ADJOINING APPROPRIATE SINGULAR ELEMENTS
TO TRANSPORT THEORY COMPUTATIONS

I. K. Abu-Shumays and E. H. Bareiss
Northwestern University
Evanston, Illinois

NOTICE

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

May 1973

PREPARED FOR THE U.S. ATOMIC ENERGY COMMISSION

, UNDER CONTRACT NO. AT(11-1)-2280

MASTER

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

ADJOINING APPROPRIATE SINGULAR ELEMENTS
TO TRANSPORT THEORY COMPUTATIONS*

by

I. K. Abu-Shumays and E. H. Bareiss

Abstract

Neutron density possesses logarithmic singularities in its derivatives near material interfaces (discontinuities) and vacuum boundaries. Singularities also exist near edges and corners, and can be shown to propagate to the interior regions. The accuracy and the rate of convergence of higher order approximations are affected by the choice of mesh spacing, and depend critically on how well the local behavior near boundaries and edges is accounted for. Considerable information can be obtained, as is done below, through the study of one-energy-group slab geometry model problems for the integral transport equation. Here, variational methods are applied to (a) the Milne problem (b) a source problem based on the Milne problem, and (c) the slab eigenvalue and the extended eigenvalue problem. In the first problem the Galerkin technique is used with the exponential integrals as base functions. The results are improved especially close to the interface by a first iteration as indicated by Kourganoff. For the source problem, linear and cubic spline bases are used. It is shown that considerable improvement is gained by adjoining to the trial function in the first interval an $x \log x$ "add-on term" to allow for the analytic behavior near the interface. In particular for small n ($n < 32$), addition of the $x \log x$ term is often better than doubling the mesh size;

*Work supported by the U.S. Atomic Energy Commission and in part by Northwestern University.

furthermore the linear results with the $x \log x$ term are better than the uncorrected cubic results without this term. The advantage of a non-uniform mesh (dense close to the interface and sparse in the regular region) is also demonstrated. In the third problem, simple trial functions for the criticality problem already used in the literature are applied, but better accuracy is achieved both with and without the $x \log x$ term.

In conclusion, a recommendation is made as to how to approach two and three dimensional problems.

1. INTRODUCTION

The objective of this work is to demonstrate the need to seek the exact analytic behavior of the neutron concentration near material discontinuities, edges and corners, and to incorporate it into accurate transport theory computational methods. This is in fact essential if one is to obtain higher-order accuracy in finite element or other computational approximations, especially in view of the practical limitations on the mesh size and storage requirements of practical two and three dimensional problems. George Fix [25] has provided the evidence in an analogous situation for the finite element approximation to the diffusion equation. The main idea here is that higher accuracy could be achieved if appropriate singular functions (describing the local behavior in certain regions near boundaries, etc.) are adjoined to the space of piecewise polynomial approximations.

In view of the large number of variables (coordinates, velocities, etc.) necessary to describe the transport of neutrons, practical computations are inevitably based on a multigroup formulation and the method of "source iterations". In this formulation [41], the equations for one-energy-group are treated one at a time. At each level of the inner iterations, the sources for the chosen group include both the external sources and the contributions from other energy groups, and are considered to be fixed known functions of the independent variables. Therefore, it is sufficient to consider the one energy group transport equation. Furthermore, for algebraic simplicity, the treatment is restricted to stationary transport with isotropic scattering and the units of length and time are chosen such that the neutron speed $v = 1$ and the mean free path $\lambda = 1/\sigma = 1$. The

integrodifferential transport equation for the vector flux $\psi(\underline{x}, \Omega)$ of neutrons at $\underline{x}$ moving in the direction of the unit vector Ω and the integral transport equation for the scalar flux $\Phi(\underline{x})$ describing the net density of neutrons at $\underline{x}$ are, respectively

$$\Omega \cdot \nabla \psi(\underline{x}, \Omega) + \psi(\underline{x}, \Omega) = \frac{c}{4\pi} \int_{\{\Omega\}} \psi(\underline{x}, \Omega') d\Omega' + \text{Source}, \quad (1.1)$$

$$\Phi(\underline{x}) = \int_v K(\underline{x}, \underline{x}') \Phi(\underline{x}') d\underline{x}' + \text{Source}, \quad (1.2)$$

where c is the number of secondaries per collision. The equivalence of these integrodifferential and integral forms is well known [12,16]. The integral equation [38] has the advantage of being self-adjoint and having fewer variables, but the disadvantages of a kernel $K(\underline{x}, \underline{x}')$ that is singular at $\underline{x} = \underline{x}'$, and of yielding full matrices in computational methods. This paper will nonetheless concentrate on the integral formulation.

Initial highlights of this paper [3] were first presented in a seminar at the University of Illinois and at a SIAM conference [8] in Madison, Wisconsin in 1971, (these highlights were also mentioned in a panel discussion at a Nuclear Society meeting in Florida, the same year). Since then considerable progress has been achieved by many researchers, most notably by Mika et al. [5], Kellog et al. [6], Birkhoff [11] and Bareiss [7], towards better understanding of the discontinuities in the derivatives of the neutron flux across characteristic surfaces passing through edges and corners.

2. BEHAVIOR NEAR VACUUM BOUNDARIES AND MATERIAL INTERFACES

The most detailed work on the behavior of the neutron flux near boundaries and interfaces was carried out by C. Mark [39] in 1944 in

connection with the free surface of the Milne problem, and by I. W. Busbridge [14] in 1960 in connection with the free surfaces of a slab. Additional relevant contributions from the early days of transport theory were made by G. Placezek [48-50], B. Davison [20,22], J. LeCaine [35-6], V. Kourganoff [34] and a few others [4,30,42-4,54].

Davison [20,1T-18] points out that differentiating the integral transport equation for the net flux $\Phi(\underline{x})$ of neutrons at $\underline{x}$ yields an inhomogeneous integral transport equation for $\text{grad } \Phi(\underline{x})$ with a free term tending to infinity logarithmically as the interface is approached along its normal. He concludes that the same is true with respect to $\Phi(\underline{x})$ itself. This can also be seen if a small layer on both sides of the interface is stretched by a change of the units of the coordinate system in the direction of the surface normal. The transformed equation takes the form of a one dimensional slab geometry equation and the behavior of the flux is similar to that of the half-space Milne problem.

Mark [39] has shown that the exact solution for the scalar flux of the Milne problem for conservative ($c = 1$) isotropic scattering near the free surface takes the form

$$\begin{aligned} \Phi(x) = \Phi(0) \{ & 1 - \frac{1}{2} x \log x + 1.278794x + \frac{1}{16} x^2 (\log x)^2 \\ & - 0.382189x^2 \log x + 0.501194x^2 + 0(x^3 (\log x)^3) \}, \quad x < .02 \end{aligned} \quad (2.1)$$

Essentially, he started from the integral equation for the Milne Problem

$$\Phi(x) = \frac{c}{2} \int_0^\infty \Phi(x') E_1(|x - x'|) dx' \equiv c \Lambda_x \{ \Phi(x') \} \quad (2.2)$$

with $c = 1$ and E_1 the exponential integral [18,26,27,37]

$$E_n(x) = \int_0^1 e^{-x/v} v^{n-2} dv. \quad (2.3)$$

He then considered the asymptotic expansion for the Laplace transform $\bar{\Phi}(s)$ of $\Phi(x)$ obtained by Wiener-Hopf and deduced the expression for $\Phi(x)$ near $x = 0$ from that of its transform $\bar{\Phi}(s)$ for large s . Dr. J. Carson Mark, jointly with the late W. H. Hendry, has recently [November, 1971, private communication] justified his expression and shown how the expansion could be continued indefinitely.

The expansion (2.1) will be justified here thus. Starting from the initial approximation of the scalar flux

$$\Phi_0(x) = x_0 + x, \quad (2.4)$$

which is the exact asymptotic expression for $\Phi(x)$, it is well known that the Neumann series

$$\Phi_{n+1}(x) \equiv \Lambda_x \{\Phi_n(x')\} = \frac{1}{2} \int_0^\infty \Phi_n(x') E_1(|x - x'|) dx' \quad (2.5)$$

converges to the exact solution. Applying the operator Λ_x to $\Phi_0(x)$ of (2.4) yields

$$\Phi_1(x) = x_0 + x - \frac{x_0}{2} E_2(x) + \frac{1}{2} E_3(x) \quad (2.6a)$$

$$\Phi_2(x) = x_0 + x - \frac{x_0}{2} E_2(x) + \frac{1}{2} E_3(x) - \frac{x_0}{2} \Lambda_x \{E_2(x')\} + \frac{1}{2} \Lambda_x \{E_3(x')\}, \quad (2.6b)$$

etc. In particular for small x , $E_n(x)$ behaves like $(x^{n-1} \log x / (n-1)!) + \text{Polynomials}$ and $\Lambda_x \{E_n(x')\}$ includes [34] a constant multiple $x^n E_1^{(2)}(x)$ where, again, for small x

$$E_1^{(2)}(x) = \frac{\pi^2}{12} + \gamma \log x + \frac{1}{2} (\log x)^2 - x + \frac{x^2}{8} - \dots \quad (2.7)$$

Thus, the expression (2.6b) for $\Phi_2(x)$ already includes the polynomial and log terms given in the expression (2.1) for $\Phi(x)$. Higher order terms can be derived by further application of the operator Λ_x as in (2.5). Furthermore, repeated applications of Λ_x yield the proper behavior near the vacuum interface. In fact, Kourganoff [34] has shown that applying the operator Λ_x to an approximate solution improves the solution and the improvement is greater near the origin. This improvement also holds for non-conservative

scattering and can be proved for $0 \leq c \leq 1$.

The same procedure followed by Mark, and the iterative process described above show that (2.1) also holds for the non-conservative subcritical case, except that the coefficients of the various terms depend on the relevant properties of the medium considered, cross-sections, number of secondaries per collision c , etc. The usefulness of including the proper analytic behavior in approximation schemes was recognized by Le Caine, Kourganoff and others [34]. They started with the exact solution for the Milne problem for conservative scattering [16,34]

$$\Phi(x) = x_0 + x + \int_0^1 A(v) e^{-x/v} dv, \quad (2.8)$$

and approximate $A(v)$ by a polynomial

$$A(v) = \sum_{i=2}^n A_i v^{i-2}. \quad (2.9)$$

Substituting this expression (2.9) for $A(v)$ in (2.8) and using the definition (2.3), it follows that

$$\Phi(x) \simeq x_0 + x + A_2 E_2(x) + A_3 E_3(x) + \dots + A_n E_n(x). \quad (2.10)$$

LeCaine, Kourganoff and others [30,34-6,42-4] used variational methods to evaluate the expansion coefficients in (2.10). Here the functions $E_n(x)$ are smooth over the entire range $x \in (0, \infty)$, behave like $(x^{n-1}(\log x)/(n-1)! + \text{Polynomial})$ for small x and vanish at $x = \infty$. Trial functions of the form (2.10) indeed yield excellent approximations to the scalar flux as is shown below. Yet, a glance at Eq. (2.1) indicates that the approximation (2.10) cannot converge to the exact solution as $n \rightarrow \infty$, as it does not account for higher powers of $\log x$. We will try to prove elsewhere that the representation is rendered complete when functions such as [31] $E_n^{(m)}(x)$, possessing higher powers of $\log x$, are adjoined to (2.10).

