ANALYTICAL SOLUTIONS TO EDDY-CURRENT PROBE COIL PROBLEMS

C. V. Dodd
W. E. Deeds

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CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>1</td>
</tr>
<tr>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>Derivation of Vector Potential</td>
<td>4</td>
</tr>
<tr>
<td>Closed Form Solutions of the Vector Potential</td>
<td>8</td>
</tr>
<tr>
<td>Coil above a Two-Conductor Plane</td>
<td>8</td>
</tr>
<tr>
<td>Coil Encircling a Two-Conductor Rod</td>
<td>13</td>
</tr>
<tr>
<td>Coils of Finite Cross Section</td>
<td>19</td>
</tr>
<tr>
<td>Calculation of Physical Phenomena</td>
<td>23</td>
</tr>
<tr>
<td>Induced Eddy Currents</td>
<td>23</td>
</tr>
<tr>
<td>Induced Voltage</td>
<td>24</td>
</tr>
<tr>
<td>Coil Impedance</td>
<td>25</td>
</tr>
<tr>
<td>Flaw Impedance</td>
<td>26</td>
</tr>
<tr>
<td>Coil Inductance</td>
<td>26</td>
</tr>
<tr>
<td>Mutual Inductance</td>
<td>27</td>
</tr>
<tr>
<td>Evaluation of Integrals</td>
<td>28</td>
</tr>
<tr>
<td>Experimental Verification</td>
<td>28</td>
</tr>
<tr>
<td>Accuracy of Calculations</td>
<td>29</td>
</tr>
<tr>
<td>Axial Symmetry</td>
<td>30</td>
</tr>
<tr>
<td>Current Sheet Approximation</td>
<td>30</td>
</tr>
<tr>
<td>High Frequency Effects</td>
<td>31</td>
</tr>
<tr>
<td>Conclusions</td>
<td>32</td>
</tr>
<tr>
<td>Acknowledgments</td>
<td>32</td>
</tr>
<tr>
<td>Appendix A</td>
<td>33</td>
</tr>
<tr>
<td>Appendix B</td>
<td>35</td>
</tr>
</tbody>
</table>
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ABSTRACT

Solutions have been obtained for axially symmetric eddy-current problems in two configurations of wide applicability. In both cases the eddy currents are assumed to be produced by a circular coil of rectangular cross section, driven by a constant amplitude alternating current. One solution is for a coil above a semi-infinite conducting slab with a plane surface, covered with a uniform layer of another conductor. This solution includes the special cases of a coil above a single infinite plane conductor or above a sheet of finite thickness, as well as the case of one metal clad on another. The other solution is for a coil surrounding an infinitely long circular conducting rod with a uniformly thick coating of another conductor. This includes the special cases of a coil around a conducting tube or rod, as well as one metal clad on a rod of another metal. The solutions are in the form of integrals of first-order Bessel functions giving the vector potential, from which the other electromagnetic quantities of interest can be obtained. The coil impedance has been calculated for the case of a coil above a two-conductor plane. The agreement between the calculated and experimental values is excellent.

INTRODUCTION

Electromagnetic problems are usually divided into three categories: low frequency, intermediate frequency, and high frequency. At low frequencies, static conditions are assumed; at high frequencies, wave equations are used. Both of these regions have been studied extensively. However, in the intermediate frequency range, where diffusion equations are used, very few problems have actually been solved. Eddy-current coil problems fall into this intermediate frequency region. This report presents an accurate technique for analyzing the problems of eddy-current testing.

1Consultant from the University of Tennessee.
Eddy-current testing has been used in industry for many years. As early as 1879, D. E. Hughes\(^2\) used an induction coil to sort metals. There have been numerous articles on the testing of materials with eddy currents. Some of the first papers dealing with both the theory and the practical aspects of eddy-current testing are by Förster,\(^3\) Förster and Stambke,\(^4\) and Förster.\(^5\) In this series of papers, analyses are made of a coil above a conducting surface, assuming the coil to be a magnetic dipole, and of an infinite coil encircling an infinite rod. Hochschild\(^6\) also gives an analysis of an infinite coil including some eddy-current distributions in the metal. Waidelich and Renken\(^7\) made an analysis of the coil impedance using an image approach. Their theoretical results agreed well with theory for relatively high frequencies. Libby\(^8\) presented a theory in which he assumed the coil was a transformer with a network tied to the secondary. This network representation gave good results when compared to experiment. The diffusion of eddy-current pulses (Atwood and Libby\(^9\) ) can be represented in this manner. Russell, Schuster and Waidelich\(^10\) gave an analysis of a cup core coil where they assumed the flux was entirely coupled into the conductor. The semi-empirical results agreed fairly well with the experimental measurements.

\(^2\)D. E. Hughes, Phil. Mag. 8(5), 50 (1879).
\(^3\)Friedrich Förster, Z. Metallk. 43, 163-171 (1952).
\(^8\)H. L. Libby, Broadband Electromagnetic Testing Methods, HW-59614 (1959).
Vein,\textsuperscript{11} Cheng,\textsuperscript{12} and Burrows\textsuperscript{13} gave treatments based on delta function coils, and Burrows continued with the development of an eddy-current flaw theory. Dodd and Deeds,\textsuperscript{14} Dodd,\textsuperscript{15} and Dodd\textsuperscript{16} gave a relaxation theory to calculate the vector potential of a coil with a finite cross section. Here we extend a "closed form" solution to such coils.

The vector potential is used as opposed to the electric and magnetic fields. The differential equations for the vector potential will be derived from Maxwell's equations, with the assumption of cylindrical symmetry. This differential equation will then be solved to obtain a "closed form" solution.

For the "closed form" solution, sinusoidal driving currents and linear, isotropic, and homogeneous media will be assumed. Solutions will be obtained for two different conductor geometries: a rectangular cross-section coil above a plane with one conductor clad on another and a rectangular cross-section coil encircling a two-conductor rod. The solutions for both geometries will be given in terms of integrals of Bessel functions. Once the vector potential has been determined, it can be used to calculate any physically observable electromagnetic quantity.

Equations to calculate eddy-current density, induced voltage, coil impedance, and effect of defects will be given. Measured values of coil impedance as compared with calculated values show excellent agreement.

\begin{itemize}
\item\textsuperscript{11} P. R. Vein, J. Electron. Control \textbf{13}, 471-494 (1962).
\item\textsuperscript{13} Michael Leonard Burrows, A Theory of Eddy Current Flaw Detection, University Microfilms, Inc., Ann Arbor, Michigan, 1964.
\item\textsuperscript{15} C. V. Dodd, A Solution to Electromagnetic Induction Problems, ORNL-TM-1185 (1965) and M.S. Thesis, the University of Tennessee, 1965.
\item\textsuperscript{16} C. V. Dodd, Solutions to Electromagnetic Induction Problems, ORNL-TM-1842 (1967) and Ph.D. Dissertation, the University of Tennessee, 1967.
\end{itemize}
DERIVATION OF VECTOR POTENTIAL

The differential equations\(^1\) for the vector potential will be derived from Maxwell's equations which are:

\[
\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad (1)
\]
\[
\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (2)
\]
\[
\nabla \cdot \vec{B} = 0 \quad (3)
\]
\[
\nabla \cdot \vec{D} = \rho \quad (4)
\]

The medium is taken to be linear and isotropic, but not homogeneous. In a linear and isotropic medium, the following relations between \(\vec{D}\) and \(\vec{E}\) and \(\vec{B}\) and \(\vec{H}\) hold:

\[
\vec{B} = \mu \vec{H} \quad (5)
\]
\[
\vec{D} = \varepsilon \vec{E} \quad (6)
\]

The current density \(\vec{J}\) can be expressed in terms of Ohm's law:

\[
\vec{J} = \sigma \vec{E} \quad (7)
\]

Equations (6) and (7) may be substituted into Equation (1) to obtain the curl of \(\vec{H}\) in terms of \(\vec{E}\):

\[
\nabla \times \vec{H} = \sigma \vec{E} + \frac{\partial \vec{D}}{\partial t} \quad (8)
\]

\(^1\)A list of symbols is given in Appendix A.
The term \( \sigma E \) is much greater than \( \frac{\partial E}{\partial t} \), so the latter may be neglected for frequencies below about 10 Mc/sec (ref. 18). The magnetic induction field \( \vec{B} \) may be expressed as the curl of a vector potential \( \vec{A} \):

\[
\vec{B} = \nabla \times \vec{A}
\]  
(9)

