all the runs was compressed, humidified air at room temperature.

2. Fluidized Particles

Nine systems of particles were employed. Table I lists some of their relevant properties. Figures 5-13 are photomicrographs of the particles used. The acetate and glass were quite spherical, and the copper shot was roughly spherical, whereas the aluminum and alumina particles

were irregular. Thus, a fairly wide spectrum of particle shapes, physical properties, and particle surface characteristics was examined.

Both residence-time and heat-transfer data were collected for only the three sizes of glass and the cellulose acetate. Only heat-transfer data were obtained for the copper, aluminum, and alumina particles. Residence-time data could not be obtained for the latter systems, since a good tracer has not yet been found for them. The glass and cellulose acetate particles were obtainable commercially in black and white, giving a workable system for obtaining residence-time data. Section VI presents details on how residence times were estimated for the copper, aluminum, and alumina particles.

TABLE I. Minimum Fluidization Properties of Fluidized Particles

Particle Type	Weight Mean Particle Diameter, ft	Minimum Fluidization Velocity (Superficial), ft/min	Void Fraction	Bed Density, lb/ft ³	Solid Density, lb/ft ³	Solid Thermal Conductivity, Btu/(hr)(ft)(°F)	Solid Specific Heat, Btu/(lb)(°F)
Cellulose acetate	0.0105	199.8	0.41	47.7	81.2	0.145	0.36
No. 1 glass	0.00265	83.5	0.40	92.0	154	0.5	0.17
No. 2 glass	0.00194	50.8	0.41	91.0	154	0.5	0.17
No. 3 glass	0.00103	19.0	0.41	91.0	154	0.5	0.17
No. 1 copper	0.00200	145.0	0.40	331.6	557	222	0.10
No. 2 copper	0.00105	57.22	0.41	325.5	557	222	0.10
Aluminum	0.00098	19.7	0.44	93.6	169	117.5	0.224
No. 1 alumina	0.00198	109.0	0.55	112.0	250	19.0	0.193
No. 2 alumina	0.00099	31.5	0.54	115.0	250	19.0	0.193

4X

Fig. 5. Photomicrograph of Cellulose Acetate

18X

Fig. 6. Photomicrograph of No. 1 Glass

18X

Fig. 7. Photomicrograph of No. 2 Glass

18X

Fig. 8. Photomicrograph of No. 3 Glass

18X

Fig. 9. Photomicrograph of No. 1 Copper

18X

Fig. 10. Photomicrograph of No. 2 Copper

50X

Fig. 11. Photomicrograph of Aluminum

18X

Fig. 12. Photomicrograph of No. 1 Alumina

18X

Fig. 13. Photomicrograph of No. 2 Alumina

IV. MEASUREMENT TECHNIQUE

This section gives the procedures for obtaining the heat-transfer and residence-time data. The method for obtaining heat-transfer coefficients is well known and is dealt with only briefly. Emphasis is placed on how the residence-time data were obtained, with some detail on machine analysis of the film.

A. General

The minimum fluidization velocity (listed in Table I) was determined for each material, using a bed depth of about $8\frac{1}{2}$ in., by obtaining the curve of bed pressure drop vs gas velocity for the unstirred bed.¹

B. Heat Transfer

The heater assembly was inserted into the cavity in the column and the column filled with the material to a depth of about $8\frac{1}{2}$ in. The air flow rate was adjusted to the required value, and the stirrer was turned on at the desired speed. Power was supplied to the main and guard heaters so

TABLE II. Heat-transfer Coefficients
for Various Heater-to-Bed
Temperature Differences

Packed bed. Cellulose acetate particles.
Superficial air velocity = 162 ft/min
No stirring

Heater-to-Bed Temperature Difference, (°F)	Overall Heat-transfer Coefficient, Btu/(hr)(ft ²)(°F)
20.9	12.79
8.5	13.71
3.3	15.60

that their temperatures were identical. The steady-state temperatures in the bed and in these heaters were recorded, along with the power input to the main heater. A heater-bed temperature difference in the range 12-25°F was used in all runs. Tests showed that the heater gave very reproducible transport coefficients at different values of the temperature difference under identical fluidization conditions (as shown in Table II). The small differences present for the two higher-temperature gradients could well be due to experimen-

tal error or variation in thermal properties. Inaccuracies increase as the temperature difference is reduced; the error is fairly high at a 3.3°F difference. The adiabatic design thus proved very satisfactory.

Overall heat-transfer coefficients were calculated from the standard defining relation,

$$h = \frac{Q}{A\Delta t'}$$

where

h = overall heat-transfer coefficient, $\text{Btu}/(\text{hr})(\text{ft}^2)(^\circ\text{F})$,

Q = power to main heater, Btu/hr ,

A = area of heater, ft^2 ,

and

Δt = temperature difference between the heater and the bed, $^\circ\text{F}$.

C. Residence Time

Residence time was studied with the transparent Lucite plate inserted in the cavity in the column. (The experimental setup for measuring residence time is shown in Fig. 4.) The conditions for the studies were identical to those of the heat-transfer runs so that heat-transfer coefficients and residence times could be related. Less than 0.5% of white tracer particles was added to the otherwise black particles in the bed. After the fluidization conditions were set, the test surface was illuminated, and then photographed with the motion-picture camera for about $1\frac{1}{2}$ min. This time was long enough for a good sampling of residence times of particles, since, on the average, these times were of the order of a few seconds. (Care was taken to ensure that, on the average, about five tracer particles were present at the test surface at any time. This ensured that a sufficiently large number of tracer particles exited and entered the test surface during the run.)

The film record for each run consisted of three sections. The first section, consisting of a few frames, recorded the run identification and conditions. The second section, about 50 frames, was a record of a few tracer particles when there was no gas flow through the bed. These will be referred to as the "reference" frames. They established an "average" particle size when the film was being analyzed (see Section 1 below). The third section, referred to as the "run" frames, and running about $1\frac{1}{2}$ min at 24 frames/sec, was of the run proper with the bed fluidized. Before the third section was exposed, the lens was capped and the camera allowed to accelerate to full speed before the cap was removed. At the end of the run, the lens was also capped before the camera was switched off. This procedure resulted in a constant time difference between each frame throughout the run by eliminating the acceleration and deceleration end effects.

1. Analysis

The two methods of analysis available to obtain residence-time data from the film record were particle tracking and total particle counting.

a. Particle Tracking. In particle tracking, individual particles were tracked for the entire length of the run. The film was projected frame by frame on a sheet of paper. Individual tracer particles were assigned numbers and tracked frame by frame as the film was advanced. The entrances and exits of individual tracer particles into and from the test surface were thus recorded. This gave the residence times of the particles directly. The sample frequency distribution of residence times was then computed, as were the mean and the variance.

This procedure, although providing a direct measurement of the particle residence times, was very time-consuming. Conveniently analyzing the large amount of film shot by this method would not have been possible. An alternative, indirect method was therefore used for analysis of the bulk of the film. The accuracy of the indirect method was checked by particle tracking for a few of the runs. There was good agreement between the two.

b. Total Particle Counts. Total particle counting has been described in Ref. 26. The quantity needed in this method is only the total tracer-particle count on each frame, individual particles being neither identified nor tracked. The time series of particle counts for the run is analyzed to obtain the mean residence time.²⁶ Koppel *et al.*¹² have shown that, in most cases of interest, only the mean residence time is necessary to predict average transport coefficients.

This procedure gave accurate estimates of the mean residence time with a minimum of effort as far as film analysis was concerned. Particle tracking was done for two runs with the cellulose acetate spheres and for one run with No. 1 glass to check the accuracy of the total-counts method. There was good agreement for all three runs.

Sometimes, to compute the average coefficient, it may be necessary¹² to know the complete age distribution of particles, as opposed to only the mean residence time. Attention was therefore focused on techniques to arrive at this distribution, knowing only the time series of total particle counts. Details are given in Section VI. (The complete age distribution is also obtained when particle tracking is performed, at the expense of considerable effort.)

2. Data Reduction

Two data-reduction approaches are theoretically possible when using the methods described in Section 1 above. Film may be examined manually, or automatic data reduction may be used.

a. Manual Data Reduction. Manual data reduction may be performed by projecting the film frame by frame onto a suitable background.

For particle tracking, particles are assigned numbers and followed on the background as the film is advanced. The frame number of the exit and entrance of the individual particles is recorded, and hence their residence times obtained in frames. This may be converted to real time when the framing speed is known. If a sufficiently large number of particles is tracked, a reasonable idea of the true residence-time density can be obtained by plotting the sample frequency histogram of residence times. The moments of the distribution may also be computed using suitable statistical estimators. The particle-tracking method concurrently yields the total tracer-particle count on each frame, since a record is kept of particles on each frame. For the total-count method, the film is similarly projected onto a suitable background. The total tracer-particle count is recorded for each frame as the film is advanced. As mentioned earlier, this is a rapid procedure compared to particle tracking.

b. Machine Data Reduction. Machine data reduction uses automatic film-scanning equipment. Several film analyzers of limited capability (usually for digitizing oscilloscope traces) are available commercially. At Argonne National Laboratory, a film-scanner computer combination, CHLOE, has been designed and built by the Applied Mathematics Division to analyze spark-chamber and other types of film. CHLOE was used to analyze some of the film obtained during this investigation. A brief description of the machine follows, a more complete exposition being given by Hodges.²⁷

CHLOE is an automatic film-scanning system for digitizing information from 35-mm film. The equipment consists of two main elements: a digital computer and an optical scanner, the latter being under control of the computer. The computer is an Advanced Scientific Instruments Model ASI-210; the optical scanner, which uses a cathode-ray tube (CRT) as the light source, was built at Argonne. The spot of light from the CRT is projected onto the film, and the light transmitted through the film is viewed by a Photomultiplier, from which a decision is made regarding the density of the film at the point in question. If the point is of interest as indicated by the Photomultiplier (for example, if it is darker than the background), its coordinates are fed into the computer before the next point is examined. If the point is not of interest, the scanner moves to the next point. The computer connects the points of interest with closed boundaries, and these data are stored on magnetic tape. If very few points occur in a neighborhood, a decision is made whether to reject this as noise. The factor determining the limiting size for noise is initially fed to the machine by the operator. CHLOE could examine one frame in about 1 sec, depending on the number of particles present in the picture.

The information on the magnetic tape is then processed by an interpretive program on a CDC-3600 computer. The interpretive program calculates the areas and coordinates of the closed boundaries found

by CHLOE. An additional area-discrimination factor is also introduced. The result is that the number, location, and areas of the particles on the frame are obtained in digital form. The final output from the interpretive program gave the number of tracer particles on each frame for the length of the run. This time series was analyzed to obtain the mean residence time and the age distribution.

The "reference" frames served an important function when automatic data reduction was used. About five of these frames were first analyzed by CHLOE before the "run" frames were examined. These reference frames fixed the size of an "average" particle on the basis of area. The area of an average particle was taken to be the arithmetic mean of the areas of all particles found by CHLOE on the reference frames. When the run frames were being analyzed, particles of area less than half the average area were rejected, while those of area greater than one and a half times the average were considered as two particles, and so on. This area factor was introduced in the interpretive program. Such an arbitrary area discrimination was unavoidable. CHLOE often could not separate two particles in contact, but would consider them as one large particle. The area factor reduced such errors. Again, often particles would not be in full contact with the wall (and would consequently give a smaller image), or only a portion of a particle would be visible at the edge of the frame. In such cases, a decision had to be made whether the particle was to be considered at the test surface. The area test was the basis of the decision. These problems also occurred when manual data reduction was used and the decision in this case was made by eye estimation.

A secondary problem arose when the area test was applied. In all cases, the same particle on different frames had slightly different areas as measured by CHLOE. This was due to random CHLOE errors, inaccurate film registration in the camera, defects in the emulsion, etc. Possibly, also, the particle was vibrating so that its distance from the lens varied slightly from frame to frame. This did not create a problem unless the area was very nearly one-half the area of an "average" particle. If this was the case, it could be rejected on one frame and accepted on the next, and so on, as its area fluctuated below and above the "average" area. Fortunately, for most of the films this did not occur. However, it is a problem and no good solution could be found.

The most serious problem with machine analysis occurred when the images on the frame were blurred. This happened at the higher gas velocities. The particles exhibited sudden very rapid motions, and the 1/192-sec shutter speed was not short enough to stop the motion on the film. As a consequence, blurred frames occurred at intervals over the entire record. These blurred images did not have sharp boundaries, and as a result, CHLOE often ignored them. In effect, the particle in question was considered to have exited from the test surface. In such runs, there

was no alternative to manual data reduction. The Arriflex camera being used could not attain an appreciably faster shutter speed. (A speed of about $1/1000$ sec should be used for future work.)

Figures 14-17 are examples of typical frames on the film record, shown alongside their CHLOE interpretations. The cellulose acetate spheres gave excellent pictures and correspondingly accurate CHLOE interpretation. All the runs with cellulose acetate particles were analyzed on CHLOE. Two of these were manually tracked, and for these, the two methods of analysis and data reduction showed good agreement. Figure 17 shows a blurred frame along with its CHLOE interpretation. Note that some of the blurred particles that were in rapid motion have been left out by CHLOE. The small dots in Figs. 14-17 are reflections from the black particles and are correctly ignored by CHLOE as noise.

In summary, the present stage in the development of machine-analysis methods gives reliable results only for very high-quality film and large particles. Reproducible results were obtained for the cellulose acetate spheres. The runs for the three sizes of glass were analyzed using manual data reduction. Attempts were made to analyze these automatically, but good results were not obtained. This was primarily due to the small size of the particles and blurring of the image because the shutter speed was too slow. The basic procedures for automatic data reduction have been worked out, and the problems remaining are not insurmountable.

Fig. 14. Original and CHLOE Frames of Cellulose Acetate Particles at an Early Time

Fig. 15. Original and CHLCE Frames of Cellulose Acetate Particles at a Later Time

Fig. 16. Original and CHLOE Frames of No. 1 Glass Particles

Fig. 17. Blurred Original and CHLOE Frames of No. 1 Glass Particles

V. THEORY AND RESULTS: HEAT TRANSFER

This section develops two models for the heat-transfer mechanism from the wall to the bed. The theoretical predictions are compared with experimental data.

A. Theory

1. Model I

The geometry of Model I is shown in Fig. 18. The following assumptions are made:

1. A "packet" of particles, initially at temperature t_b , the bulk-bed temperature, arrives at the wall surface (temperature t_w) at time zero. The packet of particles is assumed to possess the properties of the bed at minimum fluidization (as in Mickley and Fairbanks⁹).

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Fig. 18. Geometry of Model I

resistance is assumed to be $1/h_c$, corresponding to a wall-to-packet heat-transfer coefficient of h_c . Note the distinction between h_c and h ; the latter is defined by Eq. 7.

3. For the time the particle resides at the wall, heat penetrates a characteristic distance, L . Beyond this distance, the temperature is constant at the bulk-bed temperature, t_b .
4. Thermal properties of the gas and particles are constant.
5. Heat transfer from wall to packet by radiation is negligible. Baddour and Yoon²⁸ have shown that radiative transfer is negligible in packed beds below 600°C.

Under these assumptions, the differential equation and the boundary conditions for the system may be written in dimensionless form as

$$\frac{\partial T_p}{\partial a} = \frac{\partial^2 T_p}{\partial x^2}, \quad (1)$$

$$T_p(x, 0) = 0, \quad (2)$$

$$T_p(1, a) = 0, \quad (3)$$

and

$$\frac{\partial T_p}{\partial x}(0, a) = M[T_p(0, a) - 1], \quad (4)$$

where

$$\left. \begin{aligned} T_p(x, a) &= \frac{t_p(y, \theta) - t_b}{t_w - t_b}, \\ a &= \frac{\theta \alpha_m}{L^2}, \\ x &= \frac{y}{L}, \\ M &= \frac{h_c L}{k_m} \end{aligned} \right\} \quad (4a)$$

t_p = temperature of the packet,

y = distance from the wall into the bed,

α_m = thermal diffusivity of the packet,

k_m = thermal conductivity of the packet,

θ = age of the packet at the wall,

and

L = characteristic length.

This system of equations is similar to the formulation of the "film-penetration" model as set up by Toor and Marchello.²⁹ This model, as its name implies, is a combination of the film and penetration models of transport. An equation is obtained that is identical to the penetration-model equation for short ages of the packet, and becomes identical to the film-model equation for long ages.

The system of equations (Eqs. 1-4) becomes identical to that of Toor and Marchello²⁹ if Eq. 4 is replaced with

$$T_p(0, a) = 1, \quad (5)$$

or, what is equivalent, if the contact Nusselt number M in Eq. 4 is infinite.

It becomes identical to Baskakov's¹¹ system if Eq. 3 is replaced with

$$T_p(\infty, a) = 0. \quad (6)$$

It becomes identical to Mickley's⁹ system if, in addition to replacing Eq. 3 with Eq. 6, we further replace Eq. 4 with Eq. 5.

The model is thus a film-penetration model with a contact resistance at the transporting surface.

The solution of the system is given by Carslaw and Jaeger.³⁰ (See Appendix A.) The solution for the instantaneous and the average Nusselt numbers is of more interest.

a. Instantaneous Nusselt Number. Let us define an overall instantaneous heat-transfer coefficient by

$$h(a) = \frac{q(a)}{t_w - t_b}, \quad (7)$$

where $q(a)$ = instantaneous heat flux from the wall to the bed, and the instantaneous Nusselt number by

$$N_{Nu}(a) = \frac{h(a)L}{k_m}. \quad (8)$$

Then we obtain a solution for $N_{Nu}(a)$ as (see Appendix A)

$$N_{Nu}(a) = \frac{M}{M+1} + 2M^2 \sum_{n=1}^{\infty} \frac{\exp(-\lambda_n^2 a)}{M^2 + M + \lambda_n^2}, \quad (9)$$

where the eigenvalues λ_n are the positive roots of the transcendental equation

$$\lambda_n \cot \lambda_n + M = 0 \quad (10)$$

and have been tabulated.³⁰ Equation 9 is a solution that converges for long ages. A solution valid for short ages is given in Appendix A.

Koppel et al.¹² have shown that the Laplace transform of the instantaneous Nusselt number $N_{Nu}(p)$ is of interest in determining the

average Nusselt number, $\bar{N}_{Nu}$. Equation 9 follows from an inversion of $N_{Nu}(p)$ obtained from Eqs. 1-4, together with Eqs. 7 and 8 as

$$N_{Nu}(p) = \frac{M}{\sqrt{p} (\sqrt{p} + M \tanh \sqrt{p})} \quad (11)$$

b. Average Nusselt Number. In general, the instantaneous Nusselt number is of less interest than the average Nusselt number. The average Nusselt number is defined, in terms of a surface-age density function $\phi(a)$, as

$$\bar{N}_{Nu} = \int_0^{\infty} N_{Nu}(a) \phi(a) da. \quad (12)$$

In this form, the average Nusselt number is difficult to evaluate, since the instantaneous Nusselt number is given by an infinite series which converges slowly at low values of a . However, for special forms of $\phi(a)$, one of which is the exponential form, Koppel et al.¹² have derived operational techniques for arriving at the average Nusselt number using the Laplace transform of the instantaneous Nusselt number.

In particular, for an exponential age density,

$$\left. \begin{aligned} \phi(a) &= \frac{1}{\bar{\tau}} \exp(-a/\bar{\tau}) & a \geq 0 \\ &= 0 & \text{otherwise,} \end{aligned} \right\} \quad (13)$$

where $\bar{\tau}$ = mean dimensionless residence time of packets at the surface, we obtain

$$\bar{N}_{Nu} = \bar{N}_{Nu}(\bar{\tau}) = \frac{M}{1 + M \sqrt{\bar{\tau}} \tanh \sqrt{\frac{1}{\bar{\tau}}}} \quad (14)$$

For

$$\left. \begin{aligned} \phi(a) &= \left(\frac{1}{\bar{\tau}} + \frac{2a}{\bar{\tau}^2} \right) \exp(-2a/\bar{\tau}) & a \geq 0 \\ &= 0 & \text{otherwise,} \end{aligned} \right\} \quad (15)$$

we obtain

$$\bar{N}_{Nu} = \bar{N}_{Nu}(\bar{\tau}) = \frac{M}{2 + M \sqrt{2\bar{\tau}} \tanh \sqrt{\frac{2}{\bar{\tau}}}} + \frac{M \left(1 + \frac{M}{2} \sqrt{\frac{\bar{\tau}}{2}} \tanh \sqrt{\frac{2}{\bar{\tau}}} + \frac{M}{2} \operatorname{sech}^2 \sqrt{\frac{2}{\bar{\tau}}} \right)}{\left(\sqrt{2} + M \sqrt{\bar{\tau}} \tanh \sqrt{\frac{2}{\bar{\tau}}} \right)^2} \quad (16)$$

These forms, Eqs. 13 and 15, of $\phi(a)$ follow from gamma residence-time density forms as shown by Koppel *et al.*¹² (See Appendix A.)