It should be noted at this stage that representations similar to (2.1) and (2.10), including logarithmic terms, hold close to interfaces of finite slab. For example the exact solutions near an interface $x = a$ are always expected to have a term of the form

$$\int_0^1 A(\nu) e^{-(x-a)/\nu} d\nu \quad (2.11)$$

where $A(\nu)$ is continuous in the interval $[0, 1]$. Since polynomials are dense in the class of continuous functions, $A(\nu)$ can be approximated closely by a polynomial. In this case (2.11) would yield terms $E_2(x-a)$, $E_3(x-a)$, ... and would behave, for x close to a , as a polynomial in x and a polynomial times $(x-a)\log(x-a)$, etc. The results near an interface of the vector flux $\psi(x, \mu)$ of particles at x moving in direction $\cos^{-1}\mu$ with respect to the x -axis can also be deduced from the normal mode (singular eigenfunction) representation

$$\begin{aligned} \psi(x, \mu) = & a_+ e^{-x/\nu_0} \varphi_{\nu_0}(\mu) + \int_0^1 A(\nu) \varphi_{\nu}(\mu) e^{-x/\nu} d\nu \\ & + a_- e^{x/\nu_0} \varphi_{-\nu_0}(\mu) + \int_{-1}^0 A(\nu) \varphi_{\nu}(\mu) e^{-x/\nu} d\nu \end{aligned} \quad (2.12)$$

by approximating $A(\nu)$ [See 16 for an explicit expression for $\varphi_{\nu}(\mu)$ and ν_0].

This behavior can also be deduced from Chandrasakhar's $H(\mu)$ function

[1,2,17,53]

$$H(\mu) = 1 + \frac{c}{2} \mu H(\mu) \int_0^1 \frac{H(\mu') d\mu'}{\mu + \mu'} \quad (2.13)$$

In fact, McCormick and Kuscer [41] have shown that the normal modes satisfy the following half-range orthogonality relations

$$\int_0^1 \varphi_{\nu}(\mu) \varphi_{\nu'}(\mu) \mu H(\mu) d\mu = H(\nu) N(\nu) \delta(\nu - \nu'); \quad \nu, \nu' \in [0, 1], \quad (2.14a)$$

$$\int_0^1 \varphi_{\nu_0}^2(\mu) \mu H(\mu) d\mu = H(\nu_0) N(\nu_0) \quad (2.14b)$$

$$\int_0^1 \varphi_{v_0}(\mu) \varphi_v(\mu) \mu H(\mu) d\mu = 0 \quad v \neq v_0 \quad (2.14c)$$

where $N(v)$ is a known function and $\delta(v - v')$ is the Dirac delta function. Thus the H-function enters into all finite and semi-infinite slab problems. Detailed derivation of the expressions for $\psi(x, \mu)$ near interfaces will be published elsewhere together with applications which are currently in progress. It suffices to point out here that the H-function [1,2], for small μ , can be approximated by

$$\begin{aligned} H(\mu) = 1 + [\alpha_1 \mu + \alpha_2 \mu^2 + \dots] + [\beta_0 + \beta_1 \mu + \beta_2 \mu^2 + \dots] \mu \log \mu \\ + [\gamma_0 + \gamma_1 \mu + \dots] \mu^2 \log^2 \mu + \dots \end{aligned} \quad (2.15)$$

and does indeed exhibit logarithmic singularities in the derivatives at $\mu = 0$. The emerging distribution for the Milne problem is related to the H-function by

$$\psi(0, -\mu) = \frac{c_{v_0}}{2} \frac{H(\mu)}{v_0 - \mu} \quad (2.16)$$

and thus, for small μ , $\psi(0, -\mu)$ has the same behavior as (2.15).

For two adjacent half-spaces [22,54], B. Davison has shown that the angular distribution of the interfaces for the extended Milne problem of Fig. 1 is

$$\begin{aligned} \psi(0, -\mu) &= \frac{c_2}{2(1-c_2)} \frac{k_2}{1-k_2\mu} \frac{\psi_1(0, -\mu)}{\psi_2(0, -\mu)} \\ \psi(0, \mu) &= \frac{c_1}{2(1-c_1)} \frac{1-k_2\mu}{k_2} \frac{k_1^2}{1-\lambda_1^2\mu^2} \frac{\psi_2(0, -\mu)}{\psi_1(0, -\mu)} \end{aligned} \quad (2.17)$$

where ψ_i , c_i , k_i , $i = 1, 2$, are respectively the solution (2.16) of the Milne Problem, the number of secondaries per collision and the positive solution of a characteristic equation [15] in medium i . These solutions also exhibit the same behavior as (2.15) for

Fig. 1. Plane Interface

small μ . B. Davison [20,22] also points out that the boundary conditions for the interface between two homogeneous semi-infinite media, can be transferred without change to boundaries between media of finite but appreciable optical thickness.

3. VARIATIONAL METHODS AND RELATED APPLICATIONS

Variational Methods [32,33] have been popular since the early days of neutron transport theory [21,30,34-6,40]. The main reasons that they did not dominate throughout transport work are that they depend critically on the ability to choose suitable trial functions and the difficulty of carrying out the minimization process. Both of these problems are currently alleviated by the considerable progress made in Reactor Mathematics and in Computer Science and Technology. We shall show here the usefulness of incorporating the exact $x \log x$ behavior in the trial functions. The treatment below will be restricted to the self-adjoint integral transport equation

$$\Phi(x) = \int_D K(x, x') \Phi(x') dx' + S(x), \quad D = [a, b], \quad (3.1)$$

with a positive symmetric kernel $K(x, x') = \frac{c}{2} E_1(|x-x'|)$, a bounded source $S(x)$ so that $\int |S(x)| dx$ exists, and a finite or infinite interval $[a, b]$.

Note that $K(x, y) \geq 0$, is continuous in $D \times D$, is integrable as a function of y for each $x \in D$ and

$$\int_D |K(x+h, y) - K(x, y)| dy \rightarrow 0 \text{ as } h \rightarrow 0 \text{ for every } x \in D.$$

Consequently [46,52], the integral operator in (3.1) with $c \leq 1$ is a compact continuous operator.

Marshak [40] finds that the functional,

$$F_1[\Phi(x)] \equiv \frac{\int \Phi(x) \left\{ \Phi(x) - \int K(x, x') \Phi(x') dx' \right\} dx}{\left[\int \Phi(x) S(x) dx \right]^2} \quad (3.2)$$

is an extremum for the actual solution of (3.1). This functional was suggested by Schwinger, and Davison discussed the conditions under which F_1 is a minimum [21]. Le Caine [35] applied this functional to Milne problem for conservative scattering with trial function

$$\Phi(x) = x + x_0 [1 - AE_2(x) - BE_3(x)], \quad (3.3)$$

and with $x_0 = .7104457$. He evaluated A and B and thus obtained a good approximation in simple analytic form. The appearance of E_2 and E_3 in (3.3) indicate that Le Caine already incorporated correct logarithmic behavior near the vacuum boundary. Huang [30] suggested the simpler functional,

$$F_2[\Phi] \equiv \int \Phi(x) \left\{ \Phi(x) - \int K(x, x') \Phi(x') dx' \right\} dx - 2 \int \Phi(x) S(x) dx. \quad (3.4)$$

The advantage of (3.4) is that the simultaneous equations which determine the parameters of a trial function become linear. Huang improved on the work of Le Caine by using the trial function

$$\Phi(x) = x_0 + x + aE_2(x) + bE_3(x) + cE_4(x) + dE_5(x). \quad (3.5)$$

For the slab eigenvalue problem

$$\frac{1}{c} \Phi(x) = \frac{1}{2} \int_{-a}^a E_1(|x - x'|) \Phi(x') dx', \quad (3.6)$$

since the kernel is symmetric, positive, bounded from below [55], the Rayleigh-Ritz procedure yields the functional

$$F_3[\Phi(x)] \equiv \frac{\frac{1}{2} \int_{-a}^a \int_{-a}^a E_1(|x - x'|) \Phi(x) \Phi(x') dx' dx}{\int_{-a}^a \Phi^2(x) dx} \leq \lambda \equiv \frac{1}{c} \quad (3.7)$$

This functional has been extensively used in the literature [9,20,55] and does lead to a linear matrix eigenvalue problem for the parameters of the trial function [52].

Kourganoff [34] was the first to point out in the context of transport theory that several variational methods can be regarded as special cases of the method of weighted residuals. To illustrate this consider the equation

$$\Lambda \{\Phi_0(x)\} - \Phi_0(x) = 0 \quad (3.8a)$$

and define the residual

$$R(x) \equiv \Lambda \{\Phi(x)\} - \Phi(x), \quad \Phi(x) \text{ arbitrary.} \quad (3.8b)$$

Then, for a given family of approximate solutions to (3.8a), the best are those which annihilate the mean or weighted residual

$$(w(x), R(x)) \equiv \int_0^\infty w(x) R(x) dx, \quad (3.9)$$

for the widest possible variety of weighting functions $w(x)$. In practice one limits the choice of weights. In particular, the methods of least squares, collocation, Galerkin and moments correspond to the following choices of weight functions:

- (i) Least Squares: $w(x) = R(x)$ itself.

(ii) Collocations: $\{w(x) = \delta(x-x_i); i=1, \dots, n\}$, i.e., the weight functions are selected to be δ -functions. Such a choice results, for example, in the requirement that the trial functions for the scalar flux $\Phi(x)$ should satisfy (3.8a) identically for all $x = x_i$, $i = 1, \dots, n$.

(iii) Galerkin: $w(x) = u_i(x)$, $i=1, \dots, n$; u_i 's are the members of the basis of approximating functions selected. For example, corresponding to the trial function (2.10), the weight functions are

$$w(x) = 1, x, E_2(x), \dots, E_n(x). \quad (3.10)$$

(iv) Moments: In this case the residuals are made orthogonal to a system which need not be the same as that of the approximating function. The most popular system of weight functions used in this connection are $\{x^k, k=1, \dots, n\}$.