Substituting this into Equation (2) gives

\[
\nabla \times \vec{E} = -\frac{\partial}{\partial t} \nabla \times \vec{A} = -\nabla \times \frac{\partial \vec{A}}{\partial t}
\]  
(10)

or

\[
\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla \psi = \vec{E}_{\text{induced}} + \vec{E}_{\text{applied}}
\]  
(11)

\[
\sigma E = -\sigma \frac{\partial \vec{A}}{\partial t} - \sigma \nabla \psi
\]  
(12)

The term \( \psi \) is interpreted as an applied scalar potential. The coil may be driven by a voltage generator with an applied voltage \( \psi \) and an internal resistivity, \( \frac{1}{\sigma} \). However, for the purpose of this problem, the driving function is expressed as an alternating current density of constant amplitude, \( \vec{i}_0 \) rather than an applied potential, where

\[
\text{Limit } (-\sigma \nabla \psi) = \vec{i}_0
\]
\( \sigma \to 0 \).
\( \nabla \psi \to \infty \)

(13)

This provides a current which is not affected by induced voltages or the presence of other coils. Making this substitution gives:

\[
\sigma E = -\sigma \frac{\partial \vec{A}}{\partial t} + \vec{i}_0
\]  
(14)

Substituting Equations (5) and (9) into the left side of Equation (8) and Equation (14) into the right side gives:

\[\text{For sinusoidal waves, } \frac{\partial \varepsilon \vec{E}}{\partial t} = \varepsilon \vec{E} = j\omega \vec{E}. \text{ The term } \sigma E \text{ is much greater than } \varepsilon \omega \vec{E} \text{ or } \sigma \gg \varepsilon \omega. \sigma \approx 10^7 \text{ mhos/meter for metals, } \varepsilon \approx 10^{-11}. \text{ For frequencies on the order of } 10^7 \text{ cps, } \omega \approx 10^8, 10^7 \gg 10^8 \times 10^{-11}, \text{ or } \sigma \approx 10^{10} \varepsilon \omega.\]
\[ \nabla \times \vec{H} = \nabla \times \frac{\vec{B}}{\mu} = \nabla \times [(1/\mu) \nabla \times \vec{A}] = -\sigma \frac{\partial \vec{A}}{\partial t} + \vec{I}_0 \]  

(15)

The vector identities (Morse and Feshbach\textsuperscript{19})

\[ \nabla \times (\psi \vec{F}) = (\nabla \psi) \times \vec{F} + \psi \nabla \times \vec{F} \text{ and } \nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}, \]

can be used to expand the left side of Equation (15):

\[ \nabla \times (1/\mu) (\nabla \times \vec{A}) = \nabla (1/\mu) \times (\nabla \times \vec{A}) + \frac{1}{\mu} \nabla \times (\nabla \times \vec{A}) \]

\[ = \nabla (1/\mu) \times (\nabla \times \vec{A}) + \frac{1}{\mu} \nabla (\nabla \cdot \vec{A}) - \frac{1}{\mu} \nabla^2 \vec{A}. \]  

(16)

In the definition of the vector potential the divergence of the vector potential was not defined, so it can be defined to be anything convenient. For induction problems \( \nabla \cdot \vec{A} \) is set to zero. (This corresponds to the Coulomb gage.) Equation (16) will then yield the following results when substituted into Equation (15).

\[ \nabla^2 \vec{A} = -\mu \vec{I}_0 + \mu \sigma \frac{\partial \vec{A}}{\partial t} + \mu \nabla (1/\mu) \times (\nabla \times \vec{A}) \]  

(17)

This is the equation for the vector potential in an isotropic, linear, inhomogeneous medium. For most coil problems it is possible to assume axial symmetry as shown in Fig. 1. The vector potential will be symmetric about the axis of the coil. Since this assumption is valid for most problems and the alternative to this assumption is a much more complicated and impractical problem, axial symmetry is assumed. With axial symmetry, there is only a \( \theta \) component of \( \vec{I} \) and therefore of \( \vec{A} \). Expanding the \( \theta \) component of Equation (17) gives:

\[ \frac{\partial^2 \vec{A}}{\partial r^2} + \frac{1}{r} \frac{\partial \vec{A}}{\partial r} + \frac{\partial^2 \vec{A}}{\partial z^2} - \frac{\vec{A}}{r^2} = -\mu \vec{I}_0 + \mu \sigma \frac{\partial \vec{A}}{\partial t} \]

\[ -\mu \left[ \frac{\partial (1/\mu)}{\partial r} \left( \frac{1}{r} \frac{\partial \vec{A}}{\partial r} \right) + \frac{\partial (1/\mu)}{\partial z} \frac{\partial \vec{A}}{\partial z} \right] \]  

(18)

\textsuperscript{19}Philip M. Morse and Herman Feshbach, Methods of Theoretical Physics, McGraw-Hill Book Company, New York, 1953.
Assume that $i_o$ is a sinusoidal function of time, $i_o = i'_o e^{j\omega t}$. Then the vector potential is likewise a sinusoidal function of time,

$$A = A' e^{j(\omega t + \phi)} = A'' e^{j\omega t}.$$ 

Substituting into Equation (18) gives:
\[
\frac{\partial^2 A''}{\partial r^2} e^{j\omega t} + \frac{1}{r} \frac{\partial A''}{\partial r} e^{j\omega t} + \frac{\partial^2 A''}{\partial z^2} e^{j\omega t} - \frac{A''}{r^2} e^{j\omega t} = -\mu \mu' e^{j\omega t} + j\omega \sigma A'' e^{j\omega t}
\]

\[
- \mu \left[ \frac{\partial (1/\mu)}{\partial r} \left( \frac{1}{r} \frac{\partial r'}{\partial r} e^{j\omega t} \right) + \left( \frac{\partial (1/\mu)}{\partial z} \right) \frac{\partial A''}{\partial z} e^{j\omega t} \right]
\]

Canceling out the term \(e^{j\omega t}\) and dropping the prime gives:

\[
\frac{\partial^2 A}{\partial r^2} + \frac{1}{r} \frac{\partial A}{\partial r} + \frac{\partial^2 A}{\partial z^2} - \frac{A}{r^2} = -\mu i_0 + j\omega \sigma A - \mu \left[ \frac{\partial (1/\mu)}{\partial r} \left( \frac{1}{r} \frac{\partial r'}{\partial r} \right) + \left( \frac{\partial (1/\mu)}{\partial z} \right) \frac{\partial A}{\partial z} \right]
\]

This is the general differential equation for the vector potential in a linear, inhomogeneous medium with a sinusoidal driving current. We shall now obtain a "closed form" solution of Equation (19).

CLOSED FORM SOLUTIONS OF THE VECTOR POTENTIAL

We shall assume the medium to be linear, isotropic, and homogeneous. When \(I\) is the total driving current in a delta function coil at \((r_0, z_0)\), the general Equation (19) then becomes:

\[
\frac{\partial^2 A}{\partial r^2} + \frac{1}{r} \frac{\partial A}{\partial r} + \frac{\partial^2 A}{\partial z^2} - \frac{A}{r^2} = -\mu i_0 + j\omega \sigma A + \mu \delta(r - r_0) \delta(z - z_0) = 0 \tag{20}
\]

Once we have solved this linear differential equation for a particular conductor configuration, we can then superimpose any number of delta function coils to build any desired shape of coil (provided that the current in each coil is known).

We shall solve the problem for two different conductor configurations: a coil above a two-conductor plane and a coil encircling a two-conductor rod. These two configurations apply to a large number of practical problems.