Numerically, there is little difference (less than 5% over most values of M and $\bar{\tau}$) between the values of $\bar{N}_{Nu}$ calculated from Eqs. 14 and 16 for the same M and $\bar{\tau}$. This is in accordance with Koppel *et al.*,¹² who show that in typical cases, for the same $\bar{\tau}$, the shape of the age-density function does not affect the value of the average transport coefficient significantly.

Equations 13 and 14 are of particular interest since they have an equivalent in the work of Toor and Marchello.²⁹ These authors have considered the film-penetration model with zero contact resistance, or equivalently, infinite contact Nusselt number, and an exponential age density. Operating on Eq. 14, we obtain

$$\lim_{M \rightarrow \infty} [\bar{N}_{Nu}] = \frac{1}{\sqrt{\bar{\tau}} \tanh \sqrt{\frac{1}{\bar{\tau}}}} \quad (17)$$

This is the basic result of Toor and Marchello in closed form. These authors give two infinite series (for short and long average residence times) for the average Nusselt number, and use only the first term in each to approximate $\bar{N}_{Nu}$. As shown in Appendix A, both of these may be summed to give Eq. 17.

c. Asymptotic Values of Nussett Number. The asymptotic values (at $\bar{\tau} = 0$ and ∞) for the instantaneous Nusselt number may be obtained from Eq. 11 by using the limit theorems for the Laplace transform. For these values of $\bar{\tau}$, average Nusselt numbers take on the same values as the corresponding instantaneous Nusselt numbers, whatever the shape of the age distribution. Thus,

$$N_{Nu}(0) = \bar{N}_{Nu}(0) = M, \quad (18)$$

and

$$N_{Nu}(\infty) = \bar{N}_{Nu}(\infty) = \frac{M}{M+1}. \quad (19)$$

Note that the model predicts finite, nonzero values in both limiting cases, which is physically realistic.

For small values of $\bar{\tau}$, the average residence time, Eq. 14 may be approximated by

$$\bar{N}_{Nu}(\bar{\tau}) = \frac{M}{1 + M\sqrt{\bar{\tau}}} \quad (20)$$

which is identical to Danckwerts' relation³¹ for finite contact heat-transfer coefficient. For infinite M we obtain, taking the limit in Eq. 20,

$$\bar{N}_{Nu}(\bar{\tau}) = \sqrt{\frac{1}{\bar{\tau}}} \quad (21)$$

which is Danckwerts' relation.

d. Application to Fluidized-bed Heat Transfer. The model may be applied to fluidized-bed heat transfer by considering the ratio of the average Nusselt number and the Nusselt number at infinite age. For the exponential age density, combining Eqs. 14 and 19, we obtain

$$\frac{\bar{N}_{Nu}(\bar{\tau})}{\bar{N}_{Nu}(\infty)} = \frac{\bar{h}(\bar{\tau})}{\bar{h}(\infty)} = \frac{M+1}{1 + M\sqrt{\bar{\tau}} \tanh \sqrt{\frac{1}{\bar{\tau}}}} \quad (22)$$

and, noting that $h_c = \bar{h}(0)$ from Eq. 18,

$$\frac{\bar{N}_{Nu}(0)}{\bar{N}_{Nu}(\infty)} = \frac{\bar{h}(0)}{\bar{h}(\infty)} = \frac{h_c}{\bar{h}(\infty)} = M+1. \quad (23)$$

Values of the parameters M and L are necessary for the particular system in order to apply the model.

Equation 23 may be used to obtain M for an experimental situation. The data necessary is the heat-transfer coefficient at minimum fluidization. At this condition, the average age of packets at the surface is infinite since there is no movement of particles. In other words, $\bar{h}(\infty)$ is identical to the heat-transfer coefficient at minimum fluidization, h_{mf} .

Baskakov¹¹ has presented a relation for h_c by considering temperature gradients in a particle in contact with a hot wall. He postulates that temperature gradients exist in the particle only up to a depth d times the particle radius (where d is fractional). By considering the thermal resistance due to the gas and this postulated layer in the particle, he obtains

$$\frac{1}{h_c} \approx \frac{D_p}{\pi k_g \left[\ln\left(\frac{k_s}{dk_g}\right) - 1 \right]}, \quad (24)$$

where

D_p = particle diameter,

k_g = gas thermal conductivity,

and

k_s = solid thermal conductivity.

Note that h_c is not sensitive to the value of d , for which Baskakov recommends the value of 0.1.

Since h_c and h_{mf} are known, Eq. 23 yields the value of M .

The value of the characteristic length, L , is necessary since it is incorporated in the nondimensionalizing factor for $\bar{\tau}$. It is obtained from Eq. 19, recalling the definition of N_{Nu} (Eq. 8), as

$$L = \frac{M}{M+1} \frac{k_m}{h_{mf}}. \quad (25)$$

Thus the value of h_{mf} is the key quantity in the application of this model to experimental data; it is used to compute both M and L .

In practice, the mean residence times of packets at the surface is difficult to determine. Residence times of particles at the surface may be measured by tracer techniques. (See Section VI.) During the present work, particles moved in groups downwards along the wall, only a few particles (less than 5%) moving away from or towards the wall in a radial direction. This would imply that the particles are representative of the packets and that particle residence times are virtually the same as packet residence times. Since some particles do move away from and towards the surface radially, we should expect that particle residence times are in general shorter than packet residence times.

2. Model II

The geometry of Model II is shown in Fig. 19. The following assumptions are made:

1. The fluidized particles are spheres of uniform diameter, D_p .
2. Particles at temperature t_b from the bulk of the fluidized bed arrive at the heating surface, which is at temperature t_w . While the

Fig. 19. Geometry of Model II

particle is at the surface it receives heat by convection from the fluid adjacent to the surface. This fluid is assumed to be at t_w and is assumed stagnant. Additionally, the convective Nusselt number between the gas and the particle is assumed to be 2, the limiting value for an isolated sphere in an infinite, stagnant fluid. This should be approximately true since, if the particle is small enough, it is completely enclosed by the fluid boundary layer at the wall, where the velocity is near zero.

3. While the particle is at the surface, it also loses heat by conduction to a packet of particles of thickness L situated between the wall particle and the bulk of the bed. As before, the packet is assumed to have the properties of the bed at minimum fluidization.

Other assumptions regarding radiation, thermal properties, etc., are as for Model I.

Under these assumptions, the differential equation and the boundary conditions for the system may be written in dimensionless form, as (see Appendix A)

$$\frac{\partial T_p}{\partial a} = \frac{\partial^2 T_p}{\partial x^2} \quad (26)$$

$$T_p(0, a) = T(a), \quad (27)$$

$$T_p(x, 0) = 0, \quad (28)$$

$$T_p(1, a) = 0, \quad (29)$$

$$T(0) = 0, \quad (30)$$

and

$$\frac{dT}{da} = \lambda(1 - T) + \gamma \frac{\partial T_p}{\partial x}(0, a), \quad (31)$$

where

$$\left. \begin{aligned} T(a) &= \frac{t(a) - t_b}{t_w - t_b}, \\ \lambda &= \frac{12k_g}{k_m} \frac{\rho_m}{\rho_s} \left(\frac{L}{D_p}\right)^2, \\ \gamma &= \frac{3}{2} \frac{\rho_m}{\rho_s} \frac{L}{D_p}, \end{aligned} \right\} \quad (32)$$

ρ_m = density of the packet,

ρ_s = density of the solid,

and

$t(a)$ = temperature of the surface particle, assumed uniform.

Other quantities are as in Model I. Equation 31 is the heat balance for the surface particle.

This system of equations is similar to that of Ziegler *et al.*,⁸ but includes the additional heat loss from the surface particle to the bed.

This model is the film-penetration variant of the model presented briefly by Koppel *et al.*¹² It becomes identical to this latter model if Eq. 29 is replaced by

$$T_p(\infty, a) = 0 \quad (33)$$

and suitable correction is made to λ and γ . The length factor for rendering time and distance dimensionless is changed to D_p .

Defining the instantaneous Nusselt number as before, and obtaining the instantaneous heat flux by considering a heat balance for the surface particles, we obtain (see Appendix A)

$$N_{Nu}(a) = 2\pi \frac{k_g}{k_m} \frac{L}{D_p} \mu[1 - T(a)], \quad (34)$$

where

$$\frac{\mu}{D_p^2} = \text{particles per unit area of the heated surface,}$$

and

$$\begin{aligned}\mu &= 2/\sqrt{3} \text{ for hexagonal packing} \\ &= 1 \text{ for square packing.}\end{aligned}$$

Applying the Laplace transformation to Eqs. 26-31 and 34, we obtain.

$$N_{Nu}(p) = 2\pi \frac{k_g}{k_m} \frac{L}{D_p} \mu \frac{p + \gamma \sqrt{p} \coth \sqrt{p}}{p(p + \lambda + \gamma \sqrt{p} \coth \sqrt{p})} \quad (35)$$

Inversion of Eq. 35 is formidable but unnecessary for obtaining average coefficients.

a. Average Nusselt Number. For an exponential age density (Eq. 13), the average Nusselt number is obtained as

$$\bar{N}_{Nu}(\bar{\tau}) = 2\pi \frac{k_g}{k_m} \frac{L}{D_p} \mu \frac{\frac{1}{\bar{\tau}} + \gamma \sqrt{\frac{1}{\bar{\tau}}} \coth \sqrt{\frac{1}{\bar{\tau}}}}{\lambda + \frac{1}{\bar{\tau}} + \gamma \sqrt{\frac{1}{\bar{\tau}}} \coth \sqrt{\frac{1}{\bar{\tau}}}} \quad (36)$$

b. Asymptotic Values. The expressions for the instantaneous and average Nusselt numbers at arguments of zero and infinity follow in a similar manner to those for Model I. We obtain

$$N_{Nu}(0) = \bar{N}_{Nu}(0) = 2\pi \frac{k_g}{k_m} \frac{L}{D_p} \mu \quad (37)$$

and

$$N_{Nu}(\infty) = \bar{N}_{Nu}(\infty) = 2\pi \frac{k_g}{k_m} \frac{L}{D_p} \mu \frac{\gamma}{\lambda + \gamma} \quad (38)$$

Once again, finite, nonzero values are predicted.

c. Application to Fluidized-bed Heat Transfer. As before, Model II too is applied to fluidized-bed heat transfer by considering the ratio of the average Nusselt number and the Nusselt number at infinite age. Here, we obtain

$$\frac{\bar{N}_{Nu}(\bar{\tau})}{\bar{N}_{Nu}(\infty)} = \frac{\bar{h}(\bar{\tau})}{\bar{h}(\infty)} = \frac{\lambda + \gamma}{\gamma} \cdot \frac{\frac{1}{\bar{\tau}} + \gamma \sqrt{\frac{1}{\bar{\tau}}} \coth \sqrt{\frac{1}{\bar{\tau}}}}{\lambda + \frac{1}{\bar{\tau}} + \gamma \sqrt{\frac{1}{\bar{\tau}}} \coth \sqrt{\frac{1}{\bar{\tau}}}} \quad (39)$$

To apply the model, the value of L must first be determined. As before, h_{mf} is used in this determination. Using Eq. 38 and the definitions of λ and γ (Eq. 32) and solving for L , we obtain

$$L = \frac{2\pi\mu k_g k_m - k_m D_p h_{mf}}{8k_g h_{mf}} \quad (40)$$

Hence, knowing h_{mf} , we can use Eq. 40 to obtain L for a particular system. Then, $\bar{h}(\bar{\tau})/h_{mf}$ follows from Eq. 39.

d. Contact Resistance. For Model II, the contact heat-transfer coefficient is given through Eq. 37 as

$$\bar{h}(0) = h_c = 2\pi \frac{k_g}{D_p} \mu. \quad (41)$$

Equation 41 presents an alternative relation to Baskakov's relation (Eq. 24) for determining h_c . Equation 41 would imply that h_c is independent of particle thermal properties and depends on the gas conductivity and particle diameter. This is similar to the statement of Botterill *et al.*¹⁷ that heat transfer is strongly determined by a gas film between the particles and the wall. (See Section I.)

B. Results

Results from the two models were compared to data obtained in a stirred fluidized bed. Stirring speed was 440 rpm, except for cellulose acetate, where 280 rpm was used. As pointed out earlier, this system ensured that no bubbles were present at the test surface. This was necessary in the measurement of particle residence times. Residence-time data were measured for the cellulose acetate spheres and the three sizes of glass. Residence times for the copper, aluminum, and alumina particles were obtained from a dimensionless empirical plot of $\bar{\theta} v_{mf}/D_p$ vs v/v_{mf} obtained by using the data for the cellulose acetate and the glass particles. (See Section VI.) As a consequence, the results for the metal particles and the alumina are less reliable since they depend on the postulated $\bar{\theta} v_{mf}/D_p$ vs v/v_{mf} relationship. The residence-time data for the glass and acetate particles were smoothed by the postulated empirical curve. For most cases the smoothing resulted in less than 10% change in $\bar{\theta}$. (See Section VI.)

Although in general the transporting surface was clear of bubbles, occasionally a bubble would get past the stirrer and sweep the surface clear of particles. This occurred more frequently at the higher gas velocities, corresponding to short average residence times. This introduced a further source of error and may partly be why some of the data fall below the predicted values.

1. General Results

For all the particle systems, heat-transfer data were obtained for increasing superficial gas velocities for the full range of bed expansion, from the packed to the nonbubbling fluidized condition. The bed was not stirred when packed-bed data were being obtained.

Fig. 20. Heat Transfer vs Gas Velocity for Glass Systems

For all the systems, the gradient of the curve of heat-transfer coefficient vs gas velocity increased radically once the bed was fluidized. Figure 20 shows this for Nos. 1-3 glass particles. (Particle properties are given in Tables I and III.) The sharp increase in the gradient after fluidization indicates that the increase in heat transfer is not due to the increase in gas velocity. (Compare the gradient for the packed-bed condition.) The reason for the sharp increase is that the mechanism of heat transfer has changed. For the packed bed, the mechanism is one of forced convection between the gas and the particles. In the fluidized condition, the particles move in the bed and act as heat carriers. It is precisely this mechanism that the above models are designed to predict.

TABLE III. "Packet" Thermal Properties

Particle Type	Weight Mean Particle Diameter, ft	Packet Properties for Minimum Fluidization			
		Density, ρ_m , lb/ft ³	Conductivity k_m , Btu/(hr)(ft)(°F)	Specific Heat, C_m , Btu/(lb)(°F)	Heat-transfer Coefficient, h_{mf} , Btu/(hr)(ft ²)(°F)
Cellulose acetate	0.0105	47.7	0.044	0.36	14.7
No. 1 Glass	0.00265	92.0	0.105	0.17	8.5
No. 2 Glass	0.00194	91.0	0.105	0.17	6.4
No. 3 Glass	0.00103	91.0	0.103	0.17	3.7
No. 1 Copper	0.00200	331.6	0.31	0.10	17.3
No. 2 Copper	0.00105	325.5	0.31	0.10	9.4
Aluminum	0.00098	93.6	0.25	0.224	7.2
No. 1 Alumina	0.00198	112.0	0.11	0.193	10.8
No. 2 Alumina	0.00099	115.0	0.11	0.193	5.4

Thus, the data in general support the particle heat-transfer mechanisms such as those of Models I and II. The next sections compare the data with the specific predictions of these models.

2. Model I Results

Figures 21-23 compare experimental data to the theoretical predictions (assuming an exponential age density) from Eq. 14. An exponential

density was assumed since the measured residence-time data of particles showed a rough exponential form. (See Section VI.) Again, as pointed out earlier, the assumption of a different shape for the age density (Eq. 15) gave values within 5% of those from Eq. 14 over a wide range of M and $\bar{t}$. Consequently the exponential age density was adopted since the computations were simpler. Koppel *et al.*¹² have shown that this insensitivity to shape should be expected in most cases.

Fig. 21. Heat-transfer Data and Model I Predictions for Glass Systems

Values of h_c were found from Baskakov's relation (Eq. 24) for the various solids, and hence M from Eq. 23. Packet properties were estimated in terms of particle properties using methods presented by Mickley.⁹ These are presented in Table III.

Figure 21 shows the data and predicted values for three sizes of glass particles, whose properties are given in Tables I and III. The range of v/v_{mf} was from 1.0 to 2.4. Residence times were actually measured for these particles. As may be seen from Fig. 21, data for the No. 1 glass (the largest size) are roughly 20% below theory, on the average, with a maximum deviation of about 40%. The data for the No. 2 glass (the next smaller size)

Fig. 22. Heat-transfer Data and Model I Predictions for Metal Systems

exhibit more scatter from the predicted values (about 25% average deviation), whereas two of the points for the No. 3 glass (the smallest size)

Fig. 23. Heat-transfer Data and Model I Predictions for Alumina Systems

disagree radically with the model prediction. On the average, data for the No. 3 glass are about 35% below predictions, with a maximum deviation of 40-50%. Why the deviations are greater for the smaller sizes is not presently explainable. In any case, the errors are not excessive.

The gradient in the data is steeper than that predicted by the model. This is probably a velocity effect. The data at short average residence times are those for high values of the super-

ficial gas velocity. At these higher gas velocities, heat transfer to individual particles is enhanced and this shows up as a steeper slope in the data. The model, since it does not include any correction for velocity, predicts a more shallow slope. Quite possibly, the contact Nusselt number M should not be treated as a constant. At higher gas velocities, M should take on a higher value. Note that M , as calculated by Baskakov,¹¹ assumes stagnant fluid between the particles and the wall. Increases in heat transfer caused by increasing gas velocity cannot therefore be accounted for.

Figure 22 presents data and predictions for the two sizes of copper and one size of aluminum. (Particle properties are given in Tables I and III.) The range of v/v_{mf} was 1.0-1.8. The residence times were not actually measured but were obtained by postulating a "universal" relation between $\bar{\theta}v_{mf}/D_p$ and v/v_{mf} , $\bar{\theta}$ being the average residence time. The actual relation used was obtained from data on the glass particles. This postulated form may result in erroneous predictions of the mean residence time.

Figure 22 shows that the data fall below the predicted values an average of about 100% for all three particle types. In agreement with the data, the model does predict that h/h_{mf} is higher for aluminum than for the two sizes of copper. Coefficients are also correctly predicted to be higher for the smaller size of copper than for the large size. Two possible explanations for the poor predictions suggest themselves: The mean residence times are in error, or the values of M are incorrect. In the former case, assuming the model is valid in general (which appears to be true in view of Fig. 21), we may calculate the residence times

corresponding to the data points from the theoretical curves. This procedure results in residence times of the order of 50-100 sec. This is at variance with physical experience; observations of the bed surface showed residence times to be of the order of a few seconds. This leads us to believe that the values of h_c obtained using Baskakov's equation are in error. This could conceivably occur because of the presence of a thin gas film between the particles and the wall. No such film is included in Baskakov's model. The reason the gas film becomes important for the metal particles (the effect is not apparent for the glass particles) is that it becomes the controlling resistance for heat transfer. If a gas film of thickness $0.1D_p$ is assumed, following Botterill,¹⁷ we obtain $M \approx 3$ for the No. 1 copper particles. Alternately, using another expression for h_c obtained from Model II (Eq. 41), we obtain $h_c \approx 45$ and $M \approx 2$. The agreement between these two calculations of M corroborates the hypothesis that M values from Eq. 22 are too high. The theoretical curve for $M = 2$ is also plotted in Fig. 22 and shows that the data for the No. 1 copper are now in much better agreement. Similar calculations for No. 2 copper give $M \approx 9$ from Eq. 41. For the aluminum, we obtain $M \approx 8$ from Eq. 41; however, since the thermal conductivity of aluminum is smaller than that of copper, in this case the gas film is probably not as strongly controlling; the thermal resistance in the aluminum probably has some significant effect. Botterill¹⁷ has observed a similar discrepancy between theoretical predictions and experimental data for copper particles.