3A. GALERKIN AND ITERATIVE GALERKIN APPROXIMATION TO THE MILNE PROBLEM

The treatment here will be restricted to the Milne problem with conservative scattering ($c = 1$)

$$\Lambda_x \{\Phi(x')\} \equiv \frac{1}{2} \int_0^\infty E_1(|x - x'|) \Phi(x') dx' = \Phi(x) \quad (3.11)$$

The case $c = 1$ (in general $c \leq 1$) is in fact desirable for computational tests as it stands for the extreme case of a critical semi-infinite homogeneous medium occupying the half space $x \geq 0$. For this Milne problem, radiation or neutrons enter the medium from its deep interior ($x = \infty$) and eventually leave the free non-reentering surface ($x = 0$). As mentioned above, a good approximation to the scalar flux which embodies in part the correct analytic behavior at the vacuum boundary is (a multiple of)

$$\Phi(x) = x + a_1 + a_2 E_2(x) + \dots + a_n E_n(x). \quad (3.12)$$

Kourganoff [34] used this approximation for $n = 6$ and evaluated the expansion coefficients by the collocation method. He selected the first 29 coordinates

given in Table I. This choice has the advantage of giving more weight to the region close to the interface and less to the deep interior. He refrained from applying the Galerkin method for two reasons: the basis in (3.12) is not an orthogonal basis and furthermore, it is incomplete for representing arbitrary solutions. Since the main objective of this paper is to study the effect of properly accounting for the exact analytic behavior near interfaces in different computational procedures, the Galerkin method, which is very popular these days, is an illustrative case. It suffices to consider the transient part $Q(x)$ of the flux,

$$\Phi(x) \equiv x + Q(x). \quad (3.13)$$

Substituting (3.13) into (3.11) yields the inhomogeneous integral transport equation

$$Q(x) = \Lambda_x \{Q(x')\} + \frac{1}{2} E_3(x) \quad (3.14)$$

The approximation (3.12) is then

$$Q(x) = a_1 + a_2 E_2(x) + \dots + a_6 E_6(x), \quad n = 6 \quad (3.15)$$

with basis

$$u_i = 1, E_2(x), E_3(x), \dots, E_6(x) \quad (3.16)$$

Now defining an inner product

$$(u, v) \equiv \int_0^\infty u(x)v(x)dx, \quad (3.17)$$

the Galerkin procedure as applied to (3.14)-(3.16) yields the system

$$(u_i, \frac{1}{2}E_2(x)a_1 + [E_2 - \Lambda_x \{E_2\}]a_2 + \dots + [E_6 - \Lambda_x \{E_6\}]a_6) = \frac{1}{2}(u_i, E_3), \quad i=1, \dots, 6, \quad (3.18a)$$

or, in matrix notation

$$\underline{A}\underline{a} = \underline{b}. \quad (3.18b)$$

It is straightforward to show that the matrix A is a symmetric 6x6 matrix with

$$\begin{aligned}
A_{11} &= \frac{1}{4}; A_{1i} = A_{i1} = \frac{1}{2} (E_2, E_i) \equiv \frac{1}{2} J_{2i}, i > 1; \\
A_{ij} &= A_{ji} = (E_i, E_j - \Lambda_x \{E_j\}) \equiv J_{ij} - B_{ij}, i > 1, j > 1; \\
b_1 &= \frac{1}{6}; b_i = \frac{1}{2} (E_i, E_3) \equiv \frac{1}{2} J_{i3}.
\end{aligned} \tag{3.18c}$$

The integrals J_{nm} and B_{nm} in (3.18c) have in fact been evaluated analytically and the results are tabulated by Kourganoff [34, Appendix II] in connection with his application of the collocation method to this same problem.

The above very simple matrix equation (3.18b) was solved on the CDC-6400 to yield the following coefficients

$$\begin{aligned}
a_1 &= .7104485, & a_2 &= -.2828492, & a_3 &= .6174647 \\
a_4 &= -1.0966372, & a_5 &= 1.2057855, & a_6 &= -.4742459
\end{aligned} \tag{3.19}$$

The transient flux, computed from (3.15) on the basis of the above coefficients is compared in Table I with its exact value [34]. The results are reasonably good with an average error about .0004%.

The main point to note from Table I is that the solution by the Galerkin procedure with the basis (3.16) is least accurate at $x = 0$, the vacuum interface. This is in spite of the fact that (3.16) incorporates a good part of the analytic behavior near the interface. It should be pointed out however that, because the analytic behavior is accounted for in part, the result near the origin is superior [34,48-50] to that with trial function

$$Q(x) = a_1 + a_2 e^{-\beta_2 x} + \dots + a_6 e^{-\beta_6 x}$$

which has more degrees of freedom, and is also superior to the finite element results presented below. Nevertheless, the completed results are considerably improved and especially near the origin by a first iteration*, i.e. by substituting the solution (3.15) with the coefficients already evaluated, into the right-hand-side of the integral transport equation (3.14). The integrals and identities

*This corresponds to "natural interpolations" in the Nyström method.

needed for this iteration are given by Kourganoff [34]. The iterated Galerkin results are also presented in Table I.

The merging vector flux or angular distribution for the Milne problem is essentially a Laplace transform of the net flux and is

$$\psi(0, -\mu) = \frac{c}{2\mu} \int_0^{\infty} \Phi(x) e^{-x/\mu} dx, \quad c = 1. \quad (3.19)$$

The transformation (3.19) is carried out based on results by Kourganoff. The lower part of Table I gives the emerging angular distribution corresponding to the iterated Galerkin results for $\Phi(x)$. The average error in this angular distribution is 0.0000001.

3B. FINITE ELEMENT METHOD

Considerable numerical experimentations were carried out with the collocation method as applied to both the Milne integral equation and to the corresponding integral equation for conservation of current [34], and comparisons were made with the finite element method as applied to the Huang functional Eq. (3.4). In general the results with the Huang functional were superior and therefore the results for collocation will be omitted for brevity. A sample of the collocation results has already been presented at a SIAM meeting [7,8].

The Milne problem for conservative scattering is again the basis for the present application, except that attention is focussed at the part occupied by the medium between $x_1 = 0$ and $x_N = 10$ in units of the mean free path. In fact, a slab of thickness 10 mean free paths is considered to be extremely thick. Thus, an excellent approximation for the flux is

$$\Phi(x) = x + x_0 + q(x) \quad (3.20a)$$

$$q(x) = 0 \text{ for } x \geq x_N \equiv 10 \quad (3.20b)$$

and x_0 the extrapolation distance. Rather than assume x_0 known, it will be

determined from the variational method by requiring (3.20) to satisfy the first moment equation of the integral transport equation. This constraint is [34]

$$\int_0^{x_N} q(x) E_2(x) dx + \frac{x_0}{2} = \frac{1}{3}. \quad (3.21)$$

Now, substituting (3.20) into the integral transport equation and rearranging terms yield

$$q(x) = \frac{1}{2} \int_0^{x_N} E_1(|x - x'|) q(x') dx' - \frac{x_0}{2} E_2(x) + \frac{1}{2} E_3(x). \quad (3.22)$$

The corresponding Huang functional (3.4) for this source problem (3.22) is

$$\begin{aligned} H[q] = & \int_0^{x_N} q(x) \left\{ q(x) - \frac{1}{2} \int_0^{x_N} E_1(|x - x'|) q(x') dx' \right\} dx \\ & - \int_0^{x_N} q(x) [E_3(x) - x_0 E_2(x)] dx. \end{aligned} \quad (3.23)$$

The finite element method [45] as applied to the variational functional (3.23) removes the difficulty of finding base functions which satisfy the completeness and smoothness requirements over the entire domain. The slab $[0,10]$ is subdivided into $N-1$ segments $[x_1, x_2], \dots, [x_{N-1}, x_N]$ and in each segment, the transient flux $q(x)$ is approximated by a piecewise polynomial. For the present application piecewise linear and cubic "spline" approximations are adopted (results for Hermite polynomials are omitted for brevity). Spline approximations [13,28] have the advantage here of both providing sufficient smoothness in the interior of the slab and reducing the order of the resulting matrix equation to $N \times N$, N representing the number of mesh points (or knots). There are various representations for splines and a convenient one here is

$$q(x) = \sum_{i=1}^{N-1} a_i u_i(x) \quad (3.24)$$

where $u_i(x)$ are cardinal spline bases. For the linear case, these are the

"chapeau functions"

$$\begin{aligned}
 u_1(x) &= (x_2 - x)/x_2 \quad \text{for } x \in [0, x_2] \\
 &= 0 \quad \text{otherwise} \\
 u_i(x) &= (x - x_{i-1})/(x_i - x_{i-1}), \quad x \in [x_{i-1}, x_i] \\
 &= (x_{i+1} - x)/(x_{i+1} - x_i), \quad x \in [x_i, x_{i+1}] \\
 &= 0 \quad \text{otherwise.} \\
 i &= 2, \dots, N-1
 \end{aligned} \tag{3.25}$$

Similarly, for cubic splines

$$\begin{aligned}
 u_i(x) &= \sum_{j=i-3}^{i+1} (x - x_j)_+^3 / \prod_{\substack{l=i-3 \\ l \neq j}}^{i+1} (x_j - x_l), \quad x \in [x_{i-3}, x_{i+1}] \\
 &= 0 \quad \text{otherwise}
 \end{aligned} \tag{3.26}$$

where

$$\begin{aligned}
 (x - x_j)_+^3 &= (x - x_j)^3 \quad \text{for } x \geq x_j \\
 &= 0 \quad \text{for } x \leq x_j
 \end{aligned} \tag{3.27}$$

Equation (3.26) holds for $i=4, \dots, N-1$ and can be extended to hold for $i=1, 2, 3$ by defining new coordinates x_0, x_{-1}, x_{-2} , for example

$$x_0 = -x_2, \quad x_{-1} = -x_3, \quad x_{-2} = -x_4. \tag{3.28}$$

Clearly for $N \leq 5$, the above representation for cubic splines is cumbersome and can be replaced by a representation in terms of the truncated power functions (3.27). For small N , computations were carried out for both of the above cubic spline representations and the results were identical as one would expect.

The variational problem consists of substituting the respective (linear or cubic) trial function expression (3.24) into the functional H of (3.23), and setting the derivatives of H with respect to the expansion coefficients a_i equal to zero. The resulting equations are

$$\sum_{i=1}^{N-1} a_i \left[2 \int_0^{x_N} u_i(x) u_l(x) dx - \int_0^{x_N} dx \int_0^{x_N} dx' E_1(|x-x'|) u_i(x) u_l(x') \right] + x_0 \int_0^{x_N} dx E_2(x) u_l(x) = \int_0^{x_N} dx E_3(x) u_l(x), \quad l=1, \dots, n-1 \quad (3.29a)$$

Substituting (3.24) into the constraint (3.21) yields

$$\sum_{i=1}^{N-1} a_i \int_0^{x_N} dx E_2(x) u_i(x) + x_0/2 = 1/3. \quad (3.29b)$$

Since x_0 is assumed to be unknown, it is convenient to define

$$a_N \equiv x_0; \quad (3.29c)$$

then the system (3.29a,b) can be written in matrix notation

$$C \underline{a} = \underline{b}. \quad (3.29d)$$

Note from Eqs. (3.29a,b,c) that the matrix C is symmetric. The integrations in (3.29a,b) needed to evaluate the elements of the matrix C corresponding to (3.25) or (3.26) can be carried out analytically using tables already published in the literature [26,27,37]. These integrations are straightforward but numerous, and will be omitted here. A subroutine for evaluating the exponential integral is available [18]. Equation (3.29d) is solved for the expansion coefficients a_i by Gaussian elimination utilizing the symmetry of the matrix C . The representation (3.24) is then used to interpolate for the transient flux

$$Q(x) = \Phi(x) - x = q(x) + x_0 = q(x) + a_N \quad (3.30)$$

at the 32 preassigned coordinates given in the first column of Table II. This approach provides a consistent comparison with known exact results [34]. The computed results for $Q(x)$ for the linear case are presented in the first five columns of Tables IIa,b,c,d, and for the cubic case in the corresponding columns of Tables IIIa,b,c,d respectively. The four parts of each table correspond to $N = 4, 8, 16$ and 32.

Initially the computations were carried out for a uniform mesh and as the tables show, the results are poorest close to the vacuum interface $x=0$. These results are improved by

a) increasing the mesh size N (up to a certain level not clear from the tables, where either round off or ill conditioning of the matrix equation comes to play)

b) increasing the order of the approximation from linear to cubic (further comments on this point will be made in the next section).

Further improvement specifically near the origin (see also [47]) can be achieved by

c) suitably varying the mesh size (dense mesh close to the interface and sparse in the regular region) as is indeed shown in Tables II and III

d) iterating the computed results once (substituting for $q(x')$ on the right-hand-side of (3.22) to reevaluate the left-hand-side $q(x)$ as was done with the Galerkin results of Sect. 3A. Such an iteration is sometimes referred to as the method of successive approximation [35], and is essentially the same as natural interpolation. The effectiveness of such an iteration has already been demonstrated in Sect. 3A and will not be pursued here.

e) adjoining to the polynomial approximation in the first interval appropriate terms representing the analytic behavior near the interface. This is the objective in the following two sections.