Coil above a Two-Conductor Plane

The coil above a two-conductor plane is shown in Fig. 1. We have divided the problem into four regions. The differential equation in air (regions I and II) is:
The differential equation in a conductor (regions III and IV) is:

$$\frac{\partial^2 A}{\partial r^2} + \frac{1}{r} \frac{\partial A}{\partial r} + \frac{\partial^2 A}{\partial z^2} - \frac{A}{r^2} = 0$$

(21)

The differential equation in a conductor (regions III and IV) is:

$$\frac{\partial^2 A}{\partial r^2} + \frac{1}{r} \frac{\partial A}{\partial r} + \frac{\partial^2 A}{\partial z^2} - \frac{A}{r^2} - j\omega \sigma_i A = 0$$

(22)

Setting $A(r,z) = R(r) Z(z)$ and dividing by $R(r) Z(z)$ gives:

$$\frac{1}{R(r)} \frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r R(r)} \frac{\partial R(r)}{\partial r} + \frac{1}{Z(z)} \frac{\partial^2 Z(z)}{\partial z^2} - \frac{1}{r^2} - j\omega \sigma_i = 0$$

(23)

We write for the $z$ dependence:

$$\frac{1}{Z(z)} \frac{\partial^2 Z(z)}{\partial z^2} = \text{"constant"} = \alpha^2 + j\omega \sigma_i$$

(24)

or

$$Z(z) = A e^{+\sqrt{\alpha^2+j\omega \sigma_i} z} + B e^{-\sqrt{\alpha^2+j\omega \sigma_i} z}$$

(25)

We define:

$$\alpha_1 \equiv \sqrt{\alpha^2+j\omega \sigma_i}$$

(26)

Equation (23) then becomes:

$$\frac{1}{R(r)} \frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{R(r)} \frac{\partial R(r)}{\partial r} + \alpha^2 - \frac{1}{r^2} = 0$$

(27)

This is a first-order Bessel equation and has the solutions:

$$R(r) = C J_1(\alpha r) + D Y_1(\alpha r)$$

(28)

Combining the solutions we have:

$$A(r,z) = [A e^{+\alpha_1 z} + B e^{-\alpha_1 z}] [C J_1(\alpha r) + D Y_1(\alpha r)]$$

(29)

We now need to determine the constants $A$, $B$, $C$, and $D$. They are functions of the separation "constant" $\alpha$ and are usually different for each value of $\alpha$. Our complete solution would be a sum of all the individual solutions, if $\alpha$ were a discrete variable; but, since $\alpha$ is a
continuous variable, the complete solution is an integral over the entire range of $\alpha$. Thus, the general solution is:

$$A(r,z) = \int_0^\infty [A(\alpha) e^{\alpha_1 z} + B(\alpha) e^{-\alpha_1 z}] [C(\alpha) J_1(\alpha r) + D(\alpha) Y_1(\alpha r)] \, d\alpha.$$  \hspace{1cm} (30)

We must take $A(\alpha) = 0$ in region I, where $z$ goes to plus infinity. Due to the divergence of $Y_1$ at the origin, $D(\alpha) = 0$ in all regions. In region IV, where $z$ goes to minus infinity, $B(\alpha)$ must vanish. The solutions in each region then become:

$$A^{(1)}(r,z) = \int_0^\infty B_1(\alpha) e^{-\alpha z} J_1(\alpha r) \, d\alpha$$  \hspace{1cm} (31)

$$A^{(2)}(r,z) = \int_0^\infty [C_2(\alpha) e^{\alpha z} + B_2(\alpha) e^{-\alpha z}] J_1(\alpha r) \, d\alpha$$  \hspace{1cm} (32)

$$A^{(3)}(r,z) = \int_0^\infty [C_3(\alpha) e^{\alpha z} + B_3(\alpha) e^{-\alpha z}] J_1(\alpha r) \, d\alpha$$  \hspace{1cm} (33)

$$A^{(4)}(r,z) = \int_0^\infty [C_4(\alpha) e^{\alpha z} J_1(\alpha r) \, d\alpha$$  \hspace{1cm} (34)

The boundary conditions between the different regions are:

$$A^{(1)}(r,\ell) = A^{(2)}(r,\ell)$$  \hspace{1cm} (35)

$$\frac{\partial A^{(1)}}{\partial z}(r,\ell) \bigg|_{z=\ell} = \frac{\partial A^{(2)}}{\partial z}(r,\ell) \bigg|_{z=\ell} - \mu I \delta(r - r_o)$$  \hspace{1cm} (36)

$$A^{(2)}(r,0) = A^{(3)}(r,0)$$  \hspace{1cm} (37)

$$\frac{\partial A^{(2)}}{\partial z}(r,0) \bigg|_{z=0} = \frac{\partial A^{(3)}}{\partial z}(r,0) \bigg|_{z=0}$$  \hspace{1cm} (38)

$$A^{(3)}(r,-c) = A^{(4)}(r,-c)$$  \hspace{1cm} (39)

$$\frac{\partial A^{(3)}}{\partial z}(r,-c) \bigg|_{z=-c} = \frac{\partial A^{(4)}}{\partial z}(r,-c) \bigg|_{z=-c}$$  \hspace{1cm} (40)
Equation (35) gives:

\[ \int_0^\infty B_1(\alpha) e^{-\alpha l} J_1(\alpha r) \, d\alpha = \int_0^\infty [C_2(\alpha) e^{\alpha l} + B_2(\alpha) e^{-\alpha l}] J_1(\alpha r) \, d\alpha \] (41)

If we multiply both sides of Equation (41) by \( \int_0^\infty J_1(\alpha' r) \, dr \) and then reverse the order of integration, we obtain:

\[ \int_0^\infty \frac{B_1(\alpha)}{\alpha} \left[ \int_0^\infty J_1(\alpha r \) \ J_1(\alpha' r) \, dr \right] \, d\alpha = \int_0^\infty \frac{1}{\alpha} \left[ C_2(\alpha) e^{\alpha l} + B_2(\alpha) e^{-\alpha l} \right] \left[ \int_0^\infty J_1(\alpha r \) \ J_1(\alpha' r) \, dr \right] \, d\alpha \] (42)

We can simplify Equation (42) by use of the Fourier-Bessel equation, which is:

\[ \mathcal{F}(\alpha') = \int_0^\infty \mathcal{F}(\alpha) \int_0^\infty J_1(\alpha r) \ J_1(\alpha' r) \, dr \, d\alpha \] (43)

Equation (42) then becomes:

\[ \frac{B_1}{\alpha'} e^{-\alpha' l} = \frac{C_2}{\alpha'} e^{\alpha' l} + \frac{B_2}{\alpha'} e^{-\alpha' l} \] (44)

We can evaluate the other integral equations in a similar manner. We get (after dropping the primes on the \( \alpha \)):

\[ -B_1 e^{-\alpha l} = C_2 e^{\alpha l} - B_2 e^{-\alpha l} - \mu I_{0} J_1(\alpha r_0) \] (45)

\[ \frac{C_2}{\alpha} + \frac{B_2}{\alpha} = \frac{C_3}{\alpha} + \frac{B_3}{\alpha} \] (46)

\[ C_2 - B_2 = \frac{\alpha_1}{\alpha} C_3 - \frac{\alpha_1}{\alpha} B_3 \] (47)

\[ \frac{C_3}{\alpha} e^{-\alpha_1 c} + \frac{B_3}{\alpha} e^{\alpha_1 c} = \frac{C_4}{\alpha} e^{-\alpha_2 c} \] (48)

\[ \frac{\alpha_1}{\alpha} C_3 e^{-\alpha_1 c} - \frac{\alpha_1}{\alpha} B_3 e^{\alpha_1 c} = \frac{\alpha_2}{\alpha} C_4 e^{-\alpha_2 c} \] (49)
We now have six equations with six unknowns. Their solution is:

\[
B_1 = \frac{1}{2} \mu r_0 J_1(\alpha r_0) \left\{ e^{\alpha l} + \frac{(\alpha + \alpha_1)(\alpha_1 - \alpha_2) + (\alpha - \alpha_1)(\alpha_2 + \alpha_1) e^{2\alpha_1 c}}{(\alpha - \alpha_1)(\alpha_1 - \alpha_2) + (\alpha + \alpha_1)(\alpha_2 + \alpha_1) e^{2\alpha_1 c}} e^{-\alpha l} \right\}
\]

\[
C_2 = \frac{1}{2} \mu r_0 J_1(\alpha r_0) e^{\alpha l}
\]

\[
B_2 = \frac{1}{2} \mu r_0 J_1(\alpha r_0) \left\{ \frac{(\alpha + \alpha_1)(\alpha_1 - \alpha_2) + (\alpha - \alpha_1)(\alpha_2 + \alpha_1) e^{2\alpha_1 c}}{(\alpha - \alpha_1)(\alpha_1 - \alpha_2) + (\alpha + \alpha_1)(\alpha_2 + \alpha_1) e^{2\alpha_1 c}} e^{-\alpha l} \right\}
\]

\[
C_3 = \mu r_0 J_1(\alpha r_0) \left\{ \frac{\alpha(\alpha_2 + \alpha_1) e^{-\alpha l} + 2\alpha_1 c}{(\alpha - \alpha_1)(\alpha_1 - \alpha_2) + (\alpha + \alpha_1)(\alpha_2 + \alpha_1) e^{2\alpha_1 c}} \right\}
\]

\[
B_3 = \mu r_0 J_1(\alpha r_0) \left\{ \frac{\nu (\nu_1 - \nu_2) e^{\alpha l}}{(\alpha - \alpha_1)(\alpha_1 - \alpha_2) + (\alpha + \alpha_1)(\alpha_2 + \alpha_1) e^{2\alpha_1 c}} \right\}
\]

\[
C_4 = \mu r_0 J_1(\alpha r_0) \left\{ \frac{2\alpha_1 \nu e^{(\alpha_2 + \alpha_1)c} e^{-\alpha l}}{(\alpha - \alpha_1)(\alpha_1 - \alpha_2) + (\alpha + \alpha_1)(\alpha_2 + \alpha_1) e^{2\alpha_1 c}} \right\}
\]