Figure 23 shows data and theoretical curves for two sizes of alumina. (Particle properties are listed in Tables I and III.) Values of v/v_{mf} ranged from 1.0 to 1.8. As for the metal particles, residence times were estimated from their postulated relation with v/v_{mf} . The theoretical curve for No. 2 alumina has not been plotted in Fig. 23. The model correctly predicts higher coefficients for the smaller-size alumina. Errors increase with decreasing residence time. It is difficult to state whether errors in the estimates of $\bar{\tau}$ or M are the cause of the data scatter. The general trend is, however, correctly predicted.

Table IV compares the data for the cellulose acetate spheres with the theory. (See Tables I and III for properties.) The range for v/v_{mf} was 1.0-1.25, and stirring speed for these particles was 280 rpm. Because of their large size, these particles required a high gas velocity for fluidization. The predicted values have a very shallow slope at these residence times, indicating that a film mechanism is in effect. This is at variance with the data. A possible cause is the velocity effect mentioned in connection with the data on the glass particles. The lower residence times correspond to higher gas velocities, and this could be the reason for the steeper gradient in the data.

TABLE IV. Heat-transfer Data for Cellulose Acetate Spheres

$$h_{mf} = 14.7$$

$$M = 0.4$$

$\bar{\tau}$ (Dimensionless)	h/h_{mf}	
	Experimental	Theoretical (Model I)
4.84	1.26	1.02
3.24	1.51	1.03
2.46	1.68	1.03
2.06	1.80	1.04

3. Model II Results

Figures 24-26 compare the heat-transfer data to theoretical predictions from Model II, assuming an exponential age density (Eq. 36).

Fig. 24. Heat-transfer Data and Model II Predictions for Glass Systems

Reasons for choosing this form of the age density are given in Section 2 above. For all the solids used, first the characteristic length, L , was found, using Eq. 40 and experimental data for h_{mf} . Square packing of particles at the surface ($\mu = 1$) was assumed. Predictions were changed only a few percent over a wide range of $\bar{\tau}$ assuming hexagonal packing. Figure 24 shows the data and model predictions for the three sizes of glass.

(Recall that residence times

were measured for these particles.) On the average, data for the No. 1 glass are about 10% higher than the predicted values, with approximately 20% maximum deviation. For the No. 2 glass (the next smaller size), the data are also higher at an average deviation of about 30%; maximum deviation was approximately 50%. The data for the smallest size (No. 3 glass) scatter about the predicted values. Two points agree very closely, the others being about 20% different from predictions. That the data for the smallest size give best agreement with the theory is quite logical. The model assumes that the particles are small enough so that, when at the wall, they are surrounded by fluid at the wall temperature. Clearly, this assumption is valid only for small particles, and, hence, the model should give better predictions

for smaller particles. In the light of this, it is not understood why the data for No. 2 glass exhibit a greater scatter than those for No. 1 glass. There may be measurement errors for the No. 2 glass. This is substantiated by

the fact that these data also fall above the predictions of Model I, whereas the other two sizes agree quite closely. In general, the gradient of the data is higher than that predicted by the model. This is probably due to the velocity effect mentioned above.

Fig. 25. Heat-transfer Data and Model II Predictions for Metal Systems

Figure 25 shows data and model predictions for the metal particles (aluminum and No. 1 and 2 copper). On the average, data for the aluminum particles are about 10% below model predictions, with a maximum deviation of about 25%.

Since the aluminum particles were of a small size, the model was particularly applicable. Data for the copper particles are in more serious disagreement. For the No. 1 copper, the data generally are below predictions by about 70-100%. What is worse, the data appear to show that a film model is in effect (i.e., $h/h_{mf} \approx 1$, as shown by the two points at the higher $\bar{\tau}$ values), contrary to the model. Data for the No. 2 copper (the smaller size) are also at variance with model predictions (about 70-80% below). Once again, the data point at the highest value of $\bar{\tau}$ indicates a film mechanism contrary to model predictions. No firm explanation is presently available as to why data from the copper particles disagree so radically with predictions. This occurred for Model I and has also been observed by Botterill et al.¹⁷ The cause for the discrepancy could well be the residence times assumed for these data. These were obtained from the postulated $\bar{\theta} v_{mf}/D_p$ vs v/v_{mf} relationship. There is no effect of particle density in this assumed relation, and it is possible that higher densities lead to longer residence times. In general, Model II appears to give closer predictions

Fig. 26. Heat-transfer Data and Model II Predictions for Alumina Systems

for metal particles than does Model I. This is especially true for the aluminum particles, although these were nonspherical.

Figure 26 shows the data and predictions for the Nos. 1 and 2 alumina. Data for the No. 1 alumina (the larger size) scatter about the predicted values, but in general are lower by about 15%. Surprisingly, data from the smaller size alumina are in greater disagreement. (The model should be more applicable to smaller particles, as discussed in the paragraph above.) On the average, the data are below predictions by about 40%, with a maximum deviation of approximately 60%. Once again, the assumed residence times may be in error, especially for the smaller size alumina. Note too that these particles were nonspherical. In view of the many estimates made, the errors are not too great.

For the cellulose acetate particles, predictions could not be made from this model, since L as calculated from Eq. 40 was negative. The model probably fails because a basic assumption is violated. These particles are so large (1/8-in. diam) that while at the wall, they are not completely surrounded by fluid at the wall temperature, as assumed in the model. Instead, a large part of their bulk is probably in contact with fluid at lower temperature.

C. Comparison with Other Models

It is pertinent to compare the predictions from Models I and II with models that have been proposed in the literature.

The Mickley model^{9, 21} is a penetration-type model with no contact resistance and assumes the characteristic length to be infinite. The model due to Baskakov¹¹ is also of penetration type assuming infinite characteristic length, with a contact resistance included. Toor and Marchello²⁹ consider a film-penetration-type model with no contact resistance and a finite characteristic length. Table V compares the values from these three models to predictions from Models I and II for the three glass systems. The glass systems were chosen for the comparison since measured residence-time data were available for these systems. Additionally, values from Model II assuming an infinite characteristic length are also included. Note that only Models I and II and the model of Toor and Marchello require the value of h_{mf} .

For the present system (the gas-fluidized bed), residence times were sufficiently short that a pure penetration mechanism was in effect. In Eq. 14, for example, $\tanh \sqrt{1/\bar{\tau}}$ was very nearly equal to unity. Model I, therefore, gave the same results as Baskakov's model. The same was true of Model II. Values from Model II for finite and infinite L gave the same results.

TABLE V. Model Comparisons for Glass Systems

Particle System	Average Residence Time, sec	Heat-transfer Coefficient, $\bar{h}$, Btu/(hr)(ft ²)(°F)			
		Experiment	Model I and Baskakov ¹¹	Mickley ⁹ and Toor and Marchello ²⁹	Model II, L finite and L = ∞
No. 1 Glass	9.7	17.5	19.2	24.7	17.0
	2.5	26.4	31.0	48.4	25.5
	1.4	33.1	36.9	64.8	28.5
	0.63	47.8	45.5	97.0	31.6
No. 2 Glass	6.5	23.0	24.1	30.1	21.5
	2.2	44.3	36.1	51.7	30.8
	0.72	62.0	51.0	89.9	39.2
	0.42	71.8	58.8	117.7	42.1
No. 3 Glass	15.3	16.2	17.8	19.4	15.4
	1.3	37.0	50.6	65.6	45.2
	0.59	60.1	68.3	99.0	58.6
	0.48	75.0	73.2	109.6	61.9
Average Percent Absolute Deviation from Experiment			14	57	18

Table V shows that the average percent absolute deviations from experiment of values from Models I and II are almost the same. Model I is somewhat superior. Note the large deviations in the Mickley and the Toor-and-Marchello models.

These values clearly show the importance of including a contact resistance at the transporting surface. The Baskakov model, which is the Mickley model with a contact resistance, gives a percent average deviation one-fourth that of the Mickley model. These data also provide a valuable explanation for certain discrepancies in the literature. Both Baskakov and Mickley, although their models are somewhat different, showed that they could predict heat-transfer coefficients in good agreement with experiment. Table V shows that, in general, the values from Mickley's model are higher than those from Baskakov for the same average residence time. The reason these authors obtained the same predictions using different models is that they used different values for the average residence time (or the renewal frequency, as Mickley terms it). Both these authors use data on heat transfer and renewal frequency obtained by Mickley.²¹

Mickley observed two types of renewal frequency. They were determined by an examination of the transient temperature in a low-heat-capacity heater immersed in the bed. The first type of renewal frequency, the larger of the two, corresponded to changes in the derivative of the time-temperature curve for this heater. Mickley terms this frequency, the frequency of partial replacement of packets at the surface. The second type of renewal frequency corresponded to very sharp changes in the temperature of the heater. These changes were due to bubbles removing all

particles from the surface. Mickley terms this the frequency of total replacement of packets at the surface.

Mickley uses the slower frequency of total replacement in his equation for the heat-transfer coefficient, whereas Baskakov uses the faster partial-replacement frequency. This at once shows why their predictions agree. Mickley's equation would normally predict higher coefficients (Table V), but by using a lower frequency, he obtains lower coefficients, which are closer to the experimental data.

For the system used in the present work, the only kind of replacement frequency is the frequency of packets traversing the heat-transfer surface. Since no bubbles are present, there is no "total replacement." This is a simpler system and therefore more easily understood. Using the observed renewal frequency or average residence time, the Baskakov predictions are superior to those of Mickley's, as Table V shows. Clearly, then, a contact resistance present at the transporting surface must be accounted for if predictions are to be accurate.

Table V also shows that, effectively, a pure penetration mechanism, as opposed to a film-penetration mechanism, is in effect. As mentioned earlier, there is no difference between Baskakov's predictions and those of Model I, which is the film-penetration variant of Baskakov's model. This is not surprising. The results of Toor and Marchello²⁹ show that for average dimensionless residence times less than 0.5, the film-penetration model essentially becomes one of pure penetration. Figures 21-23 show that the data are in this range, implying that, for the practical situation, a pure penetration mechanism is in effect. A disadvantage of Model I is that the value of h_{mf} is necessary to determine the characteristic length, L . The values from Table V show that, in practice, Baskakov's model, which does not require h_{mf} , is as good as Model I for predicting heat-transfer coefficients. Hence, it is unnecessary to know the value of h_{mf} . Model I should give better predictions than Baskakov's models for higher $\bar{\tau}$, i.e., near the minimum fluidization conditions. These are admittedly of little practical interest. Also, in liquid-fluidized beds, the film mechanism should become more dominant.

Because Model II gives a low estimate of h_c (Eq. 41), it shows greater deviation (18%) for the glass systems. For this reason, predictions from Model II are always lower than those for Model I, and are usually lower than the experimental values. However, we have seen that Model II gives superior predictions for metal particles. Model I gives poorer predictions for high-conductivity particles because, for these particles, Baskakov's relation (Eq. 24) for h_c becomes inaccurate. Use of Eq. 41 to estimate h_c gives better results.

We therefore conclude that Model I and Baskakov's h_c give acceptable results for particles of low thermal conductivity. For high-thermal-conductivity materials, Model I should be used with h_c calculated by Eq. 41. Model II is recommended for use with particles of high thermal conductivity.

It is not possible to compare the data and results of the present work to existing heat-transfer correlations, since these correlations apply only to bubbling beds.

D. Conclusions

Film-penetration models have been presented to calculate the average heat-transfer coefficient between the wall and a fluidized bed. The only thermal data necessary, apart from solid and gas properties, are the heat-transfer coefficient at minimum fluidization. Comparison with existing models showed that predictions are as accurate as the Baskakov¹¹ model and superior to the Mickley⁹ and Toor²⁹ models. In the practical situation, the value of the heat-transfer coefficient at minimum fluidization is unnecessary since effectively a pure penetration model is in effect. It is important to include a contact resistance at the transporting surface to arrive at good predictions of the heat-transfer coefficient. Baskakov's formula for this resistance gives accurate values for most particles. For particles of high thermal conductivity, an alternative formula (Eq. 41) is recommended.

Although in this work the coefficient at minimum fluidization was determined experimentally, it should not be difficult in practice to estimate either from packed-bed data or by a theoretical approach. Edwards,³² for example, has found that the value of h_{mf} corresponds to coefficients found in fixed beds.

The models also require a knowledge of average residence times of particles at the transporting surface. This is a quantity that is not easy to measure. However, the average residence time depends on a lesser number of variables than does the heat-transfer coefficient. From a physical standpoint, the residence times should depend only on the fluid mechanics of the system; the heat-transfer coefficients additionally depend on the thermal properties. It should therefore prove simpler to correlate average residence times with the fluidization characteristics than it is to correlate the heat-transfer coefficients. Also, there should be a smaller error in the residence-time correlation. (Even though it is based on limited data, the postulated correlation of $\bar{\theta} v_{mf}/D_p$ vs v/v_{mf} gave reasonable predictions for the residence times.) From a theoretical standpoint, as well, the residence times should be easier to relate to the fluidization characteristics. Ruckenstein²² has made an attempt in this direction. (See Section I.)

Effectively, then, the model has split the complex thermal-hydrodynamic problem of fluidized-bed heat transfer into a purely thermal problem and a purely hydrodynamic problem.

Improvements to the model may be made along the following lines:

1. The validity of the "universal" relation $\bar{\theta} v_{mf}/D_p$ vs v/v_{mf} should be tested for a variety of systems. In particular, the effect of particle density should be sought. In the present work, this was not possible since good tracers could not be found for materials other than the glass and cellulose acetate particles. A theoretical approach to find the character of this relation should also prove useful.
2. The effect of gas bubbles sweeping across the transporting surface should be analyzed theoretically and experimentally. In the present work, a bubble-free surface was assumed and was experimentally enforced by breaking up the bubbles with the stirrer. Mickley²¹ approached the problem by finding the fraction of time the bubbles occupy the surface and correcting the predicted coefficient using this factor. (He assumed that the coefficient fell to zero when the bubble occupied the transporting surface.)
3. The heat-transfer models could be improved by considering the "velocity effect" mentioned in Section V.B.2 above. The approach in Model I could be to find the relation between M and the velocity. In Model II, the value of the assumed convective Nusselt number between the wall particle and the fluid surrounding it would require correction.

As shown in Appendix C, the probable overall uncertainty in h from a theoretical line (found as a combination of errors in measurement of h and $\bar{\tau}$) is about 11%. This compares favorably with the 14-16% deviation of the model predictions from the experimental values. The model predictions are therefore in good agreement with the data, considering the probable errors present in the latter.

VI. THEORY AND RESULTS: RESIDENCE TIME

This section discusses the methods used to arrive at the mean residence time and the age density function of particles at the test surface when only the time series of the total tracer count at the test surface is known. Methods for arriving at the average transport coefficients for a simple transport model are also presented. These methods are applied to computer-simulated series and to experimental data of tracer particle counts at the test surface in the fluidized bed.

For convenience, the tracer particles will be referred to simply as particles or elements. Similarly, reference to age or residence time implies the age or residence time of the tracer particles or elements at the test surface.

A. Theory

1. Mean Residence Time

The methods used to obtain the average residence time of particles at the test surface from observations of only the total tracer count at this surface have been given by Koppel *et al.*²⁶ For details, see Ref. 26. Also see Eq. 75 below.

2. Age Distribution

Koppel *et al.*²⁶ have presented one method of arriving at the age density from the time series of tracer counts at the test surface. The results of this section are based on the relations derived by these authors, the pertinent ones being stated below without proof.

It is appropriate first to differentiate between the age and the residence time of elements. The age, a , of an element implies the length of time that element has been in residence at the surface, if the surface is examined at any particular time. Residence time, τ , of an element means the total length of time that element resides at the surface. These are considered to be random variables in the usual probabilistic sense. The age density function, $\phi(a)$, and the residence-time density function, $f(\tau)$, along with their respective distribution functions, $\Phi(a)$ and $F(\tau)$, are defined in the usual probabilistic sense. Koppel *et al.*²⁶ have shown that

$$\phi(a) = \frac{1 - F(a)}{\bar{\tau}}, \quad (42)$$

where $\bar{\tau} = E\{\tau\}$, $E\{\}$ being the Expectation operator. Equation 42 indicates that $\phi(a)$ is monotone-decreasing.

Under certain independence assumptions, Koppel et al. further show that the count of elements at the surface is a Poisson-distributed random variable,

$$p_N(n) = e^{-\bar{N}} \frac{(\bar{N})^n}{n!}, \quad n = 0, 1, 2, \dots, \quad (43)$$

where

$$p_N(n) = \text{Prob}\{N = n\},$$

$$\bar{N} = E\{N\},$$

and $N(t)$ = a random function, the total count of elements at the surface, a function of time t . Hence, $E\{N\} = \text{Var}\{N\}$, $\text{Var}\{\}$ denoting the variance operator.

Equation 43 shows that the stochastic process $N(t)$ is stationary. For such a process, the autocovariance function $\psi(a)$ is defined as

$$\psi(a) = E\{N(t+a) N(t)\} - \bar{N}^2. \quad (44)$$

Note that $\psi(a)$ is a function of a only and does not depend on t . $\bar{N}$ is a constant and does not depend on t .

It may then be shown²⁶ that

$$\psi(a) = \frac{\bar{\tau}}{\bar{N}} \int_a^\infty (\tau - a) f(\tau) d\tau. \quad (45)$$

a. Age Density from Autocovariances. Equation 45 has the solution²⁶

$$f(a) = \frac{\bar{\tau}}{\bar{N}} \frac{d^2 \psi}{da^2}, \quad (46)$$

or, using Eq. 42,

$$\phi(a) = \frac{-1}{\bar{N}} \frac{d\psi}{da}. \quad (47)$$

The problem of applying Eq. 47 is that $\psi(a)$ must be estimated from data. Taking derivatives of this estimate magnifies any errors. In general, the estimators for the autocovariance have complicated

variance characteristics. Again, the estimators often predict negative values beyond a certain value of a .²⁶ This is, of course, incompatible with Eq. 45. Thus, Eq. 47 may be difficult to use in a practical situation. The problem of finding $f(a)$ from Eq. 46 is even more complex since the second derivative of the estimator for $\psi(a)$ magnifies errors even further. Fortunately, for transport problems we are interested in $\phi(a)$ rather than $f(a)$.

Using Eq. 47, Koppel et al.²⁶ further show that the mean residence time is given by

$$\bar{\tau} = \frac{-\bar{N}}{\left(\frac{d\psi}{da}\right)_{a=0}} \quad (48)$$

b. Spectral Analysis. In any time-series analysis, an approach via spectral analysis is often a fruitful alternative to the use of the autocovariance function. The motivation for using spectral analysis is threefold.³³ (See also Blackman and Tukey.³⁴) First, the spectrum is particularly useful for frequency-response studies. (The autocovariances may also be used for such studies, but the calculations are considerably more difficult.) Second, in general, the sampling properties of spectra (or the variances of spectral estimates) are considerably simpler than those for the autocovariances.³³ Third, when models are being constructed for a time series, spectral analysis frequently reveals important sources of fluctuations in the random process that would have been missed otherwise.

In this work we are primarily interested in the simpler statistical properties of spectral estimates over estimates of the autocovariance function.

c. Spectral Density. Following Solodovnikov,³⁵ the spectral density function, $S(\omega)$, a function of the frequency ω , is defined as the Fourier transform of the "autocorrelation function," $R(a) = \psi(a)/\psi(0)$:

$$S(\omega) = 2 \int_0^{\infty} R(a) \cos a\omega \, da. \quad (49)$$

Equation 49 is the definition of $S(\omega)$ for a real, stationary time series with autocorrelation function $R(a)$.

d. Relation between the Spectral Density and the Age Density. From Eq. 45 and recalling that $\psi(0) = \text{Var}\{N\} = \bar{N}$, we obtain

$$R(a) = \frac{1}{\bar{\tau}} \int_a^{\infty} (\tau - a) f(\tau) d\tau. \quad (50)$$

Using Eq. 50 in Eq. 49 and reversing the order of integration yields

$$S(\omega) = \frac{2}{\omega^2 \bar{\tau}} \int_0^{\infty} f(a)(1 - \cos \omega a) da. \quad (51)$$

Analogous to Eq. 50, we have, using Eq. 42,

$$R(a) = \int_a^{\infty} \phi(\tau) d\tau, \quad (52)$$

which yields

$$S(\omega) = \frac{2}{\omega} \int_0^{\infty} \phi(a) \sin \omega a da \quad (53)$$

the analog of Eq. 51.