3C. ADJOINING SINGULAR TERMS NEAR THE INTERFACE

From Sect. 2, it follows that the flux near the interface assumes the form

$$\begin{aligned} \Phi(x) = & a_0 + a_1x + a_2x^2 + a_3x^3 + b_1x \log x + b_2x^2 \log x + b_3x^3 \log x \\ & + c_1x^2 \log^2 x + c_2x^3 \log^2 x + d_1x^3 \log^3 x + \dots \end{aligned} \quad (3.31)$$

A proper way to adjoin the logarithmic terms to the finite element method of the previous section is in the form of "add-on terms" in the first interval.

For the linear approximation the add-on term is

$$\begin{aligned} u_{n+1}(x) &= x \log x - x \log x_2, \quad x \in [0, x_2] \\ &= 0 \quad \text{otherwise.} \end{aligned} \quad (3.32)$$

For the cubic spline approximation, again for $x \in [0, x_2]$

$$\left. \begin{aligned} u_{n+1}(x) &= x \log x - x \log x_2 + \frac{3}{2}x - 2 \frac{x^2}{x_2} + \frac{x^3}{2x_2^2} \\ u_{n+2}(x) &= x^2 \log x - x^2 \log x_2 + \frac{1}{2}x_2x - \frac{x^3}{2x_2} \\ u_{n+3}(x) &= x^3 \log x - x^3 \log x_2 - \frac{1}{2}x_2^2x + 2x_2x^2 - \frac{3}{2}x^3 \\ u_{n+4}(x) &= x^2 \log^2 x - x^2 \log^2 x_2 + xx_2 \log x_2 - \frac{x^3}{x_2} \log x_2 - xx_2 + 2x^2 - \frac{x^3}{x_2} \\ u_{n+5}(x) &= x^3 \log^2 x - x^3 \log^2 x_2 - xx_2^2 \log x_2 + 4x^2x_2 \log x_2 - 3x^3 \log x_2 \\ &\quad - x_2^2x + 2x_2x^2 - x^3 \\ u_{n+6}(x) &= x^3 \log^3 x - x^3 \log^3 x_2 - \frac{3}{2}xx_2^2 \log^2 x_2 + 6x^2x_2 \log^2 x_2 - \frac{9}{2}x^3 \log^2 x_2 \\ &\quad - 3xx_2^2 \log x_2 + 6x^2x_2 \log x_2 - 3x^3 \log x_2. \end{aligned} \right\} \quad (3.33)$$

The function $u_{n+1}(x)$ of (3.32) vanishes at both $x = 0$ and $x = x_2$. This property is also true of $u_{n+i}(x)$ of (3.33), i.e.

$$u_{n+i}(0) = u_{n+i}(x_2) = 0 \quad (3.33a)$$

In addition

$$u'_{n+i}(x_2) = u''_{n+i}(x_2) = 0, \quad i = 1, \dots, 6. \quad (3.33b)$$

Thus the above add-on terms preserve the properties of linear and cubic spline approximations in the slab $[0, x_N]$ under consideration. The functions $u_{n+1}(x)$ of (3.32) and the first of (3.33) were successfully implemented in the finite element schemes of the previous section and the computed results are presented in Tables II and III. The remaining functions in (3.33) were also partially

implemented; the procedure* is involved and the subroutine constructed is too long and, to date, not completely debugged. The outcome of this subroutine will not change the conclusions to be drawn here, namely the beneficial effect of the add-on terms. The reason these functions in (3.33) were not omitted is to recommend their use in conjunction with solutions of the integrodifferential equation where the coupling is weaker than in the integral equation (sparse matrices result, rather than full matrices) and the singular kernel $E_1(|x - x'|)$ does not arise.

Tables II and III show the overall improvement when the logarithmic add-on terms u_{n+1} are incorporated in the finite element schemes. The error at the origin and the average error for the uniform mesh case are summarized in Tables IV and V. These tables show that for the uniform mesh case

a) an improvement factor of between 1.3 and 15.8 results when the logarithmic term is used

b) the improvement factor increases as the size of the first interval is decreased up to a certain size (as the mesh spacing is decreased; shown in the third column of Table V and the rest of the computation omitted for brevity). Later, the contribution of the logarithmic terms (to the functional and to the results) becomes negligible as the size over which it is implemented shrinks to zero. This observation indicates that for a given order of approximation there is an optimum range where the additional labor involved in adding the logarithmic term is fully justified.

* Several new and old identities and quadrature formulae for logarithms and exponential integrals were derived to supplement the references [19,24,26,27,37,51]. Because of their anticipated usefulness for two and three dimensional neutronics problems and for astrophysics problems, they will be compiled and issued elsewhere.

c) the addition of one logarithmic term is often better than doubling the number of mesh points.

c) the corrected linear approximation with the logarithmic term is better than the cubic approximation without this term.

3D. EIGENVALUE AND EXTENDED EIGENVALUE PROBLEM

The critical mean secondaries per collision c for a bare slab $x \in [-a, a]$, is the inverse of the largest eigenvalue of the self-adjoint integral equation (3.6). For a given trial function

$$\phi(x) = \sum_{i=1}^n \alpha_i u_i(x), \quad (3.34)$$

maximizing the Rayleigh-Ritz functional (3.7) is equivalent to finding the maximum eigenvalue of the matrix equation [52]

$$E \underline{a} - \lambda U \underline{a} = 0. \quad (3.35)$$

Here E and U are symmetric positive definite matrices with elements

$$E_{ij} = \frac{1}{2} \int_{-a}^a \int_{-a}^a E_1(|x - x'|) u_i(x) u_j(x') dx dx' \quad (3.36a)$$

$$U_{ij} = \int_{-a}^a u_i(x) u_j(x) dx \quad (3.36b)$$

Because of symmetry, the simplest possible polynomial trial functions are

$$\phi_1 = 1, \quad (3.37)$$

$$\phi_2 = 1 - \beta x^2. \quad (3.38)$$

Odd polynomials are not admissible. The eigenvalue problems (3.35)-(3.36) are elementary in these cases and the corresponding critical mean secondaries per collision are

$$c_1 = U_{11}/E_{11} = 4a/[4a - 1 + 2 E_3(2a)] \quad (3.39)$$

and

$$c_2 = 2(U_{11}U_{22} - U_{12}^2)/(U_{11}E_{22} - 2U_{12}E_{12} + U_{22}E_{11} + [(U_{11}E_{22} - 2U_{12}E_{12} + U_{22}E_{11})^2 - 4(U_{11}U_{22} - U_{12}^2)(E_{11}E_{22} - E_{12}^2)]^{\frac{1}{2}}. \quad (3.40)$$

The integrals (3.36a,b) corresponding to (3.38) and (3.40) have simple expressions omitted for brevity. Values of c_1 and c_2 corresponding to different values of the slab half-thickness "a" appear in the second and fourth columns of Table VI. The exact results appearing in the last column of this table (except for $a = .7366, 1.2893$ and 5.6655 , computed by Bell et al. [9]) were computed by Bennett [10] by a combination of discrete ordinate method and a $1/N^2$ extrapolation. Actual values of N Bennett used, go up to $N = 161$.

Table VI shows that c_1 corresponding to $\Phi_1 = 1$ has errors in the third significant figures, while c_2 corresponding to $\Phi_2 = 1 - \beta x^2$ has errors in the sixth significant figures. In fact the extremely high accuracy of the eigenvalue corresponding to this simple trial function Φ_2 renders an elaborate procedure such as Bennett's unjustifiable. Davison [20] and Bell et al. [9] report that this trial function Φ_2 has been used to compute the critical c for slabs and the results are accurate to three or four significant figures, for thin slabs, and are not as accurate for very thick slabs (the last column in Table VI corresponds to a very thick slab). Table VI implies that Davison and Bell were too conservative in their estimates of accuracy and range of applicability of $\Phi_2(x)$ for the above eigenvalue problem.

A singular term

$$v(x) = (1 - \frac{x}{a})\log(1 - \frac{x}{a}) + (1 + \frac{x}{a})\log(1 + \frac{x}{a}), \quad (3.41)$$

which approximates the true behavior near the vacuum interface, was first adjoined to the trial function Φ_1 and then to the trial function Φ_2 . The corresponding results for the inverse eigenvalue appear in columns 3 and 5 of Table VI.

Note the resulting improvement when the singular term $v(x)$ is added to either trial function, specifically for thin slabs. For thick slabs, the improvement of column 5 over column 4 is negligible or nonexistent. This suggests that $v(x)$ approximates well the local behavior of the flux near the interface but not the global behavior throughout thick slabs.

The above trial functions were also used in conjunction with the following extended eigenvalue problem. Consider a slab of fuel in $[-a, a]$ surrounded by a moderator and assume that the number of secondaries per collision are related by

$$C_M = \alpha_o^2 C_F, \quad (3.42)$$

where α_o^2 is a preassigned fraction. The integral transport equation in this case is

$$\Phi(x) = \frac{C_F}{2} \int_{-a}^a E_1(|x - x'|) \Phi(x') dx' + \frac{C_F \alpha_o^2}{2} \int_R E_1(|x - x'|) \Phi(x') dx', \quad (3.43a)$$

$$R = (-\infty, -a] \cup [a, +\infty).$$

Now define a new function $\alpha(x) > 0$ such that

$$\begin{aligned} \alpha^2(x) &= \alpha_o^2 \quad \text{for } |x| > a \\ &= 1 \quad \text{for } |x| < a. \end{aligned} \quad (3.43b)$$

Equation (3.43a) multiplied by $\alpha(x)$ can then be written in the form

$$\alpha(x) \Phi(x) = \frac{C_F}{2} \int_{-\infty}^{+\infty} \alpha(x) E_1(|x - x'|) \alpha(x') \alpha(x') \Phi(x') dx' \quad (3.44)$$

This integral equation for $\alpha(x) \Phi(x)$ is self-adjoint and has the same properties as for a finite slab. By the Rayleigh-Ritz procedure, the following maximum principle holds

$$F[\Phi] = \frac{\frac{1}{2} \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dx' E_1(|x - x'|) \alpha^2(x) \alpha^2(x') \Phi(x) \Phi(x')}{\int_{-\infty}^{+\infty} dx \alpha^2(x) \Phi^2(x)} \leq \lambda = \frac{1}{C_F}. \quad (3.45)$$

The trial functions (3.37), (3.38) can be extended in a number of ways; for the present application we choose

$$\begin{aligned}\Phi_1(x) &= 1 \quad \text{for } |x| \leq a \\ &= E_2(|x| - a) \quad \text{for } |x| \geq a\end{aligned}\tag{3.46}$$

$$\begin{aligned}\Phi_2(x) &= 1 - \beta x^2 \quad \text{for } |x| \leq a \\ &= (1 - \beta a^2) E_2(|x| - a)\end{aligned}\tag{3.47}$$

Note that the trial functions $\Phi_1(x)$ and $\Phi_2(x)$ are continuous. Furthermore $E_2(|x| - a)$ behaves like $[(|x| - a) \log (|x| - a) + \text{polynomial}]$ as $x \rightarrow a_+$. As for the bare slab case, maximizing the functional $F[\Phi]$ of (3.45) is equivalent to the matrix eigenvalue problem of Eqs. (3.35), (3.36a,b). The only exception here is that the interval of integration $[-a, a]$ in (3.36a,b) is extended to the infinite interval $(-\infty, +\infty)$. The values of the criticality constants C_F corresponding to the trial function (3.46) and (3.47) are presented in Table VII together with the results when $v(x)$ of (3.41) is adjoined to these trial functions. The usefulness of the logarithmic terms, especially for thin central slabs, is evident from this table.