We can now write the expressions for the vector potential in each region:

\[
A^{(1)}(r,z) = \frac{\mu r_0}{2} \int_0^{\alpha r_0} J_1(\alpha r_0) J_1(\alpha r) e^{-\alpha l - \alpha z}
\]

\[
x \left\{ e^{\alpha l} + \frac{(\alpha + \alpha_1)(\alpha_1 - \alpha_2) + (\alpha - \alpha_1)(\alpha_2 + \alpha_1) e^{2\alpha_1 c}}{(\alpha - \alpha_1)(\alpha_1 - \alpha_2) + (\alpha + \alpha_1)(\alpha_2 + \alpha_1) e^{2\alpha_1 c}} \right\} d\alpha
\]

\[
A^{(2)}(r,z) = \frac{\mu r_0}{2} \int_0^{\alpha r_0} J_1(\alpha r_0) J_1(\alpha r) e^{-\alpha l}
\]

\[
x \left\{ e^{\alpha z} + \frac{(\alpha + \alpha_1)(\alpha_1 - \alpha_2) + (\alpha - \alpha_1)(\alpha_2 + \alpha_1) e^{2\alpha_1 c}}{(\alpha - \alpha_1)(\alpha_1 - \alpha_2) + (\alpha + \alpha_1)(\alpha_2 + \alpha_1) e^{2\alpha_1 c}} \right\} e^{-\alpha z} d\alpha
\]
These are the equations for the vector potential of a delta function coil above a two-conductor plane. Next we shall consider the derivation of the vector potential of a delta function coil encircling a two-conductor rod.

Coil Encircling a Two-Conductor Rod

We shall assume a delta function coil encircling an infinitely long, two-conductor rod, as shown in Fig. 2.

The general differential equation is the same as Equation (23) for a coil above a conducting plane.

$$\frac{1}{R(r)} \frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r R(r)} \frac{\partial R(r)}{\partial r} + \frac{1}{Z(z)} \frac{\partial^2 Z(z)}{\partial z^2} - \frac{1}{r^2} - j\omega \sigma = 0$$

(60)

Now, however, we shall assume the separation constant to be negative:

$$\frac{1}{Z(z)} \frac{\partial^2 Z(z)}{\partial z^2} = "\text{constant}" = -\alpha^2$$

(61)

Then

$$Z(z) = F \sin \alpha(z-z_o) + G \cos \alpha(z-z_o)$$

(62)

and Equation (60) becomes:

$$r^2 \frac{\partial^2 R(r)}{\partial r^2} + r \frac{\partial R(r)}{\partial r} - [(\alpha^2 + j\omega \sigma)r^2 + 1] R(r) = 0$$

(63)
Fig. 2. Delta Function Coil Encircling a Two-Conductor Rod.
The solution to Equation (63) in terms of modified Bessel functions is:

\[ R(r) = C I_1[(\alpha^2 + j\omega \sigma)^{1/2}r] + D K_1[(\alpha^2 + j\omega \sigma)^{1/2}r] \]  

(64)

We can now write the vector potential in each region. We shall use the fact that it is symmetric (with respect to \(z-z_0\)) to eliminate the sine terms, and the fact that \(K_1(\alpha)\) and \(I_1(\alpha)\) both diverge to eliminate their coefficients in regions I and IV, respectively. Thus we have:

\[ A^{(1)}(r,z-z_0) = \int_0^\infty C_1(\alpha) I_1[(\alpha^2 + j\omega \sigma)^{1/2}r] \cos(\alpha(z-z_0)) \, d\alpha \]  

(65)

\[ A^{(2)}(r,z-z_0) = \int_0^\infty C_2(\alpha) I_1[(\alpha^2 + j\omega \sigma)^{1/2}r] \cos(\alpha(z-z_0)) \, d\alpha \]  

(66)

\[ A^{(3)}(r,z-z_0) = \int_0^\infty C_3(\alpha) I_1(\alpha r) + D_3(\alpha) K_1(\alpha r) \cos(\alpha(z-z_0)) \, d\alpha \]  

(67)

\[ A^{(4)}(r,z-z_0) = \int_0^\infty D_4(\alpha) K_1(\alpha r) \cos(\alpha(z-z_0)) \, d\alpha \]  

(68)

The boundary conditions between the different regions are:

\[ A^{(1)}(a,z-z_0) = A^{(2)}(a,z-z_0) \]  

(69)

\[ \left. \frac{\partial}{\partial r} A^{(1)}(r,z-z_0) \right|_{r=a} = \left. \frac{\partial}{\partial r} A^{(2)}(r,z-z_0) \right|_{r=a} \]  

(70)

\[ A^{(2)}(b,z-z_0) = A^{(3)}(b,z-z_0) \]  

(71)

\[ \left. \frac{\partial}{\partial r} A^{(2)}(r,z-z_0) \right|_{r=b} = \left. \frac{\partial}{\partial r} A^{(3)}(r,z-z_0) \right|_{r=b} \]  

(72)

\[ A^{(3)}(r_0,z-z_0) = A^{(4)}(r_0,z-z_0) \]  

(73)
\[
\frac{\partial}{\partial r} A^{(3)}(r,z-z_0) \bigg|_{r=r_0} = \frac{\partial}{\partial r} A^{(4)}(r,z-z_0) \bigg|_{r=r_0} + \mu I_5(z-z_0) \tag{74}
\]

If we multiply both sides of Equation (69) by \(\cos\alpha'(z-z_0)\) and integrate from zero to infinity, we obtain:

\[
\int_0^\infty \int_0^\infty C_1(\alpha) I_1[(\alpha^2 + j\omega_1)^{\frac{1}{2}} r] \cos(\alpha z-z_0) \cos\alpha'(z-z_0) \, d\alpha = \int_0^\infty \int_0^\infty C_2(\alpha) I_1[(\alpha^2 + j\omega_2)^{\frac{1}{2}} r] \cos\alpha(z-z_0) \cos\alpha'(z-z_0) \, d\alpha \, d(z-z_0) \tag{75}
\]

We can reverse the order of integration and use the orthogonality properties of the cosine integral or use the Fourier integral theorem:

\[
\frac{1}{\pi} \int_0^\infty f(\alpha) \left[ \int_0^\infty \cos(\alpha z-z_0) \cos\alpha'(z-z_0) \, d(z-z_0) \right] \, d\alpha = f(\alpha') \tag{76}
\]

Thus, we can solve the integral equations (69) through (74). We shall use \(\alpha_1\) and \(\alpha_2\) to designate \((\alpha^2 + j\omega_1)^{\frac{1}{2}}\) and \((\alpha^2 + j\omega_2)^{\frac{1}{2}}\). We shall use primes to designate derivatives with respect to the argument. We get from the integral equations (69) through (74):

\[
C_1 I_1(\alpha_1 a) = C_2 I_1(\alpha_2 a) + D_2 K_1(\alpha_2 a) \tag{77}
\]

\[
C_1 \alpha_1 I'_1(\alpha_1 a) = C_2 \alpha_2 I'_1(\alpha_2 a) + D_2 \alpha_2 K'_1(\alpha_2 a) \tag{78}
\]

\[
C_2 I_1(\alpha_2 b) + D_2 K_1(\alpha_2 b) = C_3 I_1(\alpha b) + D_3 K_1(\alpha b) \tag{79}
\]

\[
C_2 \alpha_2 I'_1(\alpha_2 b) + D_2 \alpha_2 K'_1(\alpha_2 b) = C_3 \alpha I'_1(\alpha b) + D_3 \alpha K'_1(\alpha b) \tag{80}
\]

\[
C_3 I_1(\alpha r_0) + D_3 K_1(\alpha r_0) = D_4 K_1(\alpha r_0) \tag{81}
\]

\[
C_3 \alpha I'_1(\alpha r_0) + D_3 \alpha K'_1(\alpha r_0) = D_4 \alpha K'_1(\alpha r_0) + \frac{H_1}{\pi} \tag{82}
\]
Now we have six equations with six unknown constants. The equations may be solved to give the constants. We define:

\[
D = [\alpha_2 K_0(\alpha_2 b)K_1(\alpha b) - \alpha K_0(\alpha b)K_1(\alpha_2 b)][\alpha_1 I_1(\alpha_2 a)I_0(\alpha_1 a)
- \alpha_2 I_1(\alpha_1 a)I_0(\alpha_2 a)] + [\alpha_2 K_0(\alpha_2 a)I_1(\alpha_1 a)
+ \alpha_1 K_1(\alpha_2 a)I_0(\alpha_1 a)][\alpha_1 I_1(\alpha_2 b)K_0(\alpha b) + \alpha_2 I_0(\alpha_2 b)K_1(\alpha b)]
\]