For example, for an exponential residence-time density,

$$\left. \begin{aligned} f(\tau) &= \frac{1}{\bar{\tau}} \exp(-\tau/\bar{\tau}), & \tau \geq 0, \\ &= 0, & \text{otherwise,} \end{aligned} \right\}. \quad (54)$$

we obtain

$$\left. \begin{aligned} \phi(a) &= \frac{1}{\bar{\tau}} \exp(-a/\bar{\tau}), & a \geq 0, \\ &= 0, & \text{otherwise,} \end{aligned} \right\} \quad (55)$$

$$S(\omega) = \frac{2\bar{\tau}}{1 + (\omega\bar{\tau})^2}, \quad (56)$$

and

$$R(a) = \exp(-a/\bar{\tau}). \quad (57)$$

Equations 51 and 53 are Volterra integral equations of the first kind in f and ϕ , respectively. They may be solved by using Fourier transforms to obtain (see Appendix B)

$$f(a) = \frac{2}{\pi} \int_0^{\infty} \left[1 - \frac{\omega^2 \bar{\tau} S(\omega)}{2} \right] \cos \omega a \, d\omega, \quad a \geq 0, \quad (58)$$

and

$$\phi(a) = \frac{1}{\pi} \int_0^{\infty} \omega S(\omega) \sin \omega a \, d\omega, \quad a > 0. \quad (59)$$

Note that Eq. 59 is not valid for $a = 0$. However, since $\phi(0) = 1/\bar{\tau}$ (Eq. 42), we may obtain $\phi(0)$ from $\bar{\tau}$.

Equation 59 appears to possess computational advantages over Eq. 47. In Eq. 47 a differentiation is necessary which tends to magnify errors. Again, there is no guarantee that $\phi(a)$ will be positive for all positive a . Using Eq. 59, an integration is necessary, which should tend to minimize errors; however, once again there is no guarantee that $\phi(a)$ will be positive for all positive a . A disadvantage of Eq. 59 is that it is not valid for $a = 0$. Because of this, values computed near $a = 0$ will possibly be in error.

e. Average Transport Coefficients. In the usual case for studies of transport phenomena, the age or residence-time distributions are not of prime interest. Rather, we are interested in the average transport coefficients that are integrals of these distributions with the instantaneous coefficient

$$\bar{N}_{Nu} = \int_0^{\infty} N_{Nu}(a) \phi(a) \, da, \quad (60)$$

where

$N_{Nu}(a)$ = the instantaneous transport coefficient,

and

$\bar{N}_{Nu}$ = the average transport coefficient.

Using Eq. 59 in Eq. 60, we obtain

$$\bar{N}_{Nu} = \frac{1}{\pi} \int_0^{\infty} \int_0^{\infty} N_{Nu}(a) \omega S(\omega) \sin \omega a \, da \, d\omega, \quad (61)$$

which is an alternative to the relation presented by Koppel et al.,²⁶

$$\bar{N}_{Nu} = - \int_0^{\infty} \frac{dR}{da} N_{Nu}(a) da. \quad (62)$$

Once again, the absence of a derivative in Eq. 61 results in no magnification of errors during actual computation. For the simple transport model,

$N_{Nu}(a) = 1/\sqrt{\pi a}$ (simple penetration theory), Eq. 61 reduces to

$$\bar{N}_{Nu} = \frac{1}{\pi \sqrt{2}} \int_0^{\infty} \sqrt{\omega} S(\omega) d\omega. \quad (63)$$

We thus have an expression for $\bar{N}_{Nu}$ in terms of the spectral density.

f. Moments of the Residence-time Density. Higher moments of the residence-time density function may be obtained by considering integrals of the form

$$\int_0^{\infty} a^n \psi(a) da.$$

Operating on Eq. 45 in this manner and interchanging orders of integration, we obtain

$$\int_0^{\infty} a^n \psi(a) da = \frac{E\{\tau^{n+2}\}}{\bar{\tau}(n+1)(n+2)}, \quad n = 0, 1, 2, \dots \quad (64)$$

Similar expressions may be obtained for the even moments in terms of the spectral density by expanding the cosine term in Eq. 51 in an infinite series. In particular,

$$S(0) = \frac{E\{\tau^2\}}{\bar{\tau}}. \quad (65)$$

Equation 64 is difficult to use. In practice, estimates of $\psi(a)$ must be used, and the integral must be truncated. Further, the integral weights the autocovariances for large a more heavily, and it is precisely for these autocovariances that we have poor estimates (since the sample size decreases, as discussed in Section g below). Equation 65, which is equivalent to Eq. 64 for $n = 0$, presents a convenient method of obtaining the variance of the residence-time density.

g. Estimators. From experimental observation of the test surface we obtain the particle counts at discrete time intervals. We have $K + 1$ equally spaced observations of $N(t)$ at times $t_0 + j\Delta t$, where $j = 0, 1, 2, \dots, K$, and where Δt is the time interval separating the successive observations.

An estimator for $\bar{N}$ is constructed as

$$X_0 = \frac{1}{K+1} \sum_{j=0}^K N(t_0 + j\Delta t), \quad (66)$$

and for $\psi(j\Delta t)$ as

$$Y_j = \frac{1}{K+1-j} \sum_{i=0}^{K-j} (N_i - X_0)(N_{i+j} - X_0), \quad (67)$$

where $N_i = N(t_0 + i\Delta t)$ and Y_j are the sample autocovariances of counts $j\Delta t$ removed.

Koppel et al.²⁶ have pointed out that X_0 is probably a better estimator for $\psi(0)$ than Y_0 . This is a consequence of the Poisson nature of the particle counts. As a consequence, better estimates of $\psi(j\Delta t)$ are given by

$$Z_j = Y_j + X_0 - Y_0. \quad (68)$$

This forces $Z_0 = X_0$ as an estimate of $\psi(0)$. Tests on simulated series (which are discussed in detail below) showed that the estimates Z_j gave better results than Y_j and therefore were used throughout.

Note that the continuous variable, a , in the theoretical expressions is replaced by the discrete variable, $j\Delta t$, in the estimators. Derivatives must be approximated by differences. The variance of Y_j increases with j , since the sample size decreases, so that the estimates of the autocovariances are in greater error at higher $j\Delta t$.

Estimation of the spectral density for the time series is a complex task. Various methods exist, including prefiltering of the series, correction for trends, and so on.³⁴ In the present work, a straightforward estimation method was adopted and gave good results for computer-simulated series.

Estimators for the spectral density of a real series with sample autocovariances Z_k are of the form³⁷

$$\hat{S}(\omega) = W(0)\Delta t + \frac{2}{Z_0} \sum_{k=0}^n W(k\Delta t) Z_k \cos(\omega k\Delta t) \Delta t, \quad (69)$$

where $n = o(K)$ but tends to infinity with K . Note that Eq. 69 is a finite-sum form of Eq. 49, with a weight function $W(k\Delta t)$. For $W(k\Delta t)$ identically equal to 1, the variance of $\hat{S}(\omega)$ does not converge to 0 as $K \rightarrow \infty$.^{36,37} In other words, $\hat{S}(\omega)$ for $W(k\Delta t) = 1$ is not a "consistent" estimator. The value of n is generally chosen to be the lowest value for which Z_n approaches zero.

Various forms for $W(k\Delta t)$ have been proposed. In particular, the Bartlett estimator uses³⁷

$$\left. \begin{aligned} W(k\Delta t) &= 1 - \frac{|k|}{n}, & |k| \leq n \leq K, \\ &= 0, & |k| > n. \end{aligned} \right\} \quad (70)$$

Other weight functions have been suggested.³⁴ Actual computation, using Eq. 69 with Eq. 70 and a Tukey estimator,³⁴ did not give very different results. Equation 70 was therefore used throughout for the weight function.

The asymptotic variance of the Bartlett estimator is³⁷

$$\left. \begin{aligned} \sigma^2[S(\omega)] &\approx \frac{2}{3} \frac{n}{K} S^2(\omega), & \omega \neq 0, \\ &\approx \frac{4}{3} \frac{n}{K} S^2(0), & \omega = 0. \end{aligned} \right\} \quad (71)$$

This estimator is biased.³⁷ Furthermore, it is positive with probability one--a desirable feature since $S(\omega)$ is defined as always positive.

A further problem arises when $S(\omega)$ is estimated from a series whose elements are separated by a finite time interval (which is the practical case). The estimate of $S(\omega)$ can be obtained only up to the "Nyquist" frequency,³³ which corresponds to the shortest time interval separating two elements of the series--in this case, Δt . For frequencies higher than this, the estimate is meaningless since the elements of the series are not separated by intervals shorter than Δt . The Nyquist frequency is

$$\omega_N = \frac{\pi}{\Delta t}. \quad (72)$$

A second problem is "aliasing," which has been discussed by Jenkins³³ and Blackman and Tukey.³⁴ Aliasing occurs because estimates of

the spectral density at a particular frequency, ω , are confounded with higher frequencies, which are indistinguishable from ω as far as the data are concerned. This effect is reduced if $S(\omega)$ is small for $\omega > \omega_N$.

The age density, $\phi(a)$ is now estimated by the autocovariance method through Eqs. 47 and 66-68 as

$$\hat{\phi}_c(j\Delta t) = -\frac{1}{X_0} \frac{Z_j - Z_{j-1}}{\Delta t} \quad (73)$$

By the spectral-density method we have, using Eq. 59 in truncated form,

$$\left. \begin{aligned} \hat{\phi}_s(a) &= \frac{1}{\pi} \int_0^{\omega_N} \omega \hat{S}(\omega) \sin \omega a \, d\omega, & a > 0, \\ &= \frac{1}{\hat{\tau}}, & a = 0, \end{aligned} \right\} \quad (74)$$

where $\hat{\tau}$ is the estimate of the mean residence time derived from Eq. 48 and is given by

$$\hat{\tau} = \frac{X_0 \Delta t}{Z_0 - Z_1} \quad (75)$$

The average transport coefficients are estimated from the autocovariance method using Eqs. 73 and 60 as

$$\hat{N}_{Nu} = \sum_{j=0}^n N_{Nu}(j\Delta t) \hat{\phi}_c(j\Delta t) \Delta t, \quad (76)$$

where n is chosen as before to be the lowest value for which Z_n goes to zero.

From the spectral method, we estimate the average transport coefficients using Eq. 69 in Eq. 61, the upper limit for ω in the integral in Eq. 61 being taken as ω_N :

$$\hat{N}_{Nu} = \frac{1}{\pi} \int_0^{\omega_N} \int_0^{\infty} N_{Nu}(a) \omega \hat{S}(\omega) \sin \omega a \, da \, d\omega. \quad (77)$$

For the simple transport model, $N_{Nu}(a) = 1/\sqrt{\pi a}$, this becomes, using Eq. 63,

$$\bar{N}_{Nu} = \frac{1}{\pi\sqrt{2}} \int_0^{\omega_N} \sqrt{\omega} \hat{S}(\omega) d\omega. \quad (77a)$$

h. Computations. Problems arise when the integral in Eq. 74 is evaluated numerically.

(1) The integral is of a finite Fourier type. For larger values of a , the integrand exhibits a large number of positive and negative cycles, and numerical integration by conventional quadrature formulas results in accumulation of errors and loss of significance.

One technique for numerical integration of such an integral is the "Filon" approximation.³⁸ In this scheme, the function $\omega \hat{S}(\omega)$ is approximated by parabolic arcs over small intervals, $\Delta\omega$. The integral is then computed as a finite summation. This scheme was used throughout.

(2) When attempts were made to obtain $\phi_s(a)$ from Eq. 74, using actual values of $S(\omega)$ for a particular residence-time distribution (for example, Eqs. 56 and 54), the values of $\hat{\phi}_s(a)$ oscillated about the true $\phi(a)$ at low values of a . This could occur because the equation is not valid for $a = 0$ and may be inaccurate in the neighborhood of $a = 0$, and/or because the integral is truncated at the upper limit ω_N . The truncation actually caused the oscillations, because when the upper limit on the integral was increased sufficiently, the oscillations disappeared. The tail of the spectral density is thus important when $\phi(a)$ is estimated at low values of a . It is easy to see why this is so. Equation 73 shows that, for small a , only a few cycles of the sine term are included in the integral, since ω_N is finite. This is what causes the oscillation, since theoretically an infinite number of cycles should be included. Hence, for one value of a , we obtain more negative cycles, while for a slightly different value of a , we obtain more positive cycles, and so on, producing an oscillation in $\hat{\phi}_s(a)$. The effect diminishes as a increases since now a large number of cycles of the sine term are included.

The method adopted to overcome this problem is to extrapolate $\hat{S}(\omega)$ beyond ω_N by a negative exponential. By definition, the area under the spectral density is π .³⁵ (This may be shown from Eq. 49.) The exponential tail is chosen to match the value of $S(\omega)$ at ω_N and to have an area such that the total area under $S(\omega)$ and the tail is π . Solodovnikov³⁵ points out that for monotone-decreasing $R(a)$, $S(\omega)$ is also monotone-decreasing. In the present situation, $R(a)$ is at least monotone-decreasing (if not strictly decreasing), since the counts are less and less correlated

with one another the further removed from one another they are. Hence $S(\omega)$ is monotone-decreasing. Thus the choice of a negative exponential as the extrapolating function is consistent with the monotone-decreasing characteristic of $S(\omega)$.

In tests to check the method, theoretical values (for example, from Eq. 56) of $S(\omega)$ were used up to ω_N , and the exponential tail used beyond ω_N . The values of $\phi_s(a)$ thus obtained exhibited no oscillations and were in agreement with the theoretical values of $\phi(a)$ which, for the above example, are given by Eq. 55.

When an actual series is being analyzed, the estimator for the spectral density (Eq. 69) is such that the area from $\omega = 0$ to ω_N under $S(\omega)$ is very near π . Consequently, the surplus-area criterion cannot be used to determine the form of the exponential. Instead, the shape must be determined by considering what the area under $S(\omega)$ should be beyond ω_N for typical residence-time distributions.

It may be shown (see Appendix B) that for two residence-time densities of widely differing form (an exponential and a pulse), the area under $S(\omega)$ beyond the Nyquist frequency is $2\Delta t/\pi\bar{\tau}$ to second-order accuracy. This should hold true for most residence-time densities. This value is therefore chosen to find the shape of the exponential tail.

Therefore, when using Eq. 74, we use a Filon approximation to evaluate the integral and apply a correction to $\hat{\phi}_s(a)$ as

$$\begin{aligned} \hat{\phi}_s(a) = & \frac{1}{\pi} \int_0^{\omega_N} \omega \hat{S}(\omega) \sin \omega a \, d\omega \\ & + \frac{1}{\pi} \int_{\omega_N}^{\infty} \omega c_1 \exp(-c_2 \omega) \sin \omega a \, d\omega, \quad a > 0, \end{aligned} \quad (78)$$

where c_1 and c_2 are constants evaluated by matching the exponential to the value of $\hat{S}$ at ω_N and having an area $2\Delta t/\pi\bar{\tau}$ from ω_N to infinity. Thus,

$$c_1 = \hat{S}(\omega_N) \exp[\hat{S}(\omega_N)], \quad (79)$$

and

$$c_2 = S(\omega_N) \exp[\hat{S}(\omega_N) \omega_N / (2\Delta t / \pi \bar{\tau})]. \quad (80)$$

The second integral in Eq. 78 is an analytic form³⁹ and is evaluated as such (see Appendix B).

B. Results

The methods developed in Section A above were tested in two ways.

First, the equations were used on time series of particle counts of known residence-time density generated by a Monte Carlo technique. This verified the validity of the theoretical equations and provided a convenient means of checking the accuracy of the estimators described above. (The estimates could be compared to the known theoretical values.)

Second, the methods were applied to time series of particle counts observed at the wall of a fluidized bed. The sample age density for these particles was obtained by direct particle tracking. Thus, the sample age density should be compared to that predicted from the autocovariances and spectral density of the time series of particle counts at the surface.

1. Computer Simulation

The equations developed in Section A above were tested for a simulated time series generated on a computer by a Monte Carlo technique.²⁶ The time series of particle counts was generated for a known residence-time distribution of the gamma form. For such a distribution, the theoretical age density, autocovariance function, and spectral density were known through Eqs. 42, 45, and 51. (See Koppel²⁶ and Appendix B.) The simulated series was used to estimate these quantities by Eqs. 73-77, and these estimates were compared to the theoretical values. The individual simulated series had an inherent sampling variance (since the Monte Carlo technique samples the true residence-time density in order to generate the series of particle counts), which biased any results based on it. Several statistically independent series were therefore generated with the same theoretical residence-time distribution, and the sample means and variances of the estimated age densities, etc., were computed before being compared with the theory.

Simulated series were generated for two forms of the gamma residence-time distribution with $\alpha = 0$ and $\alpha = 1$.²⁶ (See Appendix B.) Ten independent series were generated for each α value. Other parameters were chosen as $\bar{\tau} = 1.0$, $\Delta t = 0.1$, $\bar{N} = 20$, and $K + 1 = 1500$.

a. Age Distribution. Figures 27 and 28 show the theoretical age distribution for the two α values compared to values obtained by the autocovariance technique (Eq. 73) and by the spectral-density method (Eq. 78). The means from the 10 cases, along with the "worst" and "best" cases, are plotted for each method. The worst and best cases were

arbitrarily chosen as the cases for which $\phi(a)$ had the greatest and least deviation, respectively, at $a = 0.1$.

Fig. 27

Theoretical and Estimated Age Density,
Simulated Series for $\alpha = 0$

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Fig. 28

Theoretical and Estimated Age Density,
Simulated Series for $\alpha = 1$

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Figure 27 (for $\alpha = 0$) shows that the values of $\phi(a)$ from the autocovariance method generally lie slightly below the theoretical values. The mean values for the 10 cases deviate by an average of about 10% from the theoretical values. As may be seen from the "best" case, the values tend to oscillate about the theoretical curve. The "worst" case falls consistently below the theoretical value except at high values of a .

Values from the spectral density method (Fig. 27) are generally higher than the theoretical values for small a . The tail correction does not completely remove the tendency of $\phi(a)$ to oscillate at low values of a . Although the mean values from the 10 cases are not too badly in error (except at low a), the "worst" case does deviate to a significant extent. The "best" case gives values closer to the theoretical values than the "best" case for the autocovariance method, except at low values of a .

Figure 28 shows the same data for $\alpha = 1$. The scatter appears to be greater for this alpha value, especially for the "worst" cases for the two methods. The reason for this is not known. Note that the means for the autocovariance method correctly shows the decreasing slope of $\phi(a)$ near $a = 0$, and is significantly different from the corresponding means for $\alpha = 0$ (see Fig. 27). The method therefore can distinguish between the different shapes of the theoretical distributions. The same holds true for the spectral-density method, but the distinction is less marked.

From the results of these computer-simulated time series, we conclude that the autocovariance method gives better prediction of $\phi(a)$ near $a = 0$. For higher values of a , there is not much difference between the two methods. The autocovariance method does predict a monotone-decreasing $\phi(a)$ near $a = 0$, but this is not true for the spectral method. This is not too serious a flaw in the spectral technique. The estimate of $\phi(0) = 1/\tau$ is good, and it is not too difficult to draw a monotone-decreasing function through the data points from the method. However, the autocovariance method is definitely superior in this regard.