4. TRANSPORT IN FINITE MEDIA

As mentioned above, higher order approximations are not practical for realistic two and three-dimensional problems. Furthermore, the mesh size for low order approximations cannot be made too small to achieve certain accuracies because of computer storage and other practical considerations. The analysis of the previous sections suggests that for finite media one ought to seek and incorporate the analytic behavior near interfaces, edges and corners. Contributions towards understanding singularities of transport solution due to corners, by analyzing characteristic surfaces of the differential part of the transport equation (1.1), have already been made by Arkuszewski et al. [5], Bareiss [7], Kellogg [6] and others [11]. The behavior near edges and corners can also in principle be achieved by transform techniques [Erdmann at UCLA, Leonard at Stanford, Mc Cormick at Univ. of Washington; see Bibliography on

Neutral Particle Transport Theory, LA-4287-MS], but the analytic complexities have not as yet been resolved.

A different approach based on stochastic considerations and the Neumann series expansion is recommended here. It can be shown for finite subcritical media, that the vector flux of the transport equation

$$\Omega \cdot \nabla \psi(\underline{x}, \Omega) + \psi(\underline{x}, \Omega) = \frac{c}{4\pi} \int_{\{\Omega\}} \psi(\underline{x}, \Omega) d\Omega + S(\underline{x}, \Omega), \quad (4.1)^*$$

can be expanded in the form

$$\psi(\underline{x}, \Omega) = \sum_{n=0}^{\infty} c^n \psi_n(\underline{x}, \Omega). \quad (4.2)$$

$c^n \psi_n(\underline{x}, \Omega)$ stands for the density of particles at $\underline{x}$ moving in the direction Ω due to particles that have suffered exactly n collisions after leaving the source.** Substituting (4.2) into (4.1) and equating coefficients of equal power of c yield

$$\Omega \cdot \nabla \psi_0(\underline{x}, \Omega) + \psi_0(\underline{x}, \Omega) = S(\underline{x}, \Omega) \quad (4.3a)$$

$$\Omega \cdot \nabla \psi_n(\underline{x}, \Omega) + \psi_n(\underline{x}, \Omega) = \frac{1}{4\pi} \int \psi_{n-1}(\underline{x}, \Omega') d\Omega', \quad n \geq 1 \quad (4.3b)$$

Each of Eqs. (4.3a,b) with known right-hand-side stands for transport in a purely absorbing medium. A necessary constraint, is that the flux should vanish identically in the absence of sources. This constraint implies that only the particular solutions for each of Eqs. (4.3a,b) are needed, and the solutions to the homogeneous part do not contribute. The solutions to (4.3a,b) are then

$$\psi_0(\underline{x}, \Omega) = \psi_0(\underline{x} - r_0 \Omega, \Omega) e^{-r_0} + \int_0^{r_0} S(\underline{x} - r\Omega, \Omega) e^{-r} dr \quad (4.4)$$

*The restriction to isotropic scattering is merely for the purpose of illustration and can be relaxed.

**Or belong to the n th generation, if multiplication is allowed; the term source here includes surface sources to represent incident fluxes.

and

$$\psi_n(\underline{x}, \Omega) = \int_0^{r_0} dr e^{-r} \frac{1}{4\pi} \int_{\{\Omega'\}} \psi_{n-1}(\underline{x} - r\Omega, \Omega') d\Omega', \quad n \geq 1 \quad (4.5)$$

where

$$r_0 = \{r_0(\underline{x}, \Omega) : (\underline{x} - r_0\Omega) \text{ a point on the boundary}\}. \quad (4.6)$$

For a given external source and incident flux, the first few solutions $\psi_0, \psi_1, \psi_2, \dots$ of (4.4) and (4.5) can be computed to desirable accuracies. In most situations, these first few terms describe the dominant part of the angular distribution; as such they supply the desired exact behavior near difficult regions such as edges, corners and wedges. As an illustration, consider a unit square with a uniform isotropic source $S = 1$ and no incident flux. Equation (4.4) then yields

$$\psi_0(\underline{x}, \Omega) = 1 - e^{-r_0(\underline{x}, \Omega)}. \quad (4.7)$$

The corresponding solution for a slab $[-a, a]$ as derived by Abu-Shumays and Miller (to be published) is

$$\psi_0(x, \mu) = 1 - \text{Exp.} \left\{ -\frac{a}{|\mu|} - \frac{x}{\mu} \right\} \quad (4.8)$$

and the corresponding scalar flux in this case is

$$\Phi_0(x) = 2 - E_2(a + x) - E_2(a - x) \quad (4.9)$$

We note here that the scalar flux $\Phi_0(x)$ of (4.9) already shows the $(a \pm x)$ $\log(a \pm x)$ singularity of the exact solution near an interface.

Application of the above procedure to determine the behavior of the neutron density near edges, wedges and corners will be postponed to a future publication which would include implementation in numerical schemes.

Acknowledgments

It is a pleasure to thank Garrett Birkhoff for stimulating discussions providing motivation for the present work. Special thanks are also due to C. Mark for his courtesy in clarifying his 1944 results for a plane interface and indicating how to extend them.

REFERENCES

1. I. K. Abu-Shumays, "The Albedo Problem and Chandrasekhar's H-Function," Nucl. Sci. Eng., 26, 430-433 (1966).
2. I. K. Abu-Shumays, "The Albedo Problem and Chandrasekhar's H-Function II," Nucl. Sci. Eng., 27, 607-608 (1967).
3. I. K. Abu-Shumays, "Finite Element Method in Slab Geometry," (1971 Technical Memo; not published).
4. F. T. Adler, "Theory of the Diffuse Reflection of Neutrons by a Plane Surface of a Semi-Infinite Capturing Medium," National Research Council of Canada, Report MT-72, 1944.
5. J. Arkuszewski, T. Kulikowska and J. Mika, "Effect of Singularities on Approximations in SN Methods," Institute of Nuclear Research, Swierk, Otwock, Poland, Report "P" No. 1405/XXI/PR, 1972.
6. I. Babuska and R. Kellogg, "Mathematical and Computational Problems in Reactor Calculations," Technical Note BN-760, Univ. of Maryland, College Park, Maryland, 1973.
7. E. H. Bareiss, "Finite Elements in Neutron Transport Theory," Paper presented at an AEC headquarters meeting, May 1973 (to be published).
8. E. H. Bareiss and I. K. Abu-Shumays, "Numerical Solution of the Neutron Transport Equation," Numerical Solution of Integral Equations in the Physical Sciences, B. Noble and L. Rall (eds.), SIAM (to be published).
9. G. I. Bell and S. Glasstone, Nuclear Reactor Theory, Van Nostrand Reinhold Co., New York, 1970.
10. J. H. Bennett, "Integral Equation Methods for Transport Problems. Some Numerical Results," Num. Math., 6, 49-54 (1964).
11. G. Birkhoff, "Angular Singularities of Elliptic Problems," J. Approx. Theory, 6, 215-230 (1972).
12. G. Birkhoff and I. K. Abu-Shumays, "Harmonic Solutions of Transport Equations," J. Math. Anal. Appl., 28, 211-221 (1969).
13. G. Birkhoff and C. R. DeBoor, "Piecewise Polynomial Interpolation and Approximation," Approximation of Functions, H. L. Garabedian, editor, Elsevier Pub. Co., N. Y. 1965.
14. I. W. Busbridge, The Mathematics of Radiative Transfer, Cambridge Univ. Press, 1960.
15. K. M. Case, F. DeHoffman and G. Placzek, Introduction to the Theory of Neutron Diffusion, Los Alamos Scientific Laboratory, 1953.
16. K. M. Case and P. F. Zweifel, Linear Transport Theory, Addison-Wesley, 1967.

17. S. Chandrasekhar, Radiative Transfer, Oxford, 1950.
18. W. J. Cody and H. C. Thacher, Jr., "Rational Chebyshev Approximations for the Exponential Integral $E_1(x)$," Math. Comp., (July 1968).
19. H. T. Davis, The Summation of Series, Principia Press of Trinity Univ., Texas, 1962.
20. B. Davison, Neutron Transport Theory, Oxford, 1957; see also Report LT-18.
21. B. Davison, "A Remark on the Variational Method," Phys. Rev., 71, 694-697 (1947).
22. B. Davison, "Angular Distribution of Neutrons at the Interface of Two Adjoining Media," Canad. J. Res., A28, 303-314 (1950).
23. B. Davison and G. Placzek, "Milne's Problem with Capture and Production," National Research Council of Canada Report MT-131 (No Date Given).
24. A. Erdélyi, "General Asymptotic Expansions of Laplace Integrals," Arch. Rat. Mech. And Anal., 7, 1-20 (1961).
25. G. Fix, "Higher-Order Rayleigh-Ritz Approximations," J. Math. Mech., 18, 645-657 (1969).
26. M. Geller, "A Table of Integrals Involving Powers, Exponentials, Logarithms, and the Exponential Integral," Jet Propulsion Laboratory, Technical Report No. 32-469, 1963.
27. M. Geller and E. W. Ng, "A Table of Integrals of the Exponential Integral," J. Research, NBS, 73B, 191-210 (1969).
28. J. G. Hayes, Editor, Numerical Approximation of Functions and Data, The Athlone Press, 1970.
29. E. Hopf, Mathematical Problems of Radiative Equilibrium, Cambridge tract No. 31, Cambridge Univ. Press, 1934.
30. S. Huang, "The Variational Method for Problems of Neutron Diffusion and of Radiative Transfer," Phys. Rev., 88, 50-52 (1952).
31. C. Kaplan, "On Some Functions Related to the Exponential Integral," Aerospace Research Laboratories, Report ARL 70-0097, June 1970.
32. S. Kaplan, "Variational Methods in Nuclear Engineering," in Advances in Nuclear Science and Technology, Vol. 5, Academic Press, 1969.
33. S. Kaplan and J. A. Davis, "Canonical and Involutory Transformations of the Variational Problems of Transport Theory," Nucl. Sci. Eng., 28, 166-176 (1967).
34. V. Kourganoff, Basic Methods in Transfer Problems, Oxford, 1952.
35. J. LeCaine, "Application of a Variational Method to Milne's Problem," Phys. Rev., 72, 564-566 (1947).