(83)

The constants are

\[
C_1 = \frac{\mu r_0 K_1(\alpha r_o)}{abD}
\]

(84)

\[
D_2 = \frac{\mu r_0 K_1(\alpha r_o)}{bD} \left[ (\alpha_2 I_1(\alpha_1 a)I_0(\alpha_2 a) - \alpha_1 I_1(\alpha_2 a)I_0(\alpha_1 a)) \right]
\]

(85)

\[
C_2 = \frac{\mu r_0 K_1(\alpha r_o)}{bD} \left[ \alpha_2 K_0(\alpha_2 a)I_1(\alpha_1 a) + \alpha_1 K_1(\alpha_2 a)I_0(\alpha_1 a) \right]
\]

(86)

\[
C_3 = \frac{\mu r_0 K_1(\alpha r_o)}{\pi}
\]

(87)

\[
D_3 = -\frac{\mu r_0 I_1(\alpha r_o)}{\pi} \left[ \frac{K_1(\alpha_2 b)}{K_1(\alpha b)} \right] \left[ \frac{I_1(\alpha_2 b)}{bD} \left[ (\alpha_2 K_0(\alpha_2 a)I_1(\alpha_1 a) + \alpha_1 K_1(\alpha_2 a)I_0(\alpha_1 a)) + I_1(\alpha b) \right] \right]
\]

(88)

\[
D_4 = \frac{\mu r_0 K_1(\alpha r_o)}{\pi} \left[ \frac{K_1(\alpha_2 b)[\alpha_2 I_1(\alpha_1 a)I_0(\alpha_2 a) - \alpha_1 I_1(\alpha_2 a)I_0(\alpha_1 a)]}{K_1(\alpha b)bD} \right]
\]

(89)
We can now write for the vector potential in each region:

\[ A^{(1)}(r,z-z_0) = \mu I \int_0^{\infty} \frac{r_0 K_1(\alpha r)}{\alpha} \frac{1}{\alpha} I_1(\alpha r) \cos(z-z_0) \, d\alpha \]  

\[ A^{(2)}(r,z-z_0) = \mu I \int_0^{\infty} \frac{r_0 K_1(\alpha r_0)}{bD} \left\{ \left[ \alpha_2 I_1(\alpha_1 a) I_o(\alpha_2 a) - \alpha_1 I_1(\alpha_2 a) I_o(\alpha_1 a) \right] \right\} \cos(z-z_0) \, d\alpha \]  

\[ A^{(3)}(r,z-z_0) = \mu I \int_0^{\infty} r_0 K_1(\alpha r_0) \left\{ I_1(\alpha r) - \frac{K_1(\alpha_2 b)}{bD K_1(\alpha b)} \left( \alpha_1 I_1(\alpha_2 a) I_o(\alpha_1 a) - \alpha_2 I_1(\alpha_1 a) I_o(\alpha_2 a) \right) \right\} \cos(z-z_0) \, d\alpha \]  

\[ A^{(4)} = \mu I \int_0^{\infty} r_0 K_1(\alpha r_0) K_1(\alpha r) \left\{ \frac{K_1(\alpha_2 b)}{K_1(\alpha b)} \left[ \alpha_2 I_1(\alpha_1 a) I_o(\alpha_2 a) - \alpha_1 I_1(\alpha_2 a) I_o(\alpha_1 a) \right] \right\} \cos(z-z_0) \, d\alpha \]  

Equations (90) through (93) are the equations for the vector potential of a delta function coil encircling a two-conductor rod. We will now consider the superposition of the delta function coils to form "real" coils.
Coils of Finite Cross Section

We have the equations for the vector potential produced by a single delta function coil. We can now approximate any coil such as the ones shown in Figs. 3 and 4 by the superposition of a number of delta function coils.

Fig. 3. Rectangular Cross-Section Coil above a Two-Conductor Plane.
Fig. 4. Rectangular Cross-Section Coil Encircling a Two-Conductor Rod.

In general, we have:

\[ A(r,z)_{\text{total}} = \sum_{i=1}^{n} A_1(r,z) = \sum_{i=1}^{n} A(r,z, \ell_1, r_i) \]  (94)
This equation is good for coils of any cross section. If we let the current distribution in the delta function coils approach a continuous current distribution, we obtain:

\[
A(r,z)(\text{total}) = \int_{\text{coil}} A(r,z,r_0,\ell) \, d(\text{area})
\]

where \( A(r,z,\ell, r_0) \) is the vector potential produced by an applied current density \( i_0(\ell, r_0) \). If the coil has a rectangular cross section, as in Figs. 3 and 4, we have:

\[
A(r,z)(\text{total}) = \int_{r_1}^{r_2} \int_{\ell_1}^{\ell_2} A(r,z,r_0,\ell) \, dr_0 \, d\ell
\]

We will now assume that the applied current density \( i_0(\ell, r_0) \) is a constant over the dimensions of the coil; that is, the current in each loop has the same phase and amplitude. We shall apply these results to Equation (56), the case of a probe coil above a two-conductor plane.

After reversing the order of integration, we write:

\[
A(1)(r,z) = \int_{r_1}^{r_2} \int_{\ell_1}^{\ell_2} \int_{r_0}^{r_0} \frac{\mu_0 r_0}{2} J_1(\alpha r_0)J_1(\alpha r) \, e^{-\alpha(\ell+z)} \left\{ e^{2\alpha \ell} \right. \\
+ \left. \frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2) + (\alpha-\alpha_1)(\alpha_2+\alpha_1) e^{2\alpha_1 c}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1) e^{2\alpha_1 c}} \right\} \, d\alpha \, dr_0 \, d\ell
\]

We shall express the integral over \( r_0 \) as:

\[
\int_{r_0=r_1}^{r_2} \frac{\alpha r_2}{2} J_1(\alpha r_0) \, dr_0 = \frac{1}{\alpha^2} \int_{\alpha r_0 = \alpha r_1}^{\alpha r_2} xJ_1(x) \, dx \equiv \frac{1}{\alpha^2} I(r_2, r_1)
\]
The integral over $l$ is:

$$
\int_{l_1}^{l_2} e^{-\alpha(l+z)} \left\{ e^{2\alpha l} + 1 \right\} dl = e^{-\alpha z} \int_{l_1}^{l_2} \left\{ e^{\alpha l} + e^{-\alpha l} \right\} dl
$$

$$
= \frac{e^{-\alpha z}}{\alpha} \left[ (e^{\alpha l_2} - e^{\alpha l_1}) - (e^{-\alpha l_2} - e^{-\alpha l_1}) \right] \quad (99)
$$

Upon applying Equations (98) and (99), the equations for the vector potential in the various regions for a rectangular cross-section coil become:

$$
A^{(1)}(r,z) = \frac{\mu_i}{2} \int_0^\infty \frac{1}{\alpha^3} I(r_2,r_1) J_1(\alpha r) \left\{ e^{-\alpha l_2} - e^{-\alpha l_1} - (e^{-\alpha l_1} - e^{-\alpha l_2}) \right\} \left[ (\alpha^2+\alpha) (\alpha_1^2 - \alpha_2^2) + (\alpha^{-1}+\alpha) (\alpha_2^2 + \alpha_1^2) \right] \frac{e^{2\alpha_1 c}}{(\alpha-\alpha_1) (\alpha+\alpha_1) (\alpha^2+\alpha_1 c) \alpha} \, d\alpha \quad (100)
$$

$$
A^{(2)}(r,z) = \frac{\mu_i}{2} \int_0^\infty \frac{1}{\alpha^3} I(r_2,r_1) J_1(\alpha r) \left\{ e^{-\alpha l_1} - e^{-\alpha l_2} \right\} \left\{ e^{-\alpha l_1} - e^{-\alpha l_2} \right\} \times \left[ \frac{2\alpha_1 c}{(\alpha-\alpha_1) (\alpha+\alpha_1) (\alpha^2+\alpha_1 c) \alpha} \right] \frac{e^{-\alpha z}}{(\alpha^2+\alpha) (\alpha_1^2 + \alpha_2^2) + (\alpha^{-1}+\alpha) (\alpha_2^2 + \alpha_1^2) \alpha} \, d\alpha \quad (101)
$$