For predicting transport coefficients, the age density is not the prime quantity of interest, but rather its integral with the instantaneous coefficient (Eq. 60). Thus, though the predicted age densities vary somewhat from the true density (since an integration process is being done by Eq. 60), the error in the average transport coefficients should not be too high. This was found to be true as shown in Section b below.

b. Average Coefficients. Average coefficients were found for the model $N_{Nu}(a) = 1/\sqrt{\pi a}$ using Eqs. 76 and 77a. Table VI shows the mean and variance of the 10 cases for both α values. There appears to be little difference in the values using the autocovariance relation or the spectral relation. The summation in Eq. 76 was truncated at $n\Delta t = 5.0$; beyond this value, the autocovariances are essentially zero. Both methods give values in close agreement with the theoretical value. Deviations of the means from the theoretical values for both methods are roughly the same (about 4%). However, the spectral method gives less internal scatter (4%) compared to about 8% for the autocovariance method for both alpha values. This is of some significance. Since there is less internal scatter,

we may place a tighter confidence interval on these results. This in turn enables us to state which type of age distribution actually represents the true age distribution.

TABLE VI. Results from Data Simulated by Computer

	$\alpha = 0$		$\alpha = 1$	
	Theoretical	Sample ^a	Theoretical	Sample ^a
Mean residence time	1.0	1.0104 ^b	1.0	1.0049 ^b
Second moment $E(\tau^2)$	2.0	2.0287 ^b	1.5	1.5116 ^b
$\bar{N}_{Nu}$ from autocovariances	1.0	0.9679	1.061	1.0113
Ratio of standard deviation to $\bar{N}_{Nu}$ from autocovariances	-	0.078 ^c	-	0.088 ^c
$\bar{N}_{Nu}$ from spectral density	1.0	0.9610	1.061	0.996
Ratio of standard deviation to $\bar{N}_{Nu}$ from spectral density	-	0.038	-	0.036
$E(\tau^2)$ from spectral density	2.0	1.733	1.5	1.573
Ratio of standard deviation to $E(\tau^2)$ from spectral density	-	0.173 ^c	-	0.234 ^c

^aFrom 10 cases.

^bAverage of sample means from 10 cases.

^cStandard deviation of 10 cases.

c. Second Moments of the Residence-time Density.

Equation 65 was used to compute second moments for the various cases for the two alpha values. Table VI shows the sample means and standard deviations for the 10 cases for $\alpha = 0$ and 1. The mean of the 10 cases is fairly close to the theoretical value for each value of alpha. For $\alpha = 0$, the deviation is about 15%, and for $\alpha = 1$ about 5%. For both cases, the standard error about the mean is about 20%. We thus have a reasonable estimate of the variance of the residence-time distribution, which gives us a better idea of its shape. Note that this is obtained solely from information on total particle counts at the surface.

2. Age Densities at Wall Surface of a Fluidized Bed

The methods for estimating $\phi(a)$ from particle-count data were tested on three data runs of particle residence times at the wall of a fluidized bed. For these three runs, the film was analyzed by direct tracking of individual particles so that the sample cumulative residence-time distribution, $F(\tau)$, was known, from which the sample $\phi(a)$ could be computed by Eq. 42. Concurrently, the time series of particle counts was also obtained. Table VII summarizes some of the data and results for these runs. Figures 29-31 show the results.

TABLE VII. Data and Results for Residence Times at Wall Surface

	Run 1	Run 2	Run 3
Particle type	No. 1 glass	Cellulose acetate	Cellulose acetate
Superficial gas velocity, ft/min	103.2	242	222
Stirring speed, rpm	440	280	280
<u>Particle Tracking</u>			
Number tracked	181	102	15
Sample mean residence time	28.98 frames (1/24 sec each)	16.16 frames (1/6 sec each)	84 frames (1/6 sec each)
Sample variance of residence time, fr^2	1598	160	1601
Standard deviation/mean	1.38	0.78	0.48
<u>Counts Method</u>			
Mean residence time	24 frames (1/24 sec each)	13.58 frames (1/6 sec each)	54.5 frames (1/6 sec each)
Variance of residence-time distribution Eq. 65, fr^2	1019	78	244
Standard deviation/mean	1.33	0.65	0.29

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Fig. 29. Sample and Estimated Age Density for Run 1

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Fig. 30. Sample and Estimated Age Density for Run 2

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Fig. 31. Sample and Estimated Age Density for Run 3

better results than the autocovariance method. Beyond these values of a , the scatter is considerable for both methods.

Figure 30 shows the sample $\phi(a)$ and the values estimated by the two methods for the first run with cellulose acetate. The scatter is considerably greater than that on Fig. 29. One reason for this could be that significantly fewer particles (102 vs 181) were involved, leading to higher sampling variances. The autocovariance method gives fair results near $a = 0$, and values scatter extensively beyond $a = 5$. The estimates by the spectral method are much poorer and show some oscillation. The general shape and slope of the distribution are estimated by both methods.

Figure 31 shows the results for the second run with cellulose acetate. Only 15 particles were involved in this run, and this is probably the reason for the very extensive scatter between estimated values and the sample $\phi(a)$.

Table VII compares the moments of the residence-time density obtained by the various methods. The means and variances obtained from total counts are lower than the sample means and variances obtained by particle tracking. For Run 1 the predictions are quite good, probably because of the larger number of particles tracked, giving better sampling statistics. The values become successively worse from Run 2 to Run 3, probably because of fewer samples. For Run 3, the estimates are very poor, but only 15 particles were tracked.

Figure 29 shows the sample $\phi(a)$ and those predicted by the autocovariance and spectral methods for the run with No. 1 glass. For this run, 181 particles were involved. As a consequence, we obtain a better estimate of $\phi(a)$. Figure 29 indicates that the predicted age density lies fairly close to the sample age density for both methods. As usual, the spectral method gives results that are too high for small a . However, for larger values of a , this method gives estimates closer to the sample $\phi(a)$. In the range of a from 5 to 20 frames, the spectral method gives decidedly

Although the estimates of $\phi(a)$ vary to some degree from the sample values obtained by tracking, the average transport coefficients obtained from these estimates will probably not deviate too greatly from the average coefficient obtained using the sample $\phi(a)$. Koppel *et al.*²⁶ have shown this to be true when the mean-residence-time estimates from the total-counts method are close to the sample mean residence time from particle tracking. For the simple model $N_{Nu} = 1/\sqrt{\pi a}$, the calculations for $\bar{N}_{Nu}$ were performed using the sample $\phi(a)$ in Eq. 60 and the autocovariance and spectral methods (Eqs. 76 and 77a) for Run 1 (see Table VI). The results are:

	$\bar{N}_{Nu}$
From sample $\phi(a)$	0.132
Autocovariance method	0.130
Spectral method	0.18

Agreement for the autocovariance method is good. The spectral method shows a larger deviation than the autocovariance method.

3. Average Residence Times at Wall of Fluidized Bed

Average residence times were found experimentally at different incipient gas velocities for the three sizes of glass and the cellulose acetate particles. (See Tables I and III for properties.) These average residence times were estimated from the time series of total tracer-particle counts at the surface using methods presented by Koppel *et al.*²⁶ (See Eq. 48.)

The residence times of the particles at the wall are affected by particle properties such as the diameter, density, and shape, as well as by the superficial gas velocity in excess of the minimum fluidization velocity. The superficial gas velocity, v_{mf} , is a good characterization of the shape and density of the particles. Also, the residence times should be shorter for higher gas velocities. A "universal" relation between the average residence times and the gas velocity may therefore be logically postulated to be of the dimensionless form $\bar{\theta} v_{mf}/D_p$ vs v/v_{mf} , where $\bar{\theta}$ is the average dimensional residence time. The data for the glass and cellulose acetate particles are plotted in this form in Fig. 32. Note that these data are for a bubble-free surface. The data points lie close together and show a definite trend, substantiating the postulated relation. An empirical curve has been drawn through the data for estimating residence times for other particle systems. The two data points for No. 3 glass at high v/v_{mf} are probably higher than they should be. This is probably due to a buildup of an electrostatic charge on these particles. (The smaller particles are especially susceptible to such a buildup.) The presence of the charge causes the residence times to be high.

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Fig. 32. Relation between Dimensionless Residence Time and Gas-velocity Ratio

present data cannot resolve this question since the density was not varied. (The cellulose acetate particles are of different density from the glass, but are so much larger than particles typically used in fluidization that their residence times cannot be taken to be characteristic of residence times for these smaller particles.)

C. Conclusions

The age density of particles at a transporting surface can be reasonably estimated from observations of only the total tracer-particle count with time. This time series, when suitably analyzed, gives good estimates of the age density, the average transport coefficient, and the mean and variance of the residence-time density. Errors are within acceptable limits.

In the test on a fluidized-bed system, both the autocovariance method and the spectral-density method gave satisfactory estimates of the age density, average transport coefficient, and the mean and variance of the residence-time distribution. In the present study, the spectral method gave better results than the autocovariance method did for the average transport coefficients.

A "universal" dimensionless relation is postulated relating residence times at wall surfaces in a fluidized bed to the ratio of the fluid velocity and the fluid velocity at minimum fluidization. The limited experimental data support the postulated relation.

Note that the exact numerical values of this postulated form from Fig. 32 are valid only for the particular geometry of this research. Certainly, the speed of stirring of the bed, the height of the transporting surface above the distributor, etc., would affect the numerical values of $\bar{\theta}$. What has been postulated is a unique relation between $\bar{\theta}v_{mf}/D_p$ and v/v_{mf} for a particular geometry. Also, there is some question as to whether the effect of particle density is adequately included through v_{mf} , or whether there is a direct dependence on another dimensionless parameter containing particle density. The

APPENDIX A

Derivation of Equations in Section V

1. Equations 1-4

Equations 1-4 hold for the transient temperature in an infinite sheet of finite thickness, one face having a convective boundary condition, the other being maintained at the initial uniform temperature of the sheet, t_b . The solution for the temperature for long ages is given by Carslaw and Jaeger,³⁰ p. 125, Eq. 15. Since this solution converges slowly for short ages, an alternative solution is necessary to obtain numerical values for these short ages.

By expanding the Laplace transform of the temperature in a series of negative exponentials (see Carslaw and Jaeger,³⁰ pp. 310 and 311; and also Goldstein⁴¹) and then inverting this series, we obtain the solution that converges for short ages. There is no simple series for this inverse, and the inversion of succeeding terms is complicated. However, Goldstein⁴¹ states that the first term is enough. For the temperature, we obtain

$$T_p(x, a) = \operatorname{erfc} \frac{x}{2\sqrt{a}} + \operatorname{erfc} \frac{2-x}{2\sqrt{a}} - \exp(Mx + aM^2) \operatorname{erfc} \left(\frac{x}{2\sqrt{a}} + M\sqrt{a} \right) - \exp[M(2-x) + aM^2] \operatorname{erfc} \left(\frac{2-x}{2\sqrt{a}} + M\sqrt{a} \right) + \dots \quad (\text{A.1})$$

The solution for the temperature is not of primary interest. We are more interested in the instantaneous heat-transfer coefficient and the instantaneous Nusselt number.

2. Equation 9

The instantaneous heat flux follows as

$$q(a) = h_c [t_w - t_p(0, a)]. \quad (\text{A.2})$$

Using the solutions for T_p (the dimensionless temperature) given by Carslaw and Jaeger,³⁰ along with the definitions of h and N_{Nu} (Eqs. 7 and 8), we arrive at Eq. 9, an expression for the instantaneous Nusselt number convergent for large a . The solution for short ages is obtained using Eq. A.2 for T_p and yields

$$N_{Nu}(a) = -M \operatorname{erfc}(1/\sqrt{a}) + M \exp(aM^2) \operatorname{erfc}(M\sqrt{a}) + M \exp(2M + aM^2) \operatorname{erfc} \left(\frac{1}{\sqrt{a}} + M\sqrt{a} \right) + \dots \quad (\text{A.3})$$

3. Equations 13 and 15

These forms of the age density follow from gamma forms of the residence-time densities. Equations B.17 and B.18 in Appendix B show the general forms and the method of derivation.

4. Equation 17 and Toor's²⁹ Solutions

In the notation of this work, Eq. 13a of Toor and Marchello,²⁹ which is valid for short ages for exponential age densities, becomes

$$\bar{N}_{Nu} = (1/\sqrt{\bar{\tau}}) \left[1 + 2 \sum_{n=1}^{\infty} \exp(-2n/\sqrt{\bar{\tau}}) \right]. \quad (A.4)$$

Recognizing that the summation is a binomial expansion of the type $1/(1-x)$ if the exponent n is taken outside the exponential, we obtain, after some manipulation (changing the lower limit in the summation to zero, etc.),

$$\bar{N}_{Nu} = \sqrt{\frac{1}{\bar{\tau}}} \left[\frac{2}{1 - \exp(-2/\sqrt{\bar{\tau}})} - 1 \right]. \quad (A.5)$$

By simplifying the expression in the brackets and recalling the definition of the hyperbolic tangent, we obtain Eq. 17.

For long ages, Toor²⁹ presents his Eq. 13b, which in our notation is

$$\bar{N}_{Nu} = 1 + 2 \sum_{n=1}^{\infty} \frac{1}{1 + n^2 \pi^2 \bar{\tau}}, \quad (A.6)$$

or, in an equivalent form, by manipulation of Eq. A.6,

$$\bar{N}_{Nu} = 1 + \frac{2}{\pi^2 \bar{\tau}} \sum_{n=1}^{\infty} \frac{1}{n^2 + (1/\pi\sqrt{\bar{\tau}})^2}. \quad (A.7)$$

The summation in Eq. A.7 may be expressed in terms of the hyperbolic cotangent function. Gradshteyn and Ryzhik,³⁹ p. 36, Formula 1.421 (4) give

$$\coth \pi x = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}. \quad (A.8)$$

Replacing x in Eq. A.8 by $1/\pi\sqrt{\bar{\tau}}$, we obtain an expression for the summation in Eq. A.7 as

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + (1/\pi\sqrt{\tau})^2} = \frac{\pi^2\sqrt{\tau}}{2} \left[\frac{1}{\tanh(1/\sqrt{\tau})} - \sqrt{\tau} \right]. \quad (\text{A.9})$$

Substitution of this in Eq. A.7 and some algebraic manipulation yields Eq. 17.

Hence, Toor's²⁹ series for long and short average ages are given by Eq. 17 in closed form.

5. Equations 26-31

Equations 26-29 are the standard differential equation and boundary conditions for an infinite sheet of finite thickness, one face being maintained at a constant temperature (which is given by the initial uniform temperature of the sheet t_b), the other face being maintained at a time-varying temperature $t(a)$. (See Carslaw and Jaeger.³⁰)

Equation 30 is the initial temperature condition for the surface particle; its initial temperature is also uniformly t_b .

Equation 31 is a heat balance for one surface particle. Referring to Fig. 19 and considering one particle at the surface, we obtain, for the heat balance in terms of dimensional quantities,

$$\rho_s C_s \frac{\pi D_p^3}{6} \frac{\partial t}{\partial \theta} = h_s \pi D_p^2 [t_w - t(\theta)] + \frac{\pi D_p^2}{4} k_m \left(\frac{\partial t_p}{\partial x} \right)_{x=0}, \quad (\text{A.10})$$

where h_s = convective heat-transfer coefficient between the fluid and the surface particle, and θ = dimensional time.

The first term on the left side of Eq. A.10 represents the energy accumulation in the particle. The first term on the right represents the heat input to the particle by convection from a fluid at t_w completely surrounding the particle. The second term on the right represents the heat output from the particle to the "packet" situated right behind it. The area for this heat flow is the projected area for the particle, $\pi D_p^2/4$.

Assuming that the convective Nusselt number between the fluid and the particle takes on the value 2 (see assumptions for Model II, Section V.A.2), we have

$$\frac{h_s D_p}{k_g} = 2. \quad (\text{A.11})$$

Simplification of Eq. A.10, using the definitions of Eq. 32, yields Eq. 31.

6. Equation 34

The expression for the Nusselt number (Eq. 34) follows by considering the heat input, $Q(\theta)$, from the wall to one surface particle. We obtain

$$Q(\theta) = h_s \pi D_p^2 [t_w - t(\theta)], \quad (\text{A.12})$$

or in dimensionless terms, and using Eq. A.11,

$$Q(a) = 2k_g D_p \pi (t_w - t_b) [1 - T(a)]. \quad (\text{A.13})$$

Considering the area of the wall to be unity and a packing density of particles at the wall to be μ (see definition following Eq. 34), the heat flux from the wall to the bed is given by

$$q(a) = \frac{2k_g \pi \mu}{D_p} (t_w - t_b) [1 - T(a)]. \quad (\text{A.14})$$

Using the definitions of $h(a)$ and $N_{Nu}(a)$ (Eqs. 7 and 8) in Eq. A.14, we obtain Eq. 34.

APPENDIX B

Derivation of Equations in Section VI1. Equations 58 and 59

Here we derive Eq. 59 only. The derivation of Eq. 58 proceeds in an analogous manner. To derive Eq. 59, we begin with Eq. 53, which is a Volterra integral equation of the first kind in ϕ ,

$$S(\omega) = \frac{2}{\omega} \int_0^{\infty} \phi(a) \sin \omega a \, da. \quad (\text{B.1})$$

To solve this equation, we define an odd function $\bar{\phi}(a)$ such that

$$\left. \begin{aligned} \bar{\phi}(a) &= \phi(a), & a > 0, \\ &= -\phi(-a), & a < 0. \end{aligned} \right\} \quad (\text{B.2})$$

Denoting the Fourier transform by $\mathcal{F}\{\}$, we have, taking the Fourier transform of Eq. B.2,

$$\mathcal{F}\{\bar{\phi}(a)\} = \int_{-\infty}^{\infty} \bar{\phi}(a) e^{-j\omega a} \, da, \quad (\text{B.3})$$

where $j = \sqrt{-1}$. Using De Moivre's theorem, we have

$$\begin{aligned} \mathcal{F}\{\bar{\phi}(a)\} &= \int_{-\infty}^{\infty} \bar{\phi}(a) \cos \omega a \, da - j \int_{-\infty}^{\infty} \bar{\phi}(a) \sin \omega a \, da \\ &= -2j \int_0^{\infty} \bar{\phi}(a) \sin \omega a \, da, \end{aligned} \quad (\text{B.4})$$

since $\bar{\phi}(a)$ is an odd function. Using Eq. B.2, the definition of $\bar{\phi}(a)$, in Eq. B.4, we obtain

$$\int_0^{\infty} \phi(a) \sin \omega a \, da = \frac{j}{2} \mathcal{F}\{\bar{\phi}(a)\} = \frac{\omega}{2} S(\omega). \quad (\text{B.5})$$

The second equality in Eq. B.5 follows from Eq. B.1. Designating the inverse Fourier transform by $\mathcal{F}^{-1}\{\}$, we have, from Eq. B.5,

$$\bar{\phi}(a) = \frac{1}{j} \mathcal{F}^{-1}\{\omega S(\omega)\}. \quad (\text{B.6})$$

Recalling that $S(\omega)$ is an even function by definition and that ω is odd, so that $\omega S(\omega)$ is odd, we have, from Eq. B.6, taking the inverse Fourier transform of $\omega S(\omega)$,

$$\bar{\phi}(a) = \frac{1}{2\pi j} \int_{-\infty}^{\infty} j\omega S(\omega) \sin \omega a \, d\omega. \quad (\text{B.7})$$

Using Eq. B.1 and the fact that $\omega S(\omega)$ is odd, we obtain, for $a > 0$,

$$\bar{\phi}(a) = \phi(a) = \frac{1}{\pi} \int_0^{\infty} \omega S(\omega) \sin \omega a \, d\omega, \quad a > 0, \quad (\text{B.8})$$

which is Eq. 59. It is now clear why the solution is not valid for $a = 0$.

The derivation of Eq. 58 follows from Eq. 51 by defining an even function $\bar{f}(a)$ as

$$\left. \begin{aligned} \bar{f}(a) &= f(a), & a > 0, \\ &= +f(-a), & a < 0. \end{aligned} \right\} \quad (\text{B.9})$$

The rest of the derivation is similar to that for Eq. 59. Alternatively, we may use Eq. 42 in Eq. 59.

2. Area under Spectral Density beyond the Nyquist Frequency

In this section we find the area under the spectral density beyond the Nyquist frequency, $\omega_N = \pi/\Delta t$. We find the area for spectral densities from two widely different residence-time densities--the exponential density, and a pulse density.