36. J. LeCaine, "The Neutron Density Near a Plane Surface in a Capturing Medium," Canad. J. Res., A28, 242-267 (1950).
37. J. LeCaine, "A Table of Integrals Involving the Functions $E_n(x)$," National Research Council of Canada Report MT-118, 1945.
38. E. E. Lewis, "A Polynomial Approximation for Integral Transport Calculations," J. Nucl. Energy, 23, 87-97 (1969).
39. C. Mark, "The Neutron Density Near a Plane Surface," Phys. Rev., 72, 558-564 (1947); see also MT-50 (1944).
40. R. E. Marshak, "The Variational Method for Asymptotic Neutron Densities," Phys. Rev., 71, 688-693 (1947).
41. N. McCormick and I. Kuscer, "Singular Eigenfunction Expansions in Neutron Transport Theory," Advances in Nuclear Science and Technology, E. J. Henley and J. Lewins, Eds., Vol. 7, 1972.
42. D. H. Menzel and H. K. Sen, "Transfer of Radiation," As. J., 110, 1-11 (1949).
43. D. H. Menzel and H. K. Sen, "Transfer of Radiation II," As. J., 113, 482-9 (1951).
44. D. H. Menzel and H. K. Sen, "Transfer of Radiation III," As. J., 113, 490-495 (1951).
45. W. F. Miller, Jr., E. E. Lewis and E. C. Rossow, "The Application of Phase-Space Finite Elements to the One-Dimensional Neutron Transport Equation," Nucl. Sci. Eng., (to appear).
46. A. Pazy and P. H. Rabinowitz, "A Nonlinear Integral Equation with Applications to Neutron Transport Theory," Arch. Rat. Mech. and Anal., 32, 226-246 (1969).
47. J. Pitkäranta and P. Silvennoinen, "Computational Experimentation on The Finite Element Method in Bare Slab Criticality Calculations," Helsinki Univ. of Technology Report TKK-F-A177, Finland 1972.
48. G. Placzek, "The Neutron Density Near a Plane Surface," National Research Council of Canada, MT-16, 1944.
49. G. Placzek, "The Angular Distribution of Neutrons Emerging from a Plane Surface," Phys. Rev., 72, 556-558 (1947).
50. G. Placzek and W. Seidel, "Milne's Problem in Transport Theory," Phys. Rev., 72, 550-555 (1947).
51. A. Reiz, "Quadrature Formulae for the Numerical Calculation of Mean Intensities and Fluxes in a stellar atmosphere," Ark. Astron., 1, paper 13, 147-153 (1950).
52. I. Stakgold, Boundary Value Problems of Mathematical Physics, Vol. I, Macmillan, New York, 1967.

53. D. W. N. Stibbs and R. E. Weir, "On the H-Function for Isotropic Scattering," Roy. Ast. Soc. Monthly Notices, 119, 512-525 (1959).
54. T. W. Summers, "The Two-Region Neutron Transport Equation," Report NARF-58-53T, 1952.
55. M. M. R. Williams, Mathematical Methods in Particle Transport Theory, Butterworths, London, 1971.

TABLE I. Milne Problem by the Galerkin Method

TRANSIENT SCALAR FLUXES $Q(X)=\Phi(X)-X$			
X	Galerkin	Iterated Galerkin	Q(X) EXACT
0.00	.5773832	.5773503	.5773503
.01	.5882328	.5882354	.588236
.02	.5953850	.5953907	.595391
.03	.6012363	.6012412	.601242
.04	.6062827	.6062862	.606287
.05	.6107554	.6107573	.610758
.06	.6147880	.6147887	.614789
.07	.6184677	.6184673	.618468
.08	.6218549	.6218539	.621854
.09	.6249947	.6249930	.624993
.10	.6279210	.6279189	.627919
.20	.6495517	.6495505	.649550
.30	.6633653	.6633659	.663365
.40	.6730897	.6730909	.673090
.50	.6802922	.6802933	.680293
.60	.6858005	.6858012	.685801
.70	.6901086	.6901087	.690109
.80	.6935344	.6935341	.693535
.90	.6962941	.6962935	.696294
1.00	.6985404	.6985397	.698540
1.25	.7025724	.7025718	.702572
1.50	.7051306	.7051303	.705131
1.75	.7068012	.7068017	.706802
2.00	.7079159	.7079162	.707916
2.25	.7086721	.7086736	.708673
2.50	.7091921	.7091924	.709191
2.75	.7095535	.7095527	.709551
3.00	.7098070	.7098073	.709806
3.25	.7099863	.7099864	.709985
4.00	.7102710		.710272
5.00	.7103966		.710396
10.00	.7104483		.710446
AVERAGE			
ERROR =	.0000024	.0000006	

EMERGING ANGULAR DISTRIBUTION					
μ	$\Psi(0, -\mu)$	μ	$\Psi(0, -\mu)$	μ	$\Psi(0, -\mu)$
.0	.5773503	.1	.7201581	.2	.8373608
.3	.9483107	.4	1.0561326	.5	1.1620784
.6	1.2667834	.7	1.3706150	.8	1.4738046
.9	1.5765055	1.0	1.6788253		

TABLE II. a. Finite Element Method; Linear with N = 4

N = 4	LINEAR				LINEAR WITH LOG TERM				Q(X) EXACT
	X	UNIFORM	$\Delta x_{i+1} = 1.1\Delta x_i$	$x_{i+1} = i^2\Delta x_0$	$\Delta x_{i+1} = 2\Delta x_i$	UNIFORM	$\Delta x_{i+1} = 1.1\Delta x_i$	$x_{i+1} = i^2\Delta x_0$	$\Delta x_{i+1} = 2\Delta x_i$
0.0	.654950	.653474	.630666	.637476	.618599	.615424	.593155	.596907	.577350
0.01	.655129	.653677	.631360	.637993	.621217	.618308	.598656	.601839	.588236
0.02	.655308	.653830	.632054	.638510	.623275	.620565	.602827	.605604	.595391
0.03	.655486	.654083	.632748	.639027	.625120	.622586	.606497	.608928	.601242
0.04	.655665	.654285	.633443	.639543	.626828	.624453	.609841	.611965	.606237
0.05	.655844	.654488	.634137	.640060	.628433	.626206	.612943	.614790	.610758
0.06	.656023	.654691	.634831	.640577	.629957	.627869	.615851	.617446	.614789
0.07	.656202	.654894	.635525	.641094	.631413	.629455	.618599	.619960	.618468
0.08	.656381	.655097	.636219	.641610	.632811	.630977	.621210	.622353	.621354
0.09	.656559	.655300	.636914	.642127	.634158	.632442	.623700	.624641	.624993
0.10	.656738	.655503	.637608	.642644	.635460	.633857	.626984	.626836	.627919
0.20	.658526	.657531	.644550	.647811	.646707	.646028	.645719	.645089	.649550
0.30	.660315	.659560	.651422	.652978	.655834	.655835	.660337	.658936	.663365
0.40	.662103	.661588	.658434	.658146	.663585	.664107	.671696	.669921	.673090
0.50	.663891	.663617	.665376	.663313	.670313	.671237	.680632	.678778	.680293
0.60	.665680	.665646	.672318	.668481	.676226	.677458	.687638	.685939	.685801
0.70	.667468	.667674	.679260	.673648	.681460	.682923	.693038	.691691	.690109
0.80	.669256	.669703	.686203	.678315	.686114	.687740	.697063	.696235	.693535
0.90	.671044	.671731	.693145	.683983	.690260	.691991	.699886	.699723	.696294
1.00	.672833	.673760	.700087	.689150	.693955	.695739	.701642	.702274	.693540
1.25	.677303	.678831	.707893	.702068	.701498	.703217	.702829	.705125	.702572
1.50	.681774	.683903	.708060	.711247	.707001	.708421	.703476	.704679	.705131
1.75	.686245	.688974	.708226	.711078	.710810	.711735	.704123	.705236	.706802
2.00	.690716	.694046	.708393	.710909	.713167	.713429	.704769	.705793	.707916
2.25	.695186	.699117	.708560	.710740	.714255	.713708	.705416	.706350	.708673
2.50	.699657	.704189	.708727	.710571	.714215	.712730	.706052	.706907	.709191
2.75	.704128	.709260	.708894	.710402	.713162	.710620	.706709	.707464	.709551
3.00	.708598	.714332	.709060	.710233	.711187	.707483	.707356	.708021	.709806
3.25	.713069	.714324	.709227	.710064	.708367	.707403	.708002	.708578	.709985
4.00	.713317	.712891	.709728	.709556	.707888	.708156	.709942	.710249	.710272
5.00	.711452	.710980	.710024	.709424	.708844	.709160	.711018	.710820	.710396
10.00	.709120	.709199	.710017	.709853	.710060	.710127	.710357	.710359	.710446
ERROR	.025280	.024145	.012039	.015269	.009995	.008905	.002988	.003410	

TABLE II.b. Finite Element Method; Linear with N = 8

N = 8	LINEAR				LINEAR WITH LOG TERM				Q(X) EXACT
		$\Delta x_{i+1} =$	$x_{i+1} =$	$\Delta x_{i+1} =$		$\Delta x_{i+1} =$	$x_{i+1} =$	$\Delta x_{i+1} =$	
X	UNIFORM	$1.1\Delta x_i$	$i^2\Delta x_0$	$2\Delta x_i$	UNIFORM	$1.1\Delta x_i$	$i^2\Delta x_0$	$2\Delta x_i$	
0.0	.637205	.629109	.592049	.584148	.596427	.591380	.582611	.578859	.577350
0.01	.637729	.629849	.595268	.589218	.601438	.597213	.589817	.583071	.588236
0.02	.638253	.630589	.598487	.594288	.605258	.601616	.595382	.594894	.595391
0.03	.638777	.631329	.601706	.599357	.608629	.605478	.600327	.600816	.601242
0.04	.639301	.632070	.604925	.604427	.611703	.608990	.604870	.606152	.606287
0.05	.639826	.632810	.608144	.609497	.614570	.612240	.609114	.611053	.610758
0.06	.640350	.633550	.611363	.614566	.617258	.615283	.613119	.615607	.614789
0.07	.640874	.634291	.614582	.619636	.619803	.618153	.616926	.619872	.618468
0.08	.641398	.635031	.617801	.624334	.622225	.620876	.620563	.623669	.621854
0.09	.641922	.635771	.621021	.626452	.624540	.623468	.624051	.625838	.624993
0.10	.642446	.636512	.624240	.628571	.626758	.625946	.627408	.628007	.627919
0.20	.647688	.643915	.656430	.649757	.645183	.646203	.655789	.649696	.649550
0.30	.652930	.651318	.663942	.663118	.659112	.661058	.663171	.663210	.663365
0.40	.658172	.658721	.670403	.672036	.670124	.672406	.669820	.672083	.673090
0.50	.663414	.666124	.676865	.680954	.678965	.681146	.676468	.680956	.680293
0.60	.668656	.673527	.683326	.686889	.686076	.687807	.683117	.686870	.685391
0.70	.673897	.680931	.689788	.689696	.691750	.692740	.689765	.689681	.690109
0.80	.679139	.688334	.696249	.692502	.696192	.696194	.695414	.692402	.693535
0.90	.684381	.695737	.698180	.695309	.699557	.698354	.698356	.695302	.696294
1.00	.689623	.703140	.699227	.698116	.701967	.699364	.699381	.698113	.698540
1.25	.702727	.707209	.701844	.703551	.704394	.701179	.701942	.703556	.702572
1.50	.711384	.707295	.704461	.704829	.703783	.703356	.704504	.704833	.705131
1.75	.711172	.707380	.707078	.706107	.705007	.705532	.707065	.706110	.706892
2.00	.710459	.707466	.708231	.707385	.706232	.707708	.708203	.707387	.707916
2.25	.709747	.707610	.708605	.708664	.707456	.709581	.708586	.708664	.708673
2.50	.709035	.708093	.708980	.709659	.708681	.709674	.708968	.709659	.709191
2.75	.708322	.708577	.709354	.709739	.709905	.709767	.709350	.709739	.709551
3.00	.708213	.709061	.709729	.709819	.710405	.709861	.709733	.709819	.709806
3.25	.708557	.709544	.710104	.709990	.710363	.709954	.710115	.709900	.709985
4.00	.709587	.709993	.710234	.710141	.710235	.710174	.710242	.710141	.710272
5.00	.709870	.709974	.710380	.710450	.710297	.710404	.710384	.710450	.710396
10.00	.709827	.710054	.710422	.710441	.710400	.710424	.710427	.710441	.710446
ERROR	.015144	.011403	.002271	.000920	.003082	.002086	.001303	.000543	