$$
A^{(3)}(r,z) = \mu_i \int_0^\infty \frac{1}{\alpha^3} I(r_2,r_1) J_1(\alpha r) \left\{ e^{-\alpha l_1} - e^{-\alpha l_2} \right\} \left\{ e^{-\alpha l_1} - e^{-\alpha l_2} \right\} \times \left\{ \frac{2\alpha_1 c}{(\alpha-\alpha_1) (\alpha+\alpha_1) (\alpha^2+\alpha_1 c) \alpha} \right\} \frac{e^{-\alpha z}}{(\alpha-\alpha_1) (\alpha+\alpha_1) (\alpha^2+\alpha_1 c) \alpha} \, d\alpha \quad (102)
$$

$$
A^{(4)}(r,z) = \mu_i \int_0^\infty \frac{1}{\alpha^3} I(r_2,r_1) J_1(\alpha r) \left\{ e^{-\alpha l_1} - e^{-\alpha l_2} \right\} \left\{ e^{-\alpha l_1} - e^{-\alpha l_2} \right\} \times \left\{ \frac{2\alpha_1 c}{(\alpha-\alpha_1) (\alpha+\alpha_1) (\alpha^2+\alpha_1 c) \alpha} \right\} \frac{e^{-\alpha z}}{(\alpha-\alpha_1) (\alpha+\alpha_1) (\alpha^2+\alpha_1 c) \alpha} \, d\alpha \quad (103)
$$
Equation (100) for $A^{(1)}$ is valid in the region above the coil and Equation (101) for $A^{(2)}$ is valid for the region below the coil. We have to give special treatment to region I-II, between the top and bottom of the coil. For a point $(r, z)$ in region I-II, we can use the equation $A^{(1)}(r, z)$ for the portion of the coil from $z$ down to $l_1$ and the equation $A^{(2)}(r, z)$ for the portion of the coil from $z$ up to $l_2$. If we substitute $l_2 = z$ in Equation (100) and $l_1 = z$ in Equation (101) and add the two equations, we get:

$$A^{(1,2)}(r, z) = \frac{\mu_0}{2} \int_0^\infty \frac{1}{\alpha^3} I(r_2, r_1) J_l(\alpha r) \left\{ 2 - e^{\alpha(z-l_2)} - e^{-\alpha(z-l_1)} + e^{-\alpha z} \left( e^{-\alpha l_1} - e^{-\alpha l_2} \right) \frac{[(\alpha+\alpha_1)(\alpha_1-\alpha_2) + (\alpha-\alpha_1)(\alpha_2+\alpha_1) e^{2\alpha_1 c}]}{[(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1) e^{2\alpha_1 c}]} \right\} d\alpha \quad (104)$$

We now have the equations for the vector potential in all the regions.

**CALCULATION OF PHYSICAL PHENOMENA**

Once we have determined the vector potential, we can calculate any physically observable electromagnetic induction phenomenon. We shall now give the equations and perform the calculations for some of the phenomena of interest in eddy-current testing.

**Induced Eddy Currents**

We have, from Ohm's law:

$$\vec{J} = \sigma \vec{E} = -\sigma \frac{\partial \vec{A}}{\partial t} = -j \omega \sigma \vec{A} \quad (105)$$

From the axial symmetry, Equation (105) becomes:

$$\vec{J} = -j \omega \sigma \vec{A}(r, z) \quad (106)$$

where $A(r, z)$ is given by either Equation (102) or (103), depending on the region of interest.
Induced Voltage

We have, for the voltage induced in a length of wire:

\[ V = j \omega \mathbf{A} \cdot d\mathbf{s} \]  
(107)

For an axially symmetric coil with a single loop of radius \( r \), Equation (107) becomes:

\[ V = j \omega 2\pi r A(r, z) \]  
(108)

The total voltage induced in a coil of \( n \) turns is then:

\[ V = j 2\pi \omega \sum_{i=0}^{n} r_{i} A(r_{i}, z_{i}) \]  
(109)

We can approximate the above summation by an integral over a turn density of \( N \) turns per unit cross-sectional area:

\[ V \approx j 2\pi \omega \iint_{\text{coil cross section}} rA(r, z) N \text{d}r \text{d}z \]  
(110)

For coils with a constant number of turns per unit cross-sectional area:

\[ V = \frac{j 2\pi \omega n}{\text{coil cross section}} \iint_{\text{coil cross section}} rA(r, z) \text{d}r \text{d}z \]  
(111)

This is the equation for the voltage induced in a coil by any coaxial coil.

When the two coils are one and the same, with cross-sectional area equal to \((\ell_{2} - \ell_{1})(r_{2} - r_{1})\), the self-induced voltage is:

\[ V = \frac{j 2\pi \omega n}{(\ell_{2} - \ell_{1})(r_{2} - r_{1})} \int_{\ell_{1}}^{\ell_{2}} \int_{r_{1}}^{r_{2}} rA(1, z) \text{d}r \text{d}z \]  
(112)
From the self-induced voltage, we can calculate the coil impedance

\[ V = Z I, \text{ or } Z = \frac{V}{I} \]  

(113)

The current in a single loop is related to the applied current density, \( i_o \), by:

\[ i_o = \frac{n I}{(r_2 - r_1)} \]  

(114)

The coil impedance becomes:

\[ Z = \frac{j \omega n^2}{(r_2 - r_1)^2} \int_{0}^{\infty} \frac{1}{\alpha^2} I^2(r_2, r_1) \left[ 2(r_2 - r_1) + \frac{1}{\alpha} \right] \]  

\[ \times \left( \frac{(\alpha + \alpha_1)(\alpha_1 - \alpha_2) + (\alpha - \alpha_1)(\alpha_2 + \alpha_1) e^{2\alpha_1 \alpha} e^{2\alpha_1 \alpha_1}}{(\alpha - \alpha_1)(\alpha_1 - \alpha_2) + (\alpha + \alpha_1)(\alpha_2 + \alpha_1) e^{2\alpha_1 \alpha} e^{2\alpha_1 \alpha_1}} \right) \]  

(115)

This equation can be made more general by normalizing all the dimensions in terms of a mean coil radius, \( \bar{r} \).

\[ \bar{r} = \frac{r_1 + r_2}{2} \]  

(116)

All lengths are divided by \( \bar{r} \) and all \( \alpha \)'s are multiplied by \( \bar{r} \).

Upon normalization, Equation (115) becomes:
The impedance may be normalized by dividing it by the magnitude of the air impedance. For the air impedance \( \alpha_1 = \alpha_2 = \alpha \) and:

\[
Z_{\text{air}} = \frac{2 \pi \omega \mu n^2 r}{(k_2 - k_1)^2 (r_2 - r_1)^2} \int_0^\infty \frac{1}{\alpha^5} I^2(r_2, r_1) \left\{ -2 e^{-\alpha (k_2 - k_1)} - 2 + \left( e^{-2\alpha l_2} + e^{-2\alpha l_1} - 2 e^{-\alpha (k_2 + k_1)} \right) \right. \\
\times \left. \left( \frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2) + (\alpha-\alpha_1)(\alpha_2+\alpha_1) e^{2\alpha l_1}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1) e^{2\alpha l_1}} \right) \right\} d\alpha \tag{118}
\]

Flaw Impedance

Once the eddy-current density is known, we can simulate a flaw by superimposing a small current flowing in the opposite direction. The normalized impedance change due to a small, spherical defect not too close to the surface (Burrows\textsuperscript{20}) is:

\[
Z' = -\frac{3}{2} \sigma \text{ vol} \left( \frac{A_{\text{defect}}}{I} \right)^2 \tag{119}
\]

where \( A_{\text{defect}} \) is the vector potential at the defect, given by the equations for either \( A(3) \) and \( A(4) \) and "vol" is the volume of the defect.