For the exponential density (Eq. 54), the spectral density is given through Eq. 51 as

$$S(\omega) = \frac{2\bar{\tau}}{1 + (\omega\bar{\tau})^2}. \quad (\text{B.10})$$

We wish to determine the area under this curve beyond the Nyquist frequency, or what is equivalent, the area up to the Nyquist frequency, since the area up to infinity is known to be π . Thus,

$$2 \int_0^{\pi\bar{\tau}/\Delta t} \frac{dx}{1+x^2} = 2 \tan^{-1} \frac{\pi\bar{\tau}}{\Delta t}$$

$$\approx \pi - \frac{2\Delta t}{\pi\bar{\tau}} \quad \text{for large } \pi\bar{\tau}/\Delta t. \quad (\text{B.11})$$

Equation B.11 is accurate to second order. Hence, the area beyond the Nyquist frequency is $2\Delta t/\pi\bar{\tau}$, since the total area is π .

For the pulse density,

$$f(\tau) = \delta(\tau - \bar{\tau}), \quad (\text{B.12})$$

the spectral density is given through Eq. 51 as

$$S(\omega) = \frac{2\bar{\tau}}{(\omega\bar{\tau})^2} (1 - \cos \omega\bar{\tau}). \quad (\text{B.13})$$

The area up to the Nyquist frequency is given by³⁹

$$\int_0^{\pi\bar{\tau}/\Delta t} \frac{2\bar{\tau}}{(\omega\bar{\tau})^2} (1 - \cos \omega\bar{\tau}) d\omega = 2 \left[\frac{\cos(\pi\bar{\tau}/\Delta t) - 1}{\pi\bar{\tau}/\Delta t} + \text{Si}(\pi\bar{\tau}/\Delta t) \right], \quad (\text{B.14})$$

where Si is the sine integral. We find the limit of the expression in Eq. B.14 for large $\pi\bar{\tau}/\Delta t$ using the expressions for Si(x) for large x.⁴⁰ For example, assuming $\pi\bar{\tau}/\Delta t = 2n\pi$, n being large, we obtain

$$\text{Si}(2n\pi) \approx \frac{\pi}{2} - \frac{1}{2n\pi}. \quad (\text{B.15})$$

From Eq. B.14, we obtain, for large values of n,

$$\int_0^{\pi\bar{\tau}/\Delta t} S(\omega) d\omega = 2\text{Si}(2n\pi)$$

$$\approx \pi - \frac{2\Delta t}{\pi\bar{\tau}}, \quad (\text{B.16})$$

to second-order accuracy, which is identical to Eq. B.11 for the exponential case. The same result is obtained using values of $\pi\bar{\tau}/\Delta t$ as $(2n-1)\pi$, $(2n+\frac{1}{2})\pi$, and $(2n-\frac{1}{2})\pi$.

Thus for these two widely different shapes of the spectral density, the area beyond the Nyquist frequency is approximately $2\Delta t/\pi\bar{\tau}$. The result is probably true for spectral densities resulting from all usual forms of the residence-time density.

3. Analytic Form for Integral in Eq. 78

The second integral in Eq. 78 is an analytic form and is given by Gradshteyn and Ryzhik,⁴⁰ p. 198, Formula 2.667 (5).

4. Age Density, Autocorrelation Function, and Spectral Density from Gamma Residence-time Densities

The gamma residence-time density is given by²⁶

$$f(\tau) = \frac{s^{\alpha+1}}{\alpha!} \tau^\alpha \exp(-s\tau), \quad \tau \geq 0, \\ = 0, \quad \text{otherwise,} \quad (\text{B.17})$$

where $s = (\alpha + 1)/\bar{\tau}$, $\bar{\tau}$ being the mean residence time and α , an integer, being a shape factor. The various shapes possible have been shown by Koppel *et al.*²⁶

Using Eqs. 42 and B.17, we obtain the corresponding age density,

$$\phi(a) = \frac{s \exp(-sa)}{\alpha + 1} \sum_{i=0}^{\alpha} \frac{(sa)^i}{i!}. \quad (\text{B.18})$$

The autocorrelation function of counts for this residence-time distribution is then obtained through Eq. 52 as

$$R(a) = \frac{\exp(-as)}{\alpha + 1} \sum_{i=0}^{\alpha} \frac{(\alpha + 1 - i)(sa)^i}{i!}. \quad (\text{B.19})$$

The spectral density of counts for this residence-time distribution is obtained through Eq. 51 as

$$S(\omega) = \frac{2}{\bar{\tau}\omega^2} \left\{ 1 - \left[\frac{s}{\sqrt{\omega^2 + s^2}} \right]^{\alpha+1} \cos \left[(\alpha + 1) \arctan \frac{\omega}{s} \right] \right\}. \quad (\text{B.20})$$

Note that these expressions simplify considerably for $\alpha = 0$. These simplified equations are given in Eqs. 54-57.

APPENDIX C

Experimental Data

The experimental data for heat-transfer coefficients and mean residence times at various superficial gas velocities are reported below for the different particle systems. Except where stated, stirring speed for the fluidized condition was 440 rpm. For the copper, aluminum, and alumina particles, residence-time data were not obtained. Particle properties are given in Table I.

1. Cellulose Acetate. Stirring speed, 280 rpm.

Run No.	Gas Velocity,* ft/min	Heat-transfer Coefficient, Btu/(hr)(ft ²)(°F)	Mean Residence Time, sec
AC-1	82.8 p	8.8	-
AC-2	116.1 p	11.4	-
AC-2R		9.8	
AC-3	132.8 p	12.4	-
AC-4	199.8 mf	19.5	-
AC-4R		17.0	
AC-5	199.8 f	14.5	-
AC-5R		14.8	
AC-6	221.5 f	18.4	9.1
AC-7	232.0 f	22.2	4.6
AC-7R		21.9	
AC-8	239.6 f	24.0	-
AC-9	242.5 f	24.6	2.3
AC-10	250.4 f	26.4	1.3

2. No. 1 Glass

G1-1	25.2 p	4.6	-
G1-2	68.0 p	8.2	-
G1-3	83.5 mf	8.5	-
G1-4	81.6 f	17.5	10.8

*p--denotes packed-bed conditions, no stirring;

mf--denotes minimum fluidization conditions, no stirring;

f--denotes fluidized, stirred conditions.

Run No.	Gas Velocity, ft/min	Heat-transfer Coefficient, Btu/(hr)(ft ²)(°F)	Mean Residence Time, sec
G1-5	94.4 f	26.4	3.2
G1-6	103.2 f	33.1	1.2
G1-7	123.6 f	47.8	0.58
G1-8	145.1 f	51.8	-
3. <u>No. 2 Glass</u>			
G2-1	12.4 p	3.5	-
G2-2	38.8 p	5.9	-
G2-3	50.8 mf	6.4	-
G2-4	52.8 f	23.0	3.7
G2-5	60.4 f	44.3	1.7
G2-6	76.3 f	64.7	0.7
G2-6R		59.3	1.4
G2-7	94.4 f	71.8	0.4
4. <u>No. 3 Glass</u>			
G3-1	8.1 p	2.6	-
G3-2	14.6 p	3.6	-
G3-3	19.0 mf	3.7	-
G3-4	18.7 f	16.2	12.1
G3-5	26.5 f	39.0	1.5
G3-5R		34.5	
G3-6	36.0 f	60.1	1.3
G3-7	44.8 f	75.0	0.9
5. <u>No. 1 Copper</u>			
C1-1	145.0 mf	17.3	
C1-2	145.0 f	17.9	
C1-3	165.7 f	19.6	
C1-4	182.8 f	36.1	
C1-5	211.6 f	53.7	

6. No. 2 Copper

Run No.	Gas Velocity, ft/min	Heat-transfer Coefficient, Btu/(hr)(ft ²)(°F)
C2-1	57.2 mf	9.4
C2-2	57.2 f	13.5
C2-3	73.1 f	27.5
C2-4	82.7 f	39.0
C2-5	93.0 f	55.1

7. Aluminum

Al-1	19.7 mf	7.2
Al-2	19.7 f	29.2
Al-3	25.8 f	43.5
Al-4	32.7 f	58.0
Al-5	37.1 f	68.5

8. No. 1 Alumina

Al-1	109.0 mf	10.8
Al-2	109.0 f	23.1
Al-3	114.2 f	28.9
Al-4	124.0 f	31.6
Al-5	134.2 f	34.6

9. No. 2 Alumina

A2-1	31.5 mf	5.4
A2-2	31.5 f	20.0
A2-3	40.8 f	31.9
A2-4	49.9 f	40.9
A2-5	56.5 f	48.2

10. Time Series

The time series of tracer particle counts used to obtain the above average residence times are given below. Each series is keyed to the appropriate

run by the run number. The time separation between each frame is given before the series. Successive frame-by-frame counts are given from left to right.

a. Run No. AC-6; Frame Separation, $1/6$ sec (Tracked Run; Fig. 31)

NUMBER OF COUNTS FOR 574 INTERVALS

1	1	1	2	3	3	2	3	3	3
4	4	4	4	4	4	4	4	4	4
5	5	5	4	4	4	5	5	5	5
5	5	5	5	5	5	5	5	5	5
5	5	5	4	4	4	4	4	4	4
4	4	4	5	5	5	5	4	4	4
4	4	4	4	4	4	4	5	5	5
5	5	5	4	5	5	5	5	5	5
5	5	5	4	4	4	3	3	4	4
4	4	4	4	4	4	4	4	4	3
3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	2
2	2	2	2	2	2	2	2	2	2
2	2	2	2	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1
1	1	1	2	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	0
2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2
2	2	3	3	3	3	3	3	3	2
3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5	5
4	4	4	4	4	4	4	4	4	4
5	5	5	4	5	5	5	5	5	5
5	5	5	4	5	5	5	5	5	5
4	4	4	5	5	5	5	4	4	4
4	4	4	4	4	4	4	4	4	4
3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3
2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2
0	0	1	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1
1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1
2	3	5	3	2	2	2	2	2	2

c. Run No. AC-9; Frame Separation, 1/6 sec (Tracked Run; Fig. 30)

NUMBER OF COUNTS FOR 543 INTERVALS

2	2	2	2	2	2	2	2	2	2
2	3	4	3	3	2	2	2	4	5
5	5	5	5	5	5	7	7	7	7
6	5	5	5	6	6	6	5	4	4
4	5	5	5	5	4	5	5	4	4
4	4	4	5	5	4	4	4	3	3
3	3	3	2	2	2	1	1	1	1
1	1	1	1	1	1	1	1	1	1
1	2	3	3	4	4	4	4	4	3
5	5	5	6	6	6	5	5	5	5
5	6	6	6	6	6	6	4	3	3
3	3	3	3	3	3	3	4	4	4
4	2	3	3	3	3	2	2	2	0
0	0	0	0	0	0	1	1	1	1
1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	0
0	0	0	0	0	0	0	0	0	0
0	0	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	2	2
1	1	1	2	2	2	2	3	3	3
3	3	3	2	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	2
1	1	1	1	1	1	3	3	3	4
3	3	2	3	3	3	3	3	3	3
3	3	3	2	2	2	2	2	2	2
2	3	3	3	3	3	4	4	4	2
2	2	2	2	2	2	2	2	3	3
3	3	3	3	3	3	2	2	2	2
2	2	1	1	1	1	1	1	1	2
2	2	2	2	2	5	4	4	4	6
6	6	7	7	7	7	8	7	8	6
5	5	5	5	6	5	6	6	5	6
5	5	5	4	4	4	4	5	4	3
3	3	2	2	3	6	5	4	4	4
5	5	5	4	3	3	4	5	5	4
5	5	5	5	5	5	5	5	4	4
4	4	6	6	5	5	8	8	8	7
7	8	6	6	6	6	5	5	5	5
5	5	5	7	8	7	6	6	7	8
6	7	6	6	5	5	4	4	4	2
1	1	1	1	1	1	1	1	1	3
2	2	3	4	5	4	7	6	6	6
6	5	5	5	4	4	3	3	3	3
4	5	5	5	5	6	6	6	6	6
6	6	6	5	4	4	4	4	4	3
3	5	5	5	2	4	5	5	5	6
5	4	3	3	3	3	3	2	2	2
2	2	1	1	1	1	1	0	0	0
0	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	1
1	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
2	2	2	2	1	1	2	2	2	2

d. Run No. AC-10; Frame Separation, 1/6 sec

NUMBER OF COUNTS FOR 545 INTERVALS

1	1	2	2	2	2	2	2	1	2
2	3	3	3	2	2	2	2	3	3
3	3	4	4	5	4	5	3	3	4
4	4	5	5	5	5	4	4	5	5
5	4	6	5	5	6	6	5	3	4
4	4	4	4	4	3	3	3	2	1
1	1	2	3	3	2	2	1	1	1
2	3	2	2	2	3	3	3	3	3
2	2	2	4	6	5	4	4	4	4
4	3	3	2	2	2	2	2	2	2
3	3	3	4	2	4	1	1	1	2
1	3	4	3	4	3	4	4	5	3
3	3	5	5	3	4	5	5	5	5
4	6	6	5	5	5	6	6	7	8
7	6	6	6	6	6	6	1	3	3
3	3	3	3	2	2	2	2	2	2
2	2	2	2	2	2	3	3	3	3
2	3	4	4	5	5	3	3	3	4
4	5	4	3	3	3	3	6	4	4
4	4	4	4	3	2	2	2	4	4
5	5	5	5	5	4	4	4	4	3
3	3	3	3	2	3	3	4	3	3
3	4	2	3	3	4	4	4	4	3
2	2	3	4	4	4	4	4	4	4
3	5	5	6	3	3	4	4	4	4
4	3	3	3	3	3	3	1	2	2
2	2	2	2	1	1	1	2	2	2
0	0	1	2	2	2	0	0	0	0
0	1	0	1	2	0	0	0	0	0
0	3	3	2	2	3	3	2	6	3
4	4	3	3	3	4	6	5	6	3
6	6	6	6	4	4	4	3	5	4
4	4	3	3	4	4	3	3	4	2
1	2	2	2	2	2	2	2	2	1
1	2	2	2	2	2	1	2	2	2
2	2	2	2	3	3	4	3	3	3
3	2	3	4	4	4	4	4	2	2
2	1	1	1	1	1	1	1	1	1
1	1	1	3	3	2	1	1	1	2
5	5	5	5	5	5	6	5	6	6
6	6	5	4	4	5	5	5	4	4
4	4	4	4	4	4	4	4	5	3
3	3	4	5	4	4	4	4	4	4
2	2	3	3	2	2	3	1	1	1
1	1	2	3	3	2	2	2	2	3
2	2	2	2	3	2	2	4	4	3
2	3	3	3	3	2	2	3	4	4
4	4	4	4	4	2	3	3	2	1
2	1	1	1	1	1	1	1	2	2
3	3	3	5	5	3	3	3	3	3
3	7	7	6	5	5	5	5	7	7
6	7	7	6	6	6	7	5	6	6
7	7	7	7	7	6	5	5	5	7
5	5	5	5	4	5	3	3	3	3
4	3	4	5	0					

e. Run No. G1-4; Frame Separation, 1/6 sec

NUMBER OF COUNTS FOR 583 INTERVALS

3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	3	3	3	3
3	3	3	3	3	3	3	2	2	2
2	2	2	2	2	2	2	2	2	2
2	2	2	3	3	3	2	2	1	1
1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1
1	1	1	1	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	2	2	2	2	2
2	2	2	3	3	3	3	3	3	3
3	3	3	3	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2
1	1	1	1	1	1	1	1	2	2
2	2	2	2	2	2	2	2	1	1
1	1	1	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	3
3	3	3	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	5	5	5
5	5	5	5	5	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4
4	4	3	3	3	3	3	3	3	3
3	3	3	3	2	2	2	2	2	2
2	2	3	3	3	3	3	3	3	2
3	3	3	3	3	2	2	2	2	2
1	1	1	1	1	2	2	1	1	1
1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	2
2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	3	4
4	5	5	5	5	5	5	5	5	5
4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	3	3	3
3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3
3	4	4	4	4	3	3	3	3	3
3	3	3	3	3	3	3	2	2	1
1	1	1	1	1	1	2	2	2	2
2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1
1	1	1	1	0	0	0	0	0	0

f. Run No. G1-5; Frame Separation, 1/6 sec

NUMBER OF COUNTS FOR 572 INTERVALS

2	2	2	2	2	2	3	2	1	2
2	2	2	3	3	3	3	3	3	3
2	2	2	3	2	2	1	1	1	1
1	1	1	2	2	1	1	2	2	2
2	2	2	3	3	3	3	3	3	4
3	3	3	3	3	3	4	4	5	5
5	5	5	5	5	5	5	4	5	5
5	5	5	5	6	5	5	5	5	5
5	5	5	5	4	4	4	4	3	3
3	3	3	3	3	5	5	5	5	5
6	6	6	6	6	6	6	5	5	5
6	5	5	4	4	4	5	3	3	2
2	2	2	2	4	4	4	5	5	5
4	4	4	4	4	3	4	3	4	3
5	4	4	5	5	5	5	5	3	3
4	3	3	2	2	2	2	2	2	2
2	2	2	2	3	2	3	2	2	2
2	3	3	3	4	4	4	4	4	4
4	5	5	5	5	5	6	6	6	6
7	6	5	5	5	5	5	4	4	3
3	3	3	3	3	3	3	3	3	3
3	3	3	2	3	2	2	3	3	3
4	4	4	4	6	4	4	4	4	3
3	2	2	2	2	2	2	1	1	1
2	1	1	2	2	2	2	2	3	3
3	4	6	4	4	3	3	3	3	2
2	2	2	2	2	0	1	0	2	1
4	4	5	5	4	3	3	4	3	5
5	5	5	5	5	5	5	5	6	6
6	6	6	6	6	6	6	6	7	7
7	7	8	6	7	7	8	8	7	7
7	7	7	6	6	6	6	8	7	6
6	5	6	4	5	6	6	6	6	6
4	4	6	5	6	6	6	5	5	5
5	5	5	5	4	4	5	5	6	6
6	6	5	4	4	4	4	4	4	4
5	5	5	5	6	6	7	6	7	7
7	7	7	7	7	7	7	6	5	5
5	5	5	4	5	7	6	8	8	8
8	8	7	7	7	7	7	7	7	7
7	7	8	8	10	10	11	11	11	11
11	11	10	8	8	9	9	7	8	8
8	8	8	7	7	7	7	8	7	8
6	6	6	6	5	5	5	5	5	5
5	5	5	5	5	5	5	5	5	4
4	4	4	4	4	4	4	4	3	3
3	4	3	3	4	4	4	3	3	3
3	3	4	4	4	4	4	4	4	4
4	4	3	3	3	2	2	2	2	2
2	2	4	5	5	5	5	5	5	5
4	4	3	4	3	4	4	4	3	2
2	3	3	3	3	3	3	3	3	3
3	2	2	2	2	2	1	1	2	2
2	2	2	3	3	3	3	3	3	3
3	2	2	2	2	2	2	3	2	3
3	3	3	2	1	1	1	1	1	1
1	1	2	2	2	2	2	2	2	2
2	2								

g. Run No. G1-6; Frame Separation, 1/24 sec (Tracked Run; Fig. 29)