TABLE II.c. Finite Element Method; Linear with N = 16

N = 16					LINEAR WITH LOG TERM				
X	LINEAR				LINEAR WITH LOG TERM				Q(X) EXACT
	UNIFORM	$\Delta x_{i+1} = 1.1\Delta x_i$	$x_{i+1} = i^2\Delta x_0$	$\Delta x_{i+1} = 2\Delta x_i$	UNIFORM	$\Delta x_{i+1} = 1.1\Delta x_i$	$x_{i+1} = i^2\Delta x_0$	$\Delta x_{i+1} = 2\Delta x_i$	
0.0	.616895	.600509	.580733	.577384	.585813	.581107	.579062	.577182	.577350
0.01	.618080	.602720	.587593	.588303	.592822	.589534	.587676	.588304	.588236
0.02	.619264	.604931	.594452	.595536	.598029	.595662	.594976	.595536	.595391
0.03	.620449	.607142	.601311	.601152	.602557	.600921	.601780	.601152	.601242
0.04	.621634	.609352	.608171	.606570	.606643	.605617	.608263	.606570	.606287
0.05	.622818	.611563	.612746	.610594	.610400	.609894	.612593	.610594	.610758
0.06	.624003	.613774	.615493	.614618	.613897	.613837	.615360	.614618	.614789
0.07	.625183	.615985	.618241	.618642	.617175	.617502	.618121	.618642	.618468
0.08	.626372	.618195	.620988	.622364	.620268	.620929	.620883	.622364	.621854
0.09	.627557	.620406	.623736	.625006	.623193	.624148	.623644	.625006	.624993
0.10	.628742	.622617	.626483	.627647	.625983	.627183	.626406	.627647	.627919
0.20	.640588	.644725	.650518	.649138	.648142	.650261	.650546	.649138	.649550
0.30	.652434	.666832	.662599	.664374	.663504	.664660	.662519	.664374	.663365
0.40	.664281	.674750	.674500	.672548	.674451	.672100	.674493	.672548	.673090
0.50	.676127	.680215	.679930	.679741	.682116	.678949	.679926	.679741	.680293
0.60	.687974	.685679	.685360	.686934	.687166	.685798	.685359	.686934	.685801
0.70	.696229	.690128	.690790	.690542	.690076	.690972	.690792	.690542	.690109
0.80	.697303	.692991	.693542	.692973	.692397	.693531	.693543	.692973	.693535
0.90	.698377	.695854	.695958	.695404	.694718	.696091	.695959	.695404	.696294
1.00	.699451	.698717	.698374	.697835	.697040	.698651	.698374	.697835	.698540
1.25	.702135	.702426	.702503	.703907	.702844	.702371	.702503	.703907	.702572
1.50	.704319	.705202	.705104	.705064	.705576	.705281	.705104	.705064	.705131
1.75	.706252	.706689	.706784	.706221	.706773	.706732	.706785	.706221	.706802
2.00	.708185	.707923	.707852	.707378	.707971	.707944	.707852	.707378	.707916
2.25	.708599	.708606	.708731	.708536	.708560	.708634	.708731	.708536	.708673
2.50	.709014	.709183	.709147	.709692	.709149	.709217	.709147	.709692	.709191
2.75	.709367	.709498	.709562	.709768	.709605	.709530	.709562	.709768	.709551
3.00	.709597	.709804	.709813	.709844	.709793	.709834	.709813	.709844	.709806
3.25	.709827	.709943	.709965	.709919	.709982	.709974	.709965	.709919	.709985
4.00	.710168	.710232	.710261	.710147	.710276	.710263	.710261	.710147	.710272
5.00	.710224	.710365	.710391	.710451	.710387	.710396	.710391	.710451	.710396
10.00	.710276	.710413	.710444	.710443	.710439	.710444	.710444	.710443	.710446
ERROR	.006719	.002686	.000592	.000369	.001143	.000553	.000542	.000373	

TABLE II.d. Finite Element Method; Linear with N = 32

N = 32				LINEAR			LINEAR WITH LOG TERM			
X	UNIFORM	$\Delta x_{i+1} =$	$x_{i+1} =$	UNIFORM	$\Delta x_{i+1} =$	$x_{i+1} =$	Q(X) EXACT			
		$1.1 \Delta x_i$	$i^2 \Delta x_0$		$1.1 \Delta x_i$	$i^2 \Delta x$				
0.0	.600958	.582473	.578144	.581070	.577899	.577819	.577350			
0.01	.603129	.588254	.588950	.589523	.588095	.588940	.588236			
0.02	.605300	.594035	.595054	.595664	.595245	.595035	.595301			
0.03	.607471	.599316	.600958	.600932	.601246	.600951	.601242			
0.04	.609642	.605597	.606862	.605634	.606500	.606866	.606287			
0.05	.611813	.611377	.610865	.609914	.611199	.610870	.610758			
0.06	.613984	.615764	.614500	.613858	.614972	.614503	.614739			
0.07	.616156	.618777	.618135	.617522	.618168	.618137	.618468			
0.08	.618327	.621791	.621769	.620948	.621365	.621770	.621854			
0.09	.620498	.624804	.625404	.624164	.624562	.625403	.624993			
0.10	.622669	.627818	.628239	.627195	.627759	.628238	.627919			
0.20	.644379	.649547	.649357	.650194	.649508	.649357	.649550			
0.30	.666090	.663259	.663218	.664465	.663264	.663218	.663365			
0.40	.675088	.673056	.673124	.672191	.673056	.673124	.673090			
0.50	.680378	.680287	.680461	.679107	.680287	.680461	.680293			
0.60	.685668	.685779	.685684	.686023	.685779	.685684	.685801			
0.70	.689807	.690063	.690128	.690671	.690063	.690128	.690109			
0.80	.692999	.693492	.693510	.693453	.693493	.693510	.693535			
0.90	.696191	.696306	.696260	.696234	.696307	.696260	.696294			
1.00	.698835	.698561	.698537	.698633	.698562	.698537	.698540			
1.25	.702567	.702545	.702645	.702625	.702545	.702645	.702572			
1.50	.705036	.705131	.705199	.705082	.705131	.705199	.705131			
1.75	.706741	.706833	.706843	.706766	.706833	.706843	.706802			
2.00	.707887	.707930	.707925	.707923	.707930	.707925	.707916			
2.25	.708665	.708686	.708664	.708698	.708686	.708664	.708673			
2.50	.709154	.709208	.709182	.709187	.709208	.709182	.709191			
2.75	.709512	.709554	.709553	.709546	.709554	.709553	.709551			
3.00	.709771	.709802	.709818	.709804	.709802	.709818	.709806			
3.25	.709956	.709985	.709983	.709990	.709985	.709983	.709985			
4.00	.710235	.710271	.710269	.710268	.710271	.710269	.710272			
5.00	.710360	.710395	.710396	.710393	.710395	.710396	.710396			
10.00	.710411	.710446	.710446	.710444	.710446	.710446	.710446			
ERROR	.002726	.000352	.000163	.000514	.000109	.000153				

TABLE III.a. Finite Element Method; Cubic with N = 4

N = 4	CUBIC				CUBIC WITH LOG TERM				
X,	UNIFORM	$\Delta x_{i+1} = 1.1\Delta x_i$	$x_{i+1} = i^2\Delta x_0$	$\Delta x_{i+1} = 2\Delta x_i$	UNIFORM	$\Delta x_{i+1} = 1.1\Delta x_i$	$x_{i+1} = i^2\Delta x_0$	$\Delta x_{i+1} = 2\Delta x_i$	Q(X) EXACT
0.0	.627156	.625019	.602299	.607971	.616818	.612775	.585389	.588159	.577351
0.01	.628130	.626079	.604752	.609961	.619393	.615832	.592711	.595103	.588236
0.02	.629098	.627130	.607163	.611924	.621456	.618238	.598084	.600178	.595391
0.03	.630058	.628174	.609533	.613859	.623323	.620399	.602723	.604552	.601242
0.04	.631012	.629209	.611860	.615767	.625064	.622399	.606886	.608473	.606287
0.05	.631959	.630236	.614147	.617648	.626709	.624279	.610698	.612061	.610758
0.06	.632899	.631256	.616394	.619503	.628277	.626063	.614229	.615384	.614789
0.07	.633832	.632267	.618600	.621331	.629782	.627768	.617528	.618488	.618468
0.08	.634759	.633271	.620766	.623133	.631232	.629404	.620629	.621406	.621854
0.09	.635678	.634267	.622893	.624909	.632634	.630980	.623558	.624162	.624993
0.10	.636592	.635255	.624981	.626658	.633993	.632503	.626333	.626775	.627919
0.20	.645359	.644710	.643792	.642776	.645853	.645599	.648155	.647417	.649550
0.30	.653480	.653414	.659111	.656515	.655588	.656111	.663091	.661780	.663365
0.40	.660979	.661397	.671315	.668070	.663884	.664908	.673803	.672355	.673090
0.50	.667877	.668686	.680782	.677633	.671076	.672408	.681594	.680329	.680293
0.60	.674197	.675310	.687889	.685397	.677361	.678864	.687267	.686406	.685801
0.70	.679963	.681300	.693014	.691555	.682877	.684445	.691374	.691050	.690109
0.80	.685196	.686682	.696534	.696299	.687723	.689277	.694334	.694596	.693535
0.90	.690921	.691487	.698827	.699822	.691978	.693454	.696477	.697296	.696294
1.00	.694158	.695742	.700271	.702317	.695704	.697055	.698078	.699345	.698540
1.25	.702773	.704169	.702473	.705321	.703028	.703928	.701097	.702597	.702572
1.50	.708843	.709796	.704277	.706105	.708001	.708352	.703601	.704545	.705131
1.75	.712724	.713074	.705791	.706650	.711110	.710904	.705673	.706102	.706802
2.00	.714772	.714455	.707042	.707142	.712752	.712048	.707350	.707369	.707916
2.25	.715342	.714388	.708053	.707585	.713269	.712180	.708672	.708376	.708673
2.50	.714788	.713326	.708850	.707983	.712965	.711643	.709675	.709150	.709191
2.75	.713467	.711719	.709457	.708339	.712120	.710752	.710398	.709722	.709551
3.00	.711733	.710017	.709899	.708657	.710993	.709796	.710879	.710120	.709806
3.25	.709942	.708611	.710200	.708940	.709829	.709011	.711156	.710373	.709985
4.00	.706534	.706425	.710511	.709623	.707643	.707836	.711143	.710551	.710272
5.00	.706221	.706934	.710418	.710268	.707522	.708250	.710502	.710400	.710396
10.00	.712963	.712803	.710257	.711152	.712140	.711871	.709572	.710182	.710446
ERROR	.012144	.011221	.003360	.004785	.009121	.007774	.001284	.001401	