Coil Inductance

The coil inductance is related to the magnitude of the air impedance by:

\[ \omega L = \left| Z_{\text{air}} \right| \] (120)

or

\[ L = \frac{2\pi \mu n^2 r}{(l_2-l_1)^2(r_2-r_1)^2} \int_0^\infty \frac{1}{\alpha^5} I^2(r_2,r_1) \left\{ (l_2-l_1) + \frac{1}{\alpha} \left[ e^{-\alpha(l_2-l_1)} - 1 \right] \right\} d\alpha \] (121)

**Mutual Inductance**

The voltage generated in a "pickup" coil with dimensions \( r'_2, r'_1, l'_2, l'_1 \) by a current \( I \) flowing in a "driver" coil with dimensions \( r_2, r_1, l_2, l_1 \) is:

\[ V = M \frac{dI}{dt} = j\omega MI \] (122)

or

\[ M = \frac{V}{j\omega I} \] (123)

Using Equation (111) to calculate the voltage we have:

\[ M = \frac{2\pi n'}{(\text{coil cross section})'} \int \int rA(r,z) \, drdz \] (124)

The equation for \( A \) will vary, depending on the region where the pickup coil is located. If the pickup coil is located in region I-II, the mutual inductance is:
This is the mutual inductance between the driver coil and the pickup coil in the presence of a clad conductor. By the reciprocity theorem, this is equal to the mutual inductance between the pickup coil and the driver coil.

Evaluation of Integrals

The normalized impedance has been calculated using a C-E-I-R time-sharing computer to evaluate integral equations (117) and (118). The solutions have been programmed for any rectangular coil dimensions and lift-off as well as for a metal of any conductivity clad (in varying thickness) onto a base metal of any conductivity. The programs, in "BASIC" language, and their descriptions are given in Appendix B.

Figure 5 shows how the normalized impedance varies as a function of clad thickness.

Experimental Verification

A family of four coils was constructed with different mean radii but all with the same normalized dimensions. The coil impedance was measured at various values of normalized lift-off and at various values of \(\overline{r^2}\omega_0\mu_0\sigma\). The values of the experimental normalized coil impedance and
the calculated normalized coil impedance are plotted in Fig. 6. The agreement between the calculated and measured values is excellent at the higher frequencies. At the lower frequencies the measurements are very difficult to make, and the accuracy of the measured values becomes very poor. (Because of this, few eddy-current tests are made at these frequencies.) Thus the theory is in excellent agreement with experimental values at the frequencies of interest in eddy-current testing.

ACCURACY OF CALCULATIONS

This technique, like most others used in engineering, is "exact, except for a few assumptions we have to make in order to work the problem." We will now discuss the probable errors in some of these assumptions.
Fig. 6. Variation of Experimental and Calculated Values of Normalized Impedance with Frequency and Lift-Off.

Axial Symmetry

This is a very good assumption, but we cannot easily wind coils that have perfect axial symmetry. This error will vary with the winding technique and will decrease as the number of turns on the coil and the coil-to-conductor spacing increases. This error will be effectively reduced when normalized impedance is calculated. For a typical coil it should be less than 0.01%.

Current Sheet Approximation

This error arises because we have assumed a current sheet, while we actually have a coil wound with round, insulated wire. Some correction
formulas are given by Rosa and Grover\textsuperscript{21} for the inductance of a coil in air. From Equations (87) and (93) by Rosa and Grover, we have calculated the following correction formula:

\[
\frac{\Delta L}{L} = \frac{[0.5058 r_2 - 0.2742 r_1 + 0.44 (l_2 - l_1)]}{n} \left( \ln \frac{D}{d} + 0.155 \right) \tag{126}
\]

where all dimensions are normalized by the mean coil radius. The symbols \(D\) and \(d\) are the wire diameters with and without insulation, respectively. For a typical coil with 100 turns the change in inductance is 0.19\%. The change in normalized impedance will be a small fraction of the change in inductance.

**High Frequency Effects**

These are probably the most serious sources of error in this calculation technique. As the frequency increases, the current density ceases to be uniformly distributed over the cross section of the wire but becomes concentrated near the surface. The resistance of the coil increases, and the inductance decreases. The current is capacitively coupled between the turns in the coil, tending to flow across the loops of wire rather than through them. Both the interwinding capacitance and the coil-to-metal sample capacitance increase. The coil-to-sample capacitance can be reduced by winding the coil such that the turns nearest the sample are electrically near alternating-current ground. The coil-to-sample capacitance will be much less than the interwinding capacitance. If the coil is used at frequencies where the interwinding capacitance has a small effect, the error in calculated normalized impedance will be a much smaller effect.

CONCLUSIONS

This technique presents a quick and easy way to calculate the observed effects of actual eddy-current tests to a high degree of accuracy.

ACKNOWLEDGMENTS

The authors wish to express their appreciation to W. A. Simpson for performing the experimental measurements, to J. W. Luquire and W. G. Spoeri for editing and checking the equations and their assistance in programming the stepwise solution of the integral equations.
APPENDIX A

List of Symbols

In the first column the symbol used is given, and in the second column the name. In the third column the meter-kilogram-second (MKS) units are given. In the last column the dimensions are given in terms of mass (M), length (L), time (T), and electric charge (Q).

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Name</th>
<th>MKS Units</th>
<th>Dimensions</th>
</tr>
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<tbody>
<tr>
<td>A</td>
<td>vector potential</td>
<td>webers meter</td>
<td>ML</td>
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<tr>
<td></td>
<td></td>
<td>webers meter²</td>
<td>TQ</td>
</tr>
<tr>
<td>B</td>
<td>magnetic induction</td>
<td>webers meter</td>
<td>M</td>
</tr>
<tr>
<td></td>
<td></td>
<td>webers meter²</td>
<td>TQ</td>
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<tr>
<td>c</td>
<td>clad thickness</td>
<td>meter</td>
<td>L</td>
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<td>ampere meter</td>
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<td>I</td>
<td>applied current</td>
<td>ampere</td>
<td>Q/T</td>
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<tr>
<td>i₀</td>
<td>applied current density</td>
<td>ampere meter²</td>
<td>Q/TL²</td>
</tr>
<tr>
<td>J</td>
<td>current density</td>
<td>ampere meter²</td>
<td>Q/TL²</td>
</tr>
<tr>
<td>j</td>
<td>square root of minus one</td>
<td></td>
<td></td>
</tr>
<tr>
<td>L</td>
<td>inductance</td>
<td>henries</td>
<td>ML²/Q²</td>
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<tr>
<td>ℓ</td>
<td>distance from metal to delta function coil</td>
<td>meter</td>
<td>L</td>
</tr>
<tr>
<td>ℓ₂</td>
<td>distance from metal to top of the coil</td>
<td>meter</td>
<td>L</td>
</tr>
<tr>
<td>ℓ₁</td>
<td>distance from metal to bottom of the coil</td>
<td>meter</td>
<td>L</td>
</tr>
<tr>
<td>Symbol</td>
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<td>MKS Units</td>
<td>Dimensions</td>
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<td>--------</td>
<td>-----------------------------</td>
<td>-----------</td>
<td>------------</td>
</tr>
<tr>
<td>N</td>
<td>turns per unit area</td>
<td>turns/meter²</td>
<td>1/1²</td>
</tr>
<tr>
<td>n</td>
<td>number of turns</td>
<td>turns</td>
<td></td>
</tr>
<tr>
<td>r₁</td>
<td>coil inner radius</td>
<td>meters</td>
<td>L</td>
</tr>
<tr>
<td>r₂</td>
<td>coil outer radius</td>
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<td>L</td>
</tr>
<tr>
<td>r̅</td>
<td>mean coil radius</td>
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<tr>
<td>t</td>
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<td>seconds</td>
<td>T</td>
</tr>
<tr>
<td>V</td>
<td>voltage</td>
<td>volt</td>
<td>ML²/T²Q²</td>
</tr>
<tr>
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<td>impedance</td>
<td>ohms</td>
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</tr>
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<td>1/1</td>
</tr>
<tr>
<td>α₁</td>
<td>(α²+jωµσ)⁻¹/²</td>
<td>meter⁻¹</td>
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<tr>
<td>ε</td>
<td>dielectric constant</td>
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<td>henry</td>
<td>ML</td>
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<tr>
<td>σ</td>
<td>conductivity</td>
<td>mho</td>
<td>T²Q²/ML³</td>
</tr>
<tr>
<td>ω</td>
<td>angular frequency</td>
<td>radians</td>
<td>1/T</td>
</tr>
<tr>
<td></td>
<td></td>
<td>second</td>
<td></td>
</tr>
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</table>
APPENDIX B