NUMBER OF COUNTS FOR 1000 INTERVALS

10	10	10	10	11	11	11	11	11	10
10	9	10	9	9	9	9	9	9	9
9	9	9	9	8	8	8	8	8	7
6	5	5	5	5	4	4	4	4	5
4	5	5	5	4	3	3	4	5	7
7	7	6	6	6	6	7	7	7	7
7	7	7	7	6	6	6	6	6	6
6	6	5	6	6	6	6	6	5	6
7	6	6	6	6	7	7	6	6	6
7	7	7	8	8	8	7	7	8	8
9	8	8	8	6	6	9	8	8	8
8	8	8	8	9	9	9	7	7	7
11	10	10	10	9	9	9	9	9	9
9	9	9	9	9	10	10	10	9	9
9	9	9	10	9	8	10	10	10	10
7	6	5	5	6	6	6	6	6	7
7	6	6	6	7	7	8	8	7	7
7	8	8	8	8	7	5	5	4	4
4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	3	3	3	3	3
4	4	4	4	3	3	3	4	4	4
3	3	3	3	3	3	3	3	4	4
4	4	4	4	5	5	6	6	6	6
3	3	6	5	6	7	7	5	5	5
5	4	4	5	5	5	5	6	6	6
6	6	6	6	6	6	6	7	8	7
7	8	8	5	5	5	7	6	6	6
6	6	6	6	5	5	6	5	5	5
5	5	5	5	5	5	5	5	5	5
5	5	5	5	5	5	5	5	5	5
5	6	5	5	5	5	5	5	5	7
7	7	7	7	6	6	6	6	6	6
6	6	6	6	6	6	6	6	7	7
7	7	7	7	6	6	7	7	7	7
7	7	7	6	6	6	6	6	6	5
5	4	5	5	5	4	4	3	3	6
6	6	6	6	6	7	6	7	7	7
7	7	7	7	7	6	5	6	5	5
5	5	5	5	5	5	5	5	5	5
5	5	5	6	6	6	6	6	6	6
6	6	6	4	5	5	5	5	6	6
6	6	6	6	5	6	6	6	6	6
7	7	7	6	6	6	6	6	6	6
6	5	5	5	6	6	6	7	6	5
5	5	5	7	7	6	6	6	7	7
5	7	7	7	6	5	5	5	5	5
8	8	8	8	7	7	7	8	9	9
9	9	9	9	9	9	9	9	9	5
6	7	8	8	8	8	8	8	6	6
6	6	6	6	7	6	5	9	8	8
7	8	8	8	7	7	8	8	8	8
10	10	9	8	8	8	7	5	6	6
6	6	7	8	8	7	7	7	7	7
7	8	8	8	8	7	7	7	7	7
6	6	6	6	6	6	6	5	5	6
6	6	5	5	6	7	7	8	8	8
8	7	8	7	7	5	6	7	7	8
8	8	8	8	8	8	8	8	7	6
5	5	5	5	5	6	6	6	6	6
6	6	6	6	6	6	6	6	6	6
6	6	7	7	7	7	7	7	7	7
7	7	6	7	7	7	7	6	6	6
6	7	8	8	8	8	8	8	8	7
5	5	5	7	7	6	5	4	4	4
4	3	4	5	5	4	4	4	4	4
4	4	6	6	6	6	4	4	4	4
6	6	6	4	4	4	4	4	5	5
5	5	5	5	5	5	5	5	5	5
6	6	5	5	5	5	5	5	6	6
6	6	7	7	7	7	7	6	6	6

7	7	7	7	6	5	5	5	5	5
5	5	4	4	4	4	4	4	5	5
5	5	5	5	5	5	5	5	5	5
5	5	5	5	5	5	4	4	4	4
3	3	3	3	3	3	3	3	3	3
5	5	5	3	2	2	2	2	3	3
3	3	3	3	3	3	3	4	3	3
3	3	3	3	3	2	3	3	4	3
2	2	2	2	3	3	3	3	3	3
6	6	6	6	6	6	6	6	6	6
6	6	6	6	5	6	6	6	7	7
7	7	6	6	6	6	6	6	6	6
6	6	7	6	6	7	8	9	9	7
7	8	8	8	8	8	8	8	8	7
7	7	7	7	7	7	7	7	7	8
8	8	8	8	8	8	8	8	8	8
8	6	6	6	7	7	7	7	6	6
5	5	5	6	6	6	4	5	4	4
4	4	5	5	5	5	5	5	5	5
5	5	5	5	5	5	5	5	5	6
6	7	7	8	8	8	8	8	8	8
6	6	6	6	6	6	5	5	5	5
6	5	5	6	6	6	6	6	6	5
5	6	6	5	2	2	2	4	3	3
4	4	4	4	4	6	6	5	5	4
4	4	4	6	6	5	6	6	7	7
6	5	5	7	6	7	7	7	7	7
7	7	8	8	6	6	6	6	6	6
5	5	6	6	6	6	6	5	5	5

h. Run No. G1-7; Frame Separation, 1/6 sec

NUMBER OF COUNTS FOR 572 INTERVALS

3	3	3	6	6	2	4	4	6	4
7	5	7	8	6	4	8	7	5	6
8	11	9	7	5	5	6	6	7	6
8	9	9	5	5	5	7	6	5	5
3	5	4	4	2	2	3	4	5	6
5	3	5	7	7	7	4	7	7	6
6	6	4	6	6	6	7	7	7	6
3	4	5	6	7	5	4	5	8	7
7	7	7	8	7	11	8	11	7	8
6	6	8	6	7	6	5	3	6	5
7	5	7	7	7	9	6	4	5	3
4	4	4	5	3	3	4	3	2	3
3	5	4	4	7	6	5	5	3	6
4	5	6	6	6	3	4	4	6	5
6	4	7	1	1	3	1	1	1	0
0	1	1	3	3	3	7	5	6	7
6	9	8	6	7	6	3	6	4	5
4	2	2	2	1	2	3	3	2	0
4	5	2	3	3	3	3	1	4	2
3	2	1	3	1	4	3	2	5	5
5	5	6	5	5	5	4	6	4	5
5	5	7	6	7	9	6	7	5	5
9	8	9	4	7	6	8	6	6	9
8	9	10	11	6	8	14	6	9	11
9	6	6	4	5	3	4	2	1	2
3	3	4	2	3	3	3	3	6	1
1	1	6	5	2	5	3	4	6	9
12	9	11	6	9	5	6	5	8	5
6	5	6	9	10	11	8	11	8	10
9	10	10	11	8	11	10	6	7	7
5	6	6	9	8	3	3	6	4	7
4	3	2	7	5	5	5	4	6	4
5	4	7	8	7	5	6	7	7	5
5	5	6	6	8	6	6	7	5	8
7	7	7	7	4	5	5	6	4	3
3	5	5	7	7	7	7	8	7	7
9	8	8	8	8	8	7	5	6	4
5	9	8	5	8	7	8	9	11	9
9	7	11	8	5	5	5	4	5	6
5	5	3	7	6	5	7	8	9	9
9	9	11	12	11	10	10	11	13	9

9	11	11	10	9	10	11	10	11	11
15	15	12	11	11	10	9	11	12	12
11	14	14	16	14	12	15	13	13	8
10	12	10	13	7	11	13	14	11	13
13	12	10	11	12	12	9	11	8	11
9	4	6	5	6	8	6	7	7	7
6	2	4	2	2	2	2	2	2	3
8	6	6	8	7	11	12	9	8	7
6	4	3	3	3	2	6	4	6	6
4	5	6	5	6	7	5	5	5	4
3	5	4	5	5	6	5	5	6	3
6	3	6	7	2	3	2	3	1	0
0	1	2	2	3	4	6	7	5	6
6	7	5	5	6	6	5	6	9	10
11	8	9	7	6	9	8	5	7	10
11	11	10	11	12	11	12	11	13	15
16	14								

i. Run No. G2-4; Frame Separation, 1/6 sec

NUMBER OF COUNTS FOR 621 INTERVALS

5	5	5	5	6	5	5	4	5	5
5	5	4	5	5	3	2	2	2	2
2	2	2	2	3	3	4	4	4	3
3	3	3	3	3	3	2	2	3	3
4	4	4	4	2	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3
2	2	2	2	2	2	2	2	2	2
1	1	1	1	1	1	1	1	1	1
1	2	2	2	2	2	2	2	2	2
2	2	1	1	1	1	1	1	1	1
1	1	1	1	1	2	1	1	1	1
1	1	1	1	1	1	1	1	1	0
0	0	0	0	0	0	0	0	0	2
2	2	3	3	3	3	3	3	3	3
3	3	3	3	4	3	3	3	3	3
4	4	4	3	3	3	2	2	2	2
2	2	1	2	2	2	2	2	3	4
3	3	2	2	2	2	2	2	2	3
3	3	3	3	3	4	4	4	3	3
3	3	2	1	1	1	1	1	1	1
1	1	1	1	1	1	3	4	4	4
3	3	3	3	2	2	2	2	2	2
1	2	2	2	3	3	2	3	2	4
7	7	8	7	8	8	8	8	7	8
8	8	6	8	8	8	6	5	5	5
5	5	5	5	5	5	5	5	5	5
5	5	5	5	5	5	4	4	4	4
4	4	3	3	2	5	4	4	4	3
4	3	3	3	3	3	2	3	2	3
3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	4	4	2	2	2
2	2	2	2	2	2	2	2	2	2
2	4	3	3	2	2	2	2	2	1
0	0	0	0	0	1	1	1	1	0
1	1	1	1	1	1	1	1	1	1
0	1	1	1	1	2	2	2	1	1
2	4	4	4	3	3	3	2	3	3
4	4	3	3	3	3	3	3	3	3
4	4	3	3	3	4	4	4	4	4
4	4	4	4	5	5	5	5	5	4
4	4	4	5	5	5	5	6	5	5
5	5	5	5	4	3	3	3	3	2
1	1	1	1	2	2	2	2	2	2
2	2	2	2	3	3	3	4	4	4
4	4	4	3	4	3	3	4	4	4
4	4	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3
3	4	4	4	4	4	4	4	4	4
3	3	3	3	3	4	4	5	5	5
5	5	5	5	5	5	6	5	5	5
5	6	6	5	6	6	6	6	6	7
7	8	8	10	10	10	10	10	9	9

9	9	10	10	10	10	10	10	10	9
9	10	10	10	11	11	11	11	11	11
10	9	10	10	10	9	9	9	9	9
9	9	9	8	8	8	9	8	8	8
7	7	7	6	6	5	4	5	4	5
4	4	4	3	3	3	2	2	2	3
3	3	3	3	3	3	4	4	4	4
2	2	2	3	3	3	3	4	3	3
3	3	3	3	5	5	5	5	5	5
5									

j. Run No. G2-5; Frame Separation; 1/12 sec

NUMBER OF COUNTS FOR 1140 INTERVALS

7	7	7	6	5	4	4	4	4	4
5	5	6	5	5	5	5	5	4	4
4	4	4	3	4	4	4	4	5	5
4	5	4	4	5	5	3	4	5	4
4	4	4	4	4	5	4	4	5	5
5	5	6	5	5	5	5	5	5	4
4	4	4	3	4	4	4	4	3	3
3	4	4	4	4	4	4	4	3	2
3	3	3	3	3	4	4	4	4	4
4	4	4	4	4	6	5	6	5	6
6	5	5	5	5	4	4	4	3	3
3	3	3	3	3	3	3	3	3	4
4	4	3	3	3	3	3	3	3	3
3	3	4	4	4	4	4	4	3	3
3	3	2	2	2	2	2	2	2	2
2	2	2	2	3	2	2	2	2	3
3	4	3	2	3	2	2	2	2	1
1	1	2	1	1	1	1	1	1	2
1	1	1	1	1	1	1	1	1	1
3	2	2	2	2	2	2	2	1	1
1	1	1	1	1	1	1	1	2	1
1	2	2	1	1	3	2	2	2	3
4	4	4	4	3	3	3	4	4	4
4	4	3	3	3	3	3	3	3	3
3	2	1	1	2	2	2	2	2	2
3	4	4	4	4	4	4	5	5	5
5	4	4	4	4	4	3	3	4	3
3	3	4	3	3	4	4	4	3	6
7	5	5	6	7	7	6	6	6	5
5	6	6	6	6	6	5	6	5	5
5	6	7	6	8	6	5	5	5	5
7	7	7	7	6	6	7	7	5	5
6	6	6	5	6	7	6	6	6	6
7	6	6	6	4	7	7	7	7	7
7	7	7	6	6	6	6	7	6	7
7	6	6	4	4	6	6	5	5	5
5	5	5	5	5	5	5	5	6	6
5	5	5	5	4	5	5	6	6	6
7	5	6	6	6	6	6	6	6	5
5	4	6	6	6	5	4	6	5	5
5	5	4	4	4	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3
3	3	4	4	4	4	4	4	4	3
2	2	3	3	3	3	3	3	2	3
3	3	3	4	4	4	4	4	4	3
4	4	4	5	5	5	5	4	5	5
4	4	5	5	5	5	5	5	5	5
5	5	4	5	5	5	5	5	5	4
4	5	5	6	6	6	6	6	5	5
6	6	6	7	5	5	6	5	6	6
6	4	3	4	4	4	4	4	4	4
4	4	4	4	4	5	4	4	4	5
5	5	7	6	5	4	6	5	5	5
5	5	5	5	5	4	4	4	3	4
4	4	4	4	4	4	4	6	8	7
8	9	8	10	10	9	11	11	10	10
11	10	9	9	10	10	10	10	9	9
9	9	8	8	9	10	10	10	10	10
10	9	11	9	6	5	8	7	7	8

8	8	9	9	8	10	9	9	9	8
8	8	9	9	9	9	9	9	8	9
8	8	8	9	9	9	6	7	7	6
7	9	9	6	7	7	5	6	6	6
7	7	7	6	6	7	7	7	7	7
7	7	7	8	8	8	8	8	8	8
8	8	8	8	7	6	7	5	6	5
5	5	5	5	6	5	5	5	5	5
5	5	6	6	6	6	6	6	6	6
6	6	7	6	8	7	7	8	7	6
6	6	7	5	5	5	6	6	6	5
4	4	3	4	5	5	5	7	7	5
4	9	8	4	5	7	7	7	7	7
7	7	6	6	6	7	7	7	8	8
8	8	8	7	8	8	7	6	6	6
6	8	8	8	8	8	8	8	8	8
8	8	9	8	8	8	9	8	8	8
8	9	9	9	8	8	8	9	9	10
9	11	12	11	11	10	9	9	9	8
8	9	8	8	8	8	8	8	9	7
8	8	8	8	9	9	9	9	8	8
8	8	8	8	8	8	6	6	6	6
6	6	6	6	6	6	6	6	6	6
6	5	5	5	5	5	5	6	4	4
4	4	4	4	4	4	4	4	4	4
4	4	3	3	3	3	3	3	3	3
3	3	3	3	3	2	2	2	2	2
2	2	2	2	2	1	2	1	1	1
1	1	1	1	1	1	1	1	1	1
1	1	2	1	2	2	2	2	1	1
1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	3	3	3
3	2	3	3	3	3	3	3	3	3
3	3	4	4	4	3	3	4	4	4
4	4	3	5	5	4	4	4	3	3
4	4	3	4	3	3	3	3	3	3
4	3	3	4	4	4	3	4	4	3
2	3	3	3	4	3	3	4	4	4
4	3	3	3	3	3	4	3	3	6
3	3	2	4	3	2	5	6	6	6
6	5	6	4	4	4	4	4	4	3
3	3	3	4	3	5	5	4	4	5
5	6	6	7	6	7	7	6	8	8
8	9	8	7	7	7	7	9	11	9
9	10	10	10	11	11	11	11	11	11
10	11	9	9	9	10	8	9	10	10
9	9	9	9	9	9	10	10	9	10
10	10	10	10	10	10	10	10	11	11
11	11	9	10	9	9	10	10	11	11
10	10	11	11	11	11	10	9	9	10
10	10	10	9	10	10	10	9	9	9
9	10	9	9	10	10	9	9	9	9
8	8	8	8	8	8	7	7	7	7

k. Run No. G2-6; Frame Separation, 1/6 sec

NUMBER OF COUNTS FOR 559 INTERVALS

5	7	7	6	5	5	6	6	6	6
5	5	5	3	7	2	2	4	1	1
2	3	3	2	2	2	2	2	2	2
0	1	0	1	3	4	5	7	4	4
5	3	7	6	6	5	3	5	1	4
5	4	5	6	6	4	4	3	4	4
4	4	4	6	5	5	6	5	5	6
7	6	6	6	9	7	10	2	6	7
3	7	9	9	10	11	9	9	10	10
7	5	7	5	7	7	5	5	4	7
4	4	6	6	6	8	9	8	8	10
10	8	10	8	6	2	6	6	5	6
6	5	4	4	4	4	4	4	3	4
5	4	6	7	5	8	7	7	7	7
7	4	3	3	4	3	1	3	3	4
5	4	3	4	4	4	4	3	4	0

2	2	5	5	4	2	2	4	4	5
5	4	5	3	3	4	7	10	9	8
6	6	6	7	6	7	6	6	6	6
6	6	4	4	3	3	4	6	3	5
5	6	4	4	3	4	7	8	7	5
5	5	6	1	4	4	3	4	4	4
3	2	4	3	3	3	3	1	3	1
4	1	1	1	1	1	2	1	3	3
3	3	3	4	2	4	2	4	3	3
6	4	4	4	4	4	3	5	9	8
6	4	3	5	4	4	3	4	3	4
1	1	3	1	3	3	3	3	3	4
2	3	4	2	3	3	3	3	3	2
3	3	3	3	3	4	3	1	4	4
4	5	6	2	6	4	4	5	5	4
7	6	8	4	6	5	5	4	5	8
7	10	7	8	8	6	6	5	5	6
6	4	7	7	5	5	5	6	6	6
5	5	6	4	3	1	2	1	2	2
1	1	1	0	2	2	2	2	8	6
1	7	6	4	2	5	4	5	5	5
5	8	9	9	10	9	7	11	11	11
11	13	12	11	14	12	10	13	12	12
14	9	8	11	11	10	8	9	9	6
6	5	10	9	5	8	7	7	7	8
13	7	11	9	9	5	6	6	5	6
9	8	5	4	6	4	6	3	3	2
3	3	2	4	4	4	2	5	5	3
3	4	5	5	5	5	4	4	4	7
8	12	11	12	10	10	9	9	12	12
11	10	7	9	12	10	11	7	10	11
13	14	12	14	12	8	10	7	9	6
7	8	6	5	6	7	6	7	3	4
4	4	2	5	5	6	4	4	4	5
5	4	5	8	10	8	7	6	6	6
5	4	7	5	5	3	3	2	2	2
2	3	4	4	2	2	2	1	3	4
6	6	6	7	8	9	9	6	4	8
8	8	9	9	9	9	8	8	8	7
7	6	5	8	8	6	7	6	10	

1. Run No. G2-6R; Frame Separation, 1/6 sec

NUMRER OF COUNTS FOR 573 INTERVALS

5	5	5	4	4	4	5	4	4	4
4	3	3	3	3	4	3	3	2	2
1	1	2	3	2	3	1	3	1	2
2	3	2	3	3	3	2	3	2	2
2	1	2	2	2	2	2	2	2	2
3	3	4	3	2	3	4	2	5	5
5	6	5	5	5	5	5	5	4	5
5	5	5	7	6	8	6	6	7	5
7	7	6	5	5	5	5	5	6	5
5	6	6	6	6	6	5	6	6	5
5	5	5	5	5	5	5	5	5	5
4	5	5	4	5	4	3	4	3	4
4	4	4	5	5	5	4	5	6	4
5	4	4	6	5	5	4	4	4	6
5	5	5	6	6	5	7	7	8	8
9	7	6	3	7	5	8	9	8	7
7	8	6	8	8	8	9	9	9	5
10	11	10	10	11	11	10	7	7	7
8	7	9	8	7	7	8	11	7	9
9	8	9	8	8	10	10	9	11	9
11	10	8	8	9	11	11	12	13	10
10	10	11	12	13	13	15	14	11	12
11	10	11	10	10	10	11	8	8	8
9	9	8	7	7	7	7	8	8	9
8	9	6	7	7	6	6	5	6	6
6	5	5	4	6	6	6	7	7	6
6	6	3	4	5	4	5	3	5	7
6	6	2	3	5	5	5	5	3	3
3	3	2	2	2	2	1	5	1	0