TABLE III.b. Finite Element Method; Cubic with N = 8

N = 8	CUBIC				CUBIC WITH LOG TERM				Q(X) EXACT
	UNIFORM	$\Delta x_{i+1} = 1.1\Delta x_i$	$x_{i+1} = i^2\Delta x_0$	$\Delta x_{i+1} = 2\Delta x_i$	UNIFORM	$\Delta x_{i+1} = 1.1\Delta x_i$	$x_{i+1} = i^2\Delta x_0$	$\Delta x_{i+1} = 2\Delta x_i$	
X,									
0.0	.606252	.600514	.582226	.579558	.587940	.583108	.579518	.577737	.577351
0.01	.608370	.603138	.588624	.587830	.594981	.591375	.588966	.588122	.588236
0.02	.610457	.605712	.594574	.594972	.600110	.597276	.594766	.595379	.595391
0.03	.612513	.608240	.600098	.601122	.604524	.602295	.600622	.601335	.601242
0.04	.614539	.610719	.605218	.606420	.608476	.606747	.605861	.606416	.606287
0.05	.616535	.613152	.609954	.611004	.612087	.610783	.610595	.610854	.610758
0.06	.618501	.615539	.614328	.615012	.615428	.614489	.614398	.614806	.614789
0.07	.620438	.617880	.618361	.618582	.618547	.617924	.618824	.618391	.618468
0.08	.622345	.620175	.622074	.621853	.621476	.621130	.622414	.621713	.621854
0.09	.624223	.622425	.625489	.624934	.624241	.624138	.625703	.624840	.624993
0.10	.626071	.624631	.628626	.627846	.625860	.626971	.628721	.627792	.627919
0.20	.643014	.644343	.649397	.649549	.647493	.648700	.649156	.649605	.649550
0.30	.657306	.660113	.662697	.663329	.661793	.663096	.662627	.663326	.663365
0.40	.669172	.672405	.672926	.673186	.672301	.673280	.672958	.673165	.673090
0.50	.678339	.681683	.680605	.680285	.680222	.680718	.680681	.680269	.680293
0.60	.686532	.688411	.686249	.685710	.686266	.686262	.686329	.685706	.685801
0.70	.692478	.693054	.690378	.690096	.690902	.690463	.690437	.690100	.690109
0.80	.696901	.696073	.693509	.693609	.694462	.693704	.693537	.693615	.693535
0.90	.699028	.697935	.696112	.696390	.697195	.696263	.696114	.696398	.696294
1.00	.700028	.699101	.698347	.698586	.699293	.698350	.698331	.698593	.698540
1.25	.704027	.701428	.702547	.702447	.702699	.702360	.702513	.702449	.702572
1.50	.704205	.703739	.705188	.705101	.704772	.705109	.705163	.705099	.705131
1.75	.704860	.705887	.706797	.706885	.706364	.706885	.706791	.706883	.706802
2.00	.705989	.707710	.707879	.708012	.707592	.707973	.707888	.708009	.707916
2.25	.707361	.709045	.708669	.708690	.708510	.708659	.708685	.708688	.708673
2.50	.708745	.709811	.709223	.709128	.709169	.709161	.709238	.709127	.709191
2.75	.709910	.710124	.709591	.709474	.709623	.709531	.709602	.709474	.709551
3.00	.710637	.710135	.709826	.709750	.709924	.709796	.709831	.709751	.709806
3.25	.710896	.709995	.709979	.709966	.710109	.709980	.709978	.709967	.709985
4.00	.710176	.709908	.710258	.710322	.710245	.710271	.710250	.710323	.710272
5.00	.709981	.710491	.710395	.710410	.710323	.710385	.710396	.710410	.710396
10.00	.710440	.710457	.710441	.710430	.710434	.710438	.710439	.710429	.710446
ERROR	.004392	.003061	.000430	.000160	.001311	.000597	.000310	.000079	

TABLE III.c. Finite Element Method; Cubic with N = 16

N = 16	CUBIC				CUBIC WITH LOG TERM				Q(X) EXACT
	UNIFORM	$\Delta x_{i+1} = 1.1\Delta x_i$	$x_{i+1} = i^2\Delta x_0$	$\Delta x_{i+1} = 2\Delta x_i$	UNIFORM	$\Delta x_{i+1} = 1.1\Delta x_i$	$x_{i+1} = i^2\Delta x_0$	$\Delta x_{i+1} = 2\Delta x_i$	
X.									
0.0	.593008	.585473	.578405	.577923	.579619	.578019	.577919	.577917	.577351
0.01	.596563	.590648	.582040	.588305	.589076	.588268	.588155	.588269	.588236
0.02	.600022	.595576	.595550	.595392	.595672	.595278	.595586	.595397	.595391
0.03	.603388	.600264	.601463	.601197	.601209	.601094	.601428	.601221	.601242
0.04	.606662	.604721	.606309	.606267	.606067	.606149	.606263	.606272	.606287
0.05	.609845	.608953	.610602	.610774	.610430	.610651	.610570	.610767	.610758
0.06	.612938	.612968	.614593	.614815	.614403	.614721	.614574	.614805	.614789
0.07	.615943	.616774	.618308	.618478	.618057	.618439	.618300	.618470	.618468
0.08	.618862	.620377	.621765	.621846	.621443	.621862	.621765	.621843	.621854
0.09	.621696	.623786	.624981	.624986	.624598	.625033	.624986	.624986	.624993
0.10	.624445	.627007	.627973	.627921	.627550	.627985	.627983	.627923	.627919
0.20	.647647	.650554	.649479	.649564	.649571	.649584	.649479	.649563	.649550
0.30	.664029	.663544	.663401	.663335	.663561	.663230	.663396	.663334	.663365
0.40	.675006	.672669	.673064	.673105	.673229	.673093	.673064	.673106	.673090
0.50	.681994	.679985	.680296	.680334	.680276	.680408	.680298	.680335	.680293
0.60	.686408	.685760	.685824	.685755	.685656	.685842	.685826	.685756	.685801
0.70	.689656	.690257	.690096	.690059	.689956	.690056	.690096	.690059	.690109
0.80	.692613	.693728	.693524	.693557	.693484	.693488	.693523	.693557	.693535
0.90	.695330	.696402	.696301	.696368	.696351	.696277	.696300	.696368	.696294
1.00	.697789	.698512	.698545	.698608	.698654	.698542	.698545	.698608	.698540
1.25	.702680	.702496	.702571	.702482	.702587	.702593	.702571	.702482	.702572
1.50	.705599	.705171	.705132	.705079	.705066	.705117	.705132	.705079	.705131
1.75	.706981	.706826	.706800	.706857	.706783	.706797	.706799	.706857	.706802
2.00	.707696	.707888	.707919	.708003	.707934	.707921	.707919	.708003	.707915
2.25	.708439	.708661	.708671	.708703	.708697	.708676	.708671	.708703	.708673
2.50	.709172	.709203	.709195	.709143	.709195	.709189	.709195	.709143	.709191
2.75	.709676	.709561	.709554	.709480	.709538	.709553	.709554	.709480	.709551
3.00	.709880	.709801	.709807	.709751	.709798	.709808	.709807	.709751	.709806
3.25	.709944	.709920	.709987	.709963	.709989	.709988	.709987	.709963	.709985
4.00	.710289	.710271	.710271	.710318	.710266	.710270	.710271	.710318	.710272
5.00	.710372	.710394	.710395	.710410	.710396	.710395	.710395	.710410	.710396
10.00	.710434	.710444	.710446	.710431	.710445	.710446	.710446	.710431	.710446
ERROR	.001800	.000776	.000079	.000054	.000219	.000060	.000062	.000051	

TABLE III.d. Finite Element Method; Cubic with N = 32

N = 32	CUBIC				CUBIC WITH LOG TERM				
X,	UNIFORM	$\Delta x_{i+1} = 1.1\Delta x_i$	$x_{i+1} = i^2\Delta x_0$	$\Delta x_{i+1} = 2\Delta x_i$	UNIFORM	$\Delta x_{i+1} = 1.1\Delta x_i$	$x_{i+1} = i^2\Delta x_0$	$\Delta x_{i+1} = 2\Delta x_i$	Q(X) EXACT
0.0	.585551	.578878	.570484	.577940	.577975	.577429	.574971	.577956	.577351
0.01	.590701	.587889	.589947	.588259	.588270	.588223	.588016	.588241	.588236
0.02	.595608	.595294	.594953	.595401	.595292	.595408	.595396	.595405	.595391
0.03	.600280	.601389	.600495	.601226	.601110	.601259	.601338	.601238	.601242
0.04	.604722	.606466	.605973	.606273	.606164	.606281	.606308	.606276	.606287
0.05	.608943	.610822	.610860	.610764	.610662	.610730	.610732	.610761	.610758
0.06	.612949	.614744	.615089	.614803	.614726	.614765	.614765	.614798	.614789
0.07	.616748	.618390	.618781	.618469	.618439	.618468	.618454	.618465	.618468
0.08	.620348	.621784	.622059	.621843	.621857	.621874	.621844	.621841	.621854
0.09	.623754	.624947	.625044	.624986	.625023	.625018	.624983	.624986	.624993
0.10	.626975	.627900	.627852	.627923	.627971	.627935	.627913	.627924	.627919
0.20	.650571	.649531	.649617	.649563	.649578	.649555	.649546	.649563	.649550
0.30	.663557	.663373	.663335	.663335	.663260	.663365	.663370	.663334	.663365
0.40	.672604	.673090	.673105	.673105	.673101	.673092	.673089	.673106	.673090
0.50	.679983	.680294	.680290	.680335	.680387	.680294	.680293	.680335	.680293
0.60	.685846	.685801	.685798	.685756	.685813	.685802	.685803	.685756	.685801
0.70	.690340	.690109	.690112	.690059	.690055	.690109	.690108	.690059	.690109
0.80	.693708	.693534	.693535	.693557	.693504	.693534	.693534	.693557	.693535
0.90	.696285	.696293	.696291	.696367	.696296	.696293	.696293	.696367	.696294
1.00	.698407	.698539	.698539	.698608	.698559	.698539	.698539	.698608	.698540
1.25	.702601	.702572	.702571	.702482	.702565	.702571	.702571	.702482	.702572
1.50	.705147	.705130	.705130	.705079	.705127	.705130	.705130	.705079	.705131
1.75	.706772	.706801	.706801	.706857	.706807	.706801	.706801	.706857	.706802
2.00	.707934	.707917	.707917	.708003	.707913	.707917	.707917	.708003	.707916
2.25	.708663	.708673	.708673	.708703	.708674	.708673	.708673	.708703	.708673
2.50	.709191	.709193	.709193	.709144	.709193	.709193	.709193	.709144	.709191
2.75	.709556	.709554	.709554	.709480	.709554	.709554	.709554	.709480	.709551
3.00	.709803	.709808	.709808	.709751	.709808	.709808	.709808	.709751	.709806
3.25	.709986	.709987	.709987	.709963	.709986	.709987	.709987	.709963	.709985
4.00	.710269	.710270	.710271	.710318	.710270	.710271	.710271	.710318	.710272
5.00	.710393	.710395	.710395	.710410	.710395	.710395	.710395	.710410	.710396
10.00	.710444	.710446	.710446	.710431	.710446	.710446	.710446	.710431	.710446
ERROR	.000787	.000083	.000352	.000051	.000052	.000008	.000089	.000051	

TABLE IV.

Errors at the origin in Spline Approximation of the Transient Flux $Q(x) = \Phi(x) - x$,
 $0 \leq x \leq 10$, in the Milne Problem for Conservative Scattering (Huang Functional with
 Uniform Mesh)

Mesh	Linear	Corrected Linear ^a	Improvement