This appendix contains two "BASIC" programs which were used to calculate eddy-current coil impedance using the C-E-I-R time-sharing computer system. The first program, CLADT5, is the more general program and will calculate the impedance of a coil of any rectangular cross section positioned any distance above one conductor of any conductivity clad on another conductor of any conductivity. The second program is a special case of the first program where the two metals have the same conductivity. While the integral over $\alpha$ is from 0 to $\infty$, the integrals converge to within about 0.03% of their final value for the integral of 0 to $\alpha=35$. 
REM THIS IS A PROGRAM TO CALCULATE EDDY CURRENT COIL IMPEDANCE
REM FOR A COIL ABOVE A CONDUCTING PLANE. THE COIL INNER AND
REM OUTER RADII, R1 AND R2, AND THE SPACING OF THE BOTTOM AND
REM TOP OF THE COIL ABOVE THE PLANE, L1 AND L2, MUST BE GIVEN.
REM THE VALUE OF R2*2*PI*FREQ*COND MUST BE GIVEN FOR BOTH THE BASE
REM MATERIAL, M1, AND THE CLAD MATERIAL, M2. THE THICKNESS, C,
REM OF THE CLAD MATERIAL MUST ALSO BE GIVEN.
10 LET R1=.3333
20 LET R2=1.1667
30 LET L1=.0476
40 LET L2=.3809
50 LET M1=77.05
60 LET M2=40
70 LET C=.05
80 PRINT "R1=","R1","R2=","R2","L1=","L1","L2=","L2"
90 PRINT "CLAD THICKNESS IS ";C,"M1=","M1","M2=","M2"
100 LET S1=1E-2
110 LET S2=5
120 LET I6=0
130 LET I7=0
140 LET I8=0
150 LET I9=0
160 LET I1=0
170 LET R2=S2
180 FOR X = R1 +S1/2 TO R2 STEP S1
190 LET Z=R2*X
200 LET C=Z
210 LET C=R1
220 GOSUB 796
230 LET I2=F2
240 LET I3=I2+I1
250 LET I3=I2-I1
260 LET I3=I2+I1
270 LET I1=F2
280 LET I3=I2-I1
290 IF 2*X*L1>16 THEN 630
300 LET Y2=.767107*SQRT(SQR(X**X*X*X+M2*M2)-X**X)
310 LET X1 = .767107*SQRT(SQR(X**X*X*X+M1*M1)+X**X)
320 LET Y1 = .767107*SQRT(SQR(X**X*X*X+M1*M1)+X**X)
330 IF 2*X*C>30 THEN 510
340 LET Y2 = .767107*SQRT(SQR(X**X*X*X+M2*M2)-X**X)
350 LET X2 = .767107*SQRT(SQR(X**X*X*X+M2*M2)-X**X)
360 LET Y2 = .767107*SQRT(SQR(X**X*X*X+M2*M2)-X**X)
370 LET X3=EXP(X2*X1*C)
380 LET Y3=COS(2*Y1*C)
390 LET Y4=SIN(2*Y1*C)
400 LET A6=(X-X1)*(X1+X2)+Y1*(Y1+Y2)
410 LET A7=(X-X1)*(Y1+Y2)-Y1*(X1+X2)
420 LET A5=(X1+X2)*(X1-X2)+Y1*(Y1-Y2)+(A6+Y3+A7+Y4)*X3
430 LET A5=(Y1*(X1-X2)+(X+X1)*(Y1-Y2)+(A7+Y3*A6+Y4)*X3
LET C6 = (X1 + X2) * X1 + X2 - Y1 * Y1 + Y2
LET C7 = (X1 + X2) * (X1 + X2) - (Y1 + Y2)
LET C5 = (X1 - X2) * X1 - X2 - Y1 * Y1 - Y2 + C6 * Y3 - C7 * Y4 * X3
LET D5 = (X1 - X2) * Y1 - Y1 * X1 - X2 + C7 * Y3 + C6 * Y4 * X3
LET K1 = (A5 * C5 + B5 * D5) / (C5 * C5 + D5 * D5)
LET K2 = (C5 * E5 - A5 * D5) / (C5 * C5 + D5 * D5)
GOTO 560
LET A5 = X - X1
LET E5 = -Y1
LET C5 = X + X1
LET D5 = Y1
GOTO 480
LET G = EXP(-2 * X * L1) + EXP(-2 * X * L2) - 2 * EXP(-X * (L1 + L2))
LET G1 = G * K1
LET G2 = G * K2
LET A1 = S3 / G1 / (2 * X)
LET A2 = S3 / G2 / (2 * X)
LET I6 = I6 + A1
LET I8 = I8 + A2
LET G3 = EXP(-X * (L2 - L1)) - 1
LET A3 = S3 / (G3 / X + L2 - L1)
LET I9 = I9 + A3
NEXT X.
LET B1 = B1 + S2
LET B2 = B2 + S2
LET O3 = X + S1 / 2
LET I7 = I7 + 16
PRINT O3, I9, I7, I8
IF X < 3 THEN 190
LET S1 = 5E-2
IF X < 30 THEN 190
LET G1 = -I8 / I9
LET G2 = I7 / I9
PRINT "NORMALIZED IMAG PART"; G2, "NORMALIZED REAL PART"; G1
GOTO 940
IF Z > 3 THEN
LET P1 = 0.869 * Z + 4.97 + 1.738 * EXP(-2.675 * Z)
LET P2 = SIN(Z - 2.346 + 0.66 * EXP(-0.68 * Z) + 0.32 * EXP(-3 * Z))
LET P3 = 1.56 * EXP(-0.9 * Z) * SIN(2.2 * Z - 31)
LET P4 = 1
LET F2 = (F1 * P2 + P3 + P4) / (Z * Z)
RETURN
END
REM THIS IS A PROGRAM TO CALCULATE EDDY CURRENT COIL IMPEDANCE
REM FOR A COIL ABOVE A CONDUCTING PLANE. THE COIL INNER AND
REM OUTER RADII, R1 AND R2, AND THE SPACING OF THE BOTTOM AND
REM TOP OF THE COIL ABOVE THE PLANE, L1 AND L2, MUST BE GIVEN.
REM THE VALUE OF R*T2*FREQ*MU*COND MUST ALSO BE GIVEN.
10 LET R1=.8333
20 LET R2=1.1667
30 LET L1=.0952
40 LET L2=.4285
50 LET M=77.05
60 LET S1=1E-2
70 LET S2=1
80 PRINT"R1=";R1,"R2=";R2,"L1=";L1,"L2=";L2,"M=";M
90 PRINT "X","AIR VALUE","REAL PART","MAG PART"
100 LET I3=0
110 LET I4=0
120 LET I7=0
130 LET I8=0
140 LET I9=0
150 LET B1=0
160 LET B2=S2
170 FOR X = R1 +S1/2 TO B2 STEP S1
180 LET Z=R2*X
190 LET O1=R2
200 GOSUB 680
210 LET I2=F2
220 LET Z=R1*X
230 LET O1=R1
240 GOSUB 680
250 LET I1=F2
260 LET I3=I2-11
270 LET S3=S1*13*I3/X
430 LET K1=1.41421*X*SQR(X1-X*X)/M-1
450 LET K2=(2*X-1.41421*SQR(X1+X*X))/X/M
460 LET G=EXP(-2*X*L1)+EXP(-2*X*L2)-2*EXP(-X*(L1+L2))
470 LET G1=G*K1
480 LET G3=EXP(-X*(L2-L1))-1
490 LET G2=G*K2
500 LET A1=S3*G1/(2*X)
510 LET A2=S3*(G3/X+L2-L1)
520 LET A3=S3*G2/(2*X)
530 LET I7=I7+A1+A3
540 LET I9=I9+A3
550 LET I8=I8+A2
560 NEXT X
570 LET B1=B1+S2
580 LET B2=B2+S2
590 LET G3=X+S1/2
600 PRINT 03,19,17,18
610 IF X < 3 THEN 170
620 LET S1=3E-2
630 IF X < 30 THEN 170
640 LET O1=-18/19
650 LET O2=17/19
660 PRINT"NORMALIZED IMAG PART";O2,"NORMALIZED REAL PART";O1
670 GOTO820
680 IF Z>3 THEN 770
690 LET L5=INT(2*Z)+3
700 LET F1=5*O1*O1*Z
710 LET F2=F1/3
720 FOR N=1 TO L5
730 LET F1=-F1*256*Z*Z/(N*N+N)
740 LET F2=F2+F1/(2*N+3)
750 NEXT N
760 GOTO820
770 LET P1=.8669*Z+.497+.1738*EXP(-.2675*Z)
780 LET P2=3IN(Z-2.346+.106*EXP(-.068*Z)+.32*EXP(-.3*Z))
790 LET P3=.156*EXP(-.9*Z)*3IN(2.2*Z-.31)
800 LET P4=1
810 LET F2=(P1*P2+P3+P4)/(X*X)
820 RETURN
830 END
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