0	0	1	2	1	1	4	4	3	4
3	4	4	4	4	4	4	3	3	3
2	3	5	4	5	5	4	5	5	6
6	5	5	5	5	4	4	5	3	7
5	5	5	3	4	3	7	6	5	6
6	6	5	6	7	5	5	5	5	6
6	6	6	5	5	5	5	5	5	4
5	3	5	5	5	6	5	4	4	3
5	2	6	3	4	4	6	7	7	7
8	8	6	6	5	7	6	6	6	6
6	7	5	7	6	7	5	7	5	5
4	4	5	5	6	4	7	10	7	7
7	8	5	5	6	5	5	6	6	4
3	3	3	5	4	4	7	5	5	5
6	7	7	7	7	7	7	7	8	9
7	6	3	5	4	6	4	4	6	5
6	7	5	4	5	6	7	5	6	6
7	6	5	6	4	4	4	3	5	4
4	5	4	4	5	4	5	5	4	7
5	5	5	6	6	6	6	6	5	5
5	5	4	4	3	4	4	3	6	6
7	7	7	6	6	6	8	7	6	6
7	6	5	5	5	5	5	4	4	4
5	5	5	5	5	5	5	4	5	5
2	1	1	1	1	1	1	1	1	1
1	1	1	1	1	2	1	2	1	1
2	2	1	1	1	1	1	1	1	1
1	1	1	1	2	3	5	5	8	4
6	5	6							

m. Run No. G2-7; Frame Separation, 1/24 sec

NUMBER OF COUNTS FOR 496 INTERVALS

11	11	13	10	10	10	9	10	12	12
12	11	11	13	13	14	14	14	17	17
18	18	17	16	17	16	14	14	15	18
18	17	16	16	16	16	16	15	12	9
11	15	14	15	15	13	11	11	9	10
11	13	12	13	13	13	13	12	11	11
12	10	10	11	11	11	10	1	12	13
14	14	15	14	15	13	14	14	14	13
14	14	13	12	12	15	14	14	15	12
11	14	14	13	12	11	12	12	12	12
11	14	16	15	16	15	13	13	15	13
15	12	12	12	10	8	10	1	10	11
13	12	11	10	11	12	12	11	11	12
14	15	13	13	15	16	14	13	14	14
15	14	17	18	16	17	17	17	15	14
13	12	13	13	13	1	11	13	11	8
11	10	10	8	9	9	8	8	6	9
8	8	9	9	10	9	9	1	7	8
9	9	11	12	14	12	15	13	11	13
12	14	13	14	12	13	12	11	13	13
11	12	12	11	12	12	11	13	13	12
13	13	12	13	14	12	11	13	16	16
16	16	16	17	16	15	12	14	14	15
15	14	16	16	15	16	15	13	12	17
18	17	15	15	15	14	13	16	16	15
18	17	15	16	16	15	17	14	15	15
13	15	14	14	15	13	11	14	14	13
13	14	15	16	16	16	15	13	14	13
13	14	11	8	7	7	7	6	3	3
5	6	8	8	6	6	6	5	5	6
7	10	11	9	10	7	8	9	11	8
8	8	7	10	9	7	8	8	9	7
8	9	8	3	3	3	5	8	8	10
9	11	11	11	10	8	8	9	9	7
7	4	5	6	5	7	8	9	12	14
11	10	11	12	13	12	12	15	11	10
11	12	12	12	12	14	14	14	13	12
13	12	12	15	14	14	16	15	13	16
14	12	12	12	12	13	12	12	12	11
13	11	9	8	9	10	9	8	8	7
8	9	11	8	10	12	8	1	10	12
13	14	11	13	11	10	11	10	9	10

1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1
1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2
1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1
1	1	2	3	3	3	3	3	3	3
3	2	2	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1

o. Run No. G3-5; Frame Separation, 1/12 sec

NUMBER OF COUNTS FOR 936 INTERVALS

4	4	4	3	4	3	4	4	4	8
4	5	10	9	9	13	10	10	12	11
11	13	16	17	17	17	16	16	14	14
14	15	15	15	17	18	16	15	15	14
17	14	13	13	14	14	15	14	16	16
16	16	16	15	18	16	18	19	19	18
19	19	19	18	19	18	18	17	13	17
17	17	17	18	20	20	18	21	24	23
26	26	26	26	27	24	23	24	23	23
23	23	20	17	17	18	19	18	20	20
20	20	20	20	23	23	22	22	23	21
22	23	23	23	23	23	23	20	19	18
18	17	17	16	15	16	16	16	14	15
12	14	14	14	15	14	15	13	13	13
12	13	12	13	9	11	9	9	10	10
11	10	10	14	13	13	12	11	12	14
13	13	14	15	16	16	16	15	13	15
15	14	11	10	10	12	12	12	12	12
9	9	9	11	9	8	8	7	7	8
9	9	9	13	13	12	11	11	10	8
7	7	9	9	9	10	10	10	10	10
12	11	13	14	14	14	16	17	13	15
14	14	13	14	13	14	14	13	12	10
9	9	9	9	10	10	9	10	10	14
12	12	13	13	12	10	10	9	9	9
8	8	9	8	10	9	9	10	10	9
9	10	10	10	11	12	13	13	13	13
14	15	14	17	17	17	15	15	16	18
16	16	17	19	17	17	15	15	15	12
13	12	12	10	10	12	10	9	11	11
10	10	10	11	17	11	11	11	10	11
11	11	10	9	7	4	5	5	7	7
7	7	8	7	6	6	6	6	8	7
7	9	9	9	10	10	10	8	9	8
9	10	10	11	11	11	9	11	11	11
11	11	10	9	10	10	8	11	11	9
9	8	10	9	12	13	13	13	14	15
12	12	13	13	13	12	12	12	11	10
9	9	8	8	7	8	8	8	7	7
8	8	7	7	8	8	7	7	7	7
8	10	11	13	13	13	13	13	14	13
13	12	12	12	12	12	12	11	11	12
12	12	11	11	9	9	8	9	9	9
9	7	7	7	8	7	7	7	7	7
6	6	6	7	8	8	5	8	9	9
8	9	8	9	10	9	10	8	7	6
7	6	6	6	8	8	8	7	8	7
8	8	8	8	9	9	10	9	9	10
11	11	12	12	12	12	12	12	12	11
11	11	12	11	11	12	12	11	12	12
13	13	15	15	16	17	19	16	15	16
16	16	17	15	16	16	15	16	17	17
16	15	17	17	16	15	13	11	12	12

11	13	13	17	17	16	15	15	16	12
10	11	11	10	10	8	8	8	8	8
8	9	7	8	7	8	8	8	8	8
9	9	12	12	12	10	9	9	9	10
9	9	9	9	8	7	9	9	11	11
12	13	14	12	13	13	13	10	11	11
11	12	12	12	12	12	12	10	12	12
11	10	10	13	13	13	13	12	14	12
11	12	12	12	13	13	11	12	10	10
10	11	11	11	8	8	8	8	8	8
6	8	8	7	7	7	7	6	6	6
6	7	7	8	8	9	9	8	11	10
10	10	9	9	9	9	9	9	10	11
11	12	10	15	15	15	14	13	12	12
12	12	13	12	14	14	13	12	11	10
9	8	9	7	7	7	5	5	5	6
7	6	6	8	10	9	7	7	7	8
8	7	10	10	9	10	9	9	9	9
9	8	7	7	7	8	8	8	8	9
8	8	7	7	6	7	7	7	6	13
9	6	11	11	10	11	11	9	9	9
9	9	9	9	9	9	9	9	10	10
9	9	9	8	8	7	8	8	9	6
11	12	12	10	10	10	10	12	12	11
12	13	14	16	17	17	16	16	16	18
18	17	17	18	18	19	19	18	14	14
14	13	13	13	12	12	13	13	13	13
12	14	14	14	13	13	13	13	13	13
12	9	9	9	8	9	9	9	10	10
10	8	8	8	8	7	7	6	6	8
8	8	8	7	10	10	10	9	9	9
9	10	11	11	11	13	11	11	12	12
13	11	13	14	13	13	17	17	18	19
17	18	15	15	15	14	15	14	14	16
16	16	15	15	14	17	17	18	15	16
17	17	18	17	17	16	15	14	14	13
13	13	17	16	15	15	15	14	14	13
12	10	10	11	11	11	12	12	13	13
13	12	12	12	10	10	9	9	9	9
9	11	11	11	11	11	12	10	9	8
8	8	8	8	7	7				

p. Run No. G3-6; Frame Separation, 1/12 sec

NUMBER OF COUNTS FOR 93c INTERVALS

7	5	7	8	8	7	7	7	6	5
6	7	7	6	8	5	5	7	7	8
7	8	8	9	7	7	7	8	8	8
8	11	11	10	9	8	7	8	10	9
9	7	6	6	7	7	7	7	6	7
7	6	6	8	8	8	7	7	4	4
4	4	4	4	4	1	1	1	1	1
1	1	0	0	0	0	0	2	3	3
2	4	4	4	4	3	2	3	4	4
3	4	3	3	3	3	3	4	5	3
5	5	5	4	5	5	4	5	6	5
8	7	6	8	9	9	7	7	7	6
6	6	6	6	6	3	4	4	4	4
4	3	4	4	4	4	4	4	4	5
6	6	6	6	4	5	6	5	5	5
5	5	5	5	5	4	3	4	6	5
5	4	3	2	2	2	4	4	4	6
6	6	6	6	5	5	6	6	5	6
6	4	4	6	6	6	3	3	4	3
5	5	6	8	8	9	9	9	6	7
5	4	5	6	5	5	5	4	4	4
5	5	6	5	5	3	3	3	3	3
3	4	4	3	4	4	4	6	5	5
5	6	5	4	4	6	6	5	5	5
6	6	6	5	5	5	6	5	6	8
8	8	8	5	7	7	8	10	9	9
11	11	11	9	11	11	10	8	9	10
10	10	10	10	10	10	10	10	8	9
9	9	9	7	7	7	5	5	4	4
5	5	6	6	7	5	5	5	4	4

4	4	4	4	3	4	3	2	8	8
9	9	9	9	10	10	9	9	9	9
11	11	9	9	10	8	10	11	11	12
11	12	13	13	13	13	13	12	12	11
10	11	11	11	11	11	11	10	11	12
11	11	12	12	12	13	11	11	13	13
12	12	10	10	10	9	11	11	11	8
9	9	9	9	9	7	6	7	4	4
4	4	4	5	4	4	3	4	5	6
3	5	5	5	5	5	5	5	4	4
5	5	5	4	3	6	6	5	5	6
7	6	7	7	9	9	8	9	9	9
10	10	10	11	11	12	11	12	12	11
11	8	8	9	10	7	8	7	7	6
6	9	8	8	8	8	8	7	7	9
9	8	8	7	8	9	8	7	7	8
11	11	11	12	12	13	12	14	14	12
11	13	11	12	12	11	12	11	12	12
12	12	12	12	12	11	11	12	13	14
12	12	13	13	14	14	12	11	11	12
10	10	8	9	8	8	8	8	9	9
9	9	8	9	6	6	6	6	8	9
9	6	6	5	5	5	5	5	5	5
5	5	6	6	6	7	6	6	5	7
7	7	7	7	7	7	8	7	7	7
7	7	6	6	7	7	7	7	7	7
5	5	5	4	3	3	4	5	4	3
4	5	5	5	6	6	6	6	6	4
6	6	5	6	7	6	4	3	6	7
6	5	8	9	8	10	10	8	9	9
8	7	8	8	12	13	14	14	14	14
14	14	11	11	12	12	11	11	11	10
10	10	10	9	8	10	9	9	9	8
8	8	9	9	8	8	7	7	7	7
9	9	9	9	9	8	8	8	9	7
7	6	6	6	7	6	5	5	8	8
8	8	8	8	8	8	7	7	7	6
7	7	7	7	7	6	8	9	9	9
9	9	9	12	12	11	10	10	12	13
13	13	12	13	13	13	12	11	11	13
13	15	15	14	14	13	14	11	11	12
11	9	12	11	11	12	12	9	10	7
8	9	9	9	9	8	8	8	8	10
8	8	7	8	7	7	7	9	9	8
8	9	9	6	5	6	6	6	6	7
6	8	9	9	7	7	7	6	6	6
7	7	7	7	8	8	8	8	9	7
7	7	8	8	8	7	7	7	7	7
10	9	8	8	8	8	8	6	6	6
8	8	8	9	9	9	8	8	8	8
8	7	8	8	7	8	10	11	11	11
10	10	10	10	9	9	10	10	9	9
9	10	10	10	10	10	9	10	11	11
10	7	7	7	8	7	9	9	10	10
8	9	8	8	9	8	9	9	9	9
9	4	4	4	4	4	4	7	7	7
6	6	7	7	8	7	7	6	6	7
7	7	7	7	7	6	6	5	5	5
6	7	7	6	6	7	7	6	6	6
5	5	6	6	7	9	10	8	8	9
9	9	9	9	9	10	9	10	8	8
8	7	7	8	6	8	8	8	7	9
9	11	10	10	11	11	10	11	11	11
7	7	7	7	7	7				

q. Run No. G3-7; Frame Separation, 1/24 sec

NUMBER OF COUNTS FOR 75c INTERVALS

11	11	11	11	10	8	8	9	11	10
9	7	6	6	6	9	9	7	7	9
13	12	11	10	12	13	13	14	14	14
13	9	12	11	11	10	10	9	8	8
7	8	8	9	9	8	7	9	8	7
7	7	7	7	8	7	9	10	9	10
11	11	10	10	12	13	9	10	10	9

8	10	8	9	9	9	10	10	9	7
7	7	7	8	9	8	8	7	7	7
9	10	12	11	10	9	9	10	10	8
10	10	10	9	10	11	9	10	11	11
11	11	12	14	16	16	15	15	16	14
16	16	17	18	17	16	18	19	19	18
18	19	20	18	17	18	19	19	19	21
20	19	19	19	18	18	19	18	16	16
16	16	17	17	17	17	14	14	14	17
16	18	19	20	19	17	16	16	16	16
16	16	16	16	13	12	12	12	12	11
10	9	11	11	11	9	10	10	11	11
11	11	13	12	11	12	14	13	13	13
13	13	13	14	14	15	15	15	15	16
16	16	15	13	14	14	15	16	17	17
18	18	18	17	19	20	22	22	23	21
18	18	19	22	21	20	20	20	19	18
16	16	17	17	16	14	15	15	15	15
14	13	12	12	14	14	14	15	14	15
13	13	13	14	13	12	14	15	15	15
16	15	15	14	14	12	12	12	11	11
12	12	11	13	15	14	16	16	16	16
17	15	15	14	14	15	17	17	16	17
17	17	18	18	17	15	20	20	20	17
18	18	18	19	20	19	18	17	18	20
20	20	20	22	22	21	20	20	21	20
21	21	20	19	18	18	19	18	19	17
15	16	18	16	18	18	19	21	22	22
21	22	23	23	22	23	23	22	22	22
22	23	23	20	22	22	22	20	16	17
18	21	20	19	19	20	21	18	18	19
19	18	18	18	18	18	15	11	10	10
10	9	9	10	12	15	14	14	16	15
13	11	10	10	11	11	10	10	10	10
11	11	11	10	10	11	13	13	11	9
6	5	5	7	9	9	9	8	8	9
10	11	10	9	9	9	10	10	8	9
8	11	12	12	12	12	12	12	11	9
8	7	6	6	8	9	9	9	10	10
9	9	9	9	10	11	11	11	10	10
11	12	13	14	14	13	13	13	13	13
13	12	12	12	11	12	13	14	14	13
12	12	12	13	12	12	12	12	11	9
9	8	9	9	9	8	9	9	8	7
7	7	7	7	6	6	6	6	5	6
7	8	8	8	9	9	9	9	8	8
9	9	9	8	8	7	6	7	7	7
8	7	8	6	6	6	7	7	6	6
6	5	7	7	7	7	6	5	4	7
7	8	8	7	8	8	9	10	11	12
12	12	11	10	11	11	11	11	11	12
12	12	12	12	11	11	10	10	10	11
11	9	9	9	10	10	10	10	8	8
9	10	10	11	10	10	11	12	12	10
9	9	9	9	9	9	10	10	10	10
12	11	11	12	12	9	10	10	10	9
8	8	8	8	8	9	9	10	11	10
13	13	13	12	12	13	14	13	13	11
10	10	11	10	10	11	11	10	10	10
10	11	11	11	10	10	12	13	13	14
14	13	14	13	12	13	12	12	11	10
10	12	13	10	12	12	12	13	13	14
14	13	14	16	17	17	14	13	12	9
9	9	10	10	10	11	12	14	13	12
12	12	13	13	14	14	11	10	10	11
13	13	13	12	12	11	11	13	13	12
15	15	12	13	13	13	13	14	13	12
12	12	12	12	11	11	11	11	11	13
14	13	13	14	15	15				

11. Error Analysis

The maximum and probable errors present in the above data are roughly analyzed below.

a. Heat-transfer Coefficients

The heat-transfer coefficients were calculated from the relation

$$h = \frac{3.413iV}{At_d}, \quad (C.1)$$

where

h = heat-transfer coefficient, Btu/(hr)(ft²)(°F);

i = current to heater, Amp;

V = voltage across heater, V;

A = area of heater, ft² (= 1/48 ft²);

and

t_d = overall temperature difference, °F.

Following standard procedures (see, for example, Mickley,⁴² p. 57) for computing the errors, we obtain

$$\frac{\Delta h}{h} \approx \frac{\Delta i}{i} + \frac{\Delta V}{V} - \frac{\Delta t_d}{t_d}, \quad (C.2)$$

where

$$\frac{\Delta x}{x} = \text{fractional error in } x.$$

The maximum error is obtained from C.2 by considering the signs of the various errors on the right-hand side to be such that $\Delta h/h$ is maximum.⁴²

The probable error is given by

$$\frac{\sigma^2(h)}{h^2} \approx \frac{\sigma^2(i)}{i^2} + \frac{\sigma^2(V)}{V^2} + \frac{\sigma^2(t_d)}{t_d^2}, \quad (C.3)$$

(Mickley,⁴² p. 64), from which $\sigma(h)/h$ may be computed.

Errors in measured quantities are as follows:

The ammeter was calibrated to 0.5% of full-scale reading (2 Amp). Average ammeter reading was 0.5 Amp. This gives $\Delta i/i = 0.02$.

The voltmeter was calibrated to 0.5% of full-scale reading (10 V). Average voltmeter reading was 6 V. This gives $\Delta V/V = 0.008$.

The thermocouples may deviate from the true reading by about 0.5°F (in the 0 - 100°F range). This gives a combined error in the temperature difference to be 1°F . The average difference was 15°F so that $\Delta t_d/t_d = 0.067$.

From Eqs. C.2 and C.3, the errors are computed as

Maximum error in $h \approx 10\%$;

Probable error in $h \approx 7\%$.

b. Mean Residence Times

The probable errors in the mean residence times may be estimated from the results of Koppel *et al.*²⁶ These authors found that for a simulated series involving approximately 3000 particles, the fractional error in the mean residence time was approximately 0.05. In the experiments, roughly 125 particles were involved, on the average. (This number may be estimated from $\bar{N}$ and $\bar{\tau}$; see Koppel *et al.*²⁶) Assuming an independent sampling process, we may roughly estimate the error as

$$\frac{\sigma(\bar{\tau})}{\bar{\tau}} \approx 0.05 \sqrt{\frac{3000}{125}}$$

$$= 0.25.$$

The probable error in $\bar{\tau} \approx 25\%$.

c. Overall Error

The deviation of the experimental data from the model predictions is a result of errors in h combined with errors in $\bar{\tau}$. It is of interest to estimate this deviation.

Assuming, as a first approximation, a linear relationship between h and $\bar{\tau}$, we may show that

$$e = e_h - e_{\bar{\tau}} \frac{dh}{d\bar{\tau}}, \quad (\text{C.4})$$

where

e = overall error in h (deviation from true value),

e_h = error in measurement of h ,

and

$e_{\bar{\tau}}$ = error in measurement of $\bar{\tau}$.

The probable overall error is given by

$$\frac{\sigma^2(e)}{h^2} = \frac{\sigma^2(e_h)}{h^2} + \left(\frac{\bar{\tau}}{h} \frac{dh}{d\bar{\tau}} \right)^2 \frac{\sigma^2(e_{\bar{\tau}})}{\bar{\tau}^2}. \quad (C.5)$$

The term in large parentheses in Eq. C.5 may be approximately obtained from Eq. 20 as a function of M and $\bar{\tau}$ for small $\bar{\tau}$. Using values of $M = 20$ and $\bar{\tau} = 0.1$, the square of this term is $\approx 1/9$. Values of $M = 60$ and $\bar{\tau} = 0.001$ give $1/10$.

Using Eq. C.5 and the error estimates previously obtained, we get:

Probable overall error in heat-transfer coefficient $\approx 11\%$.

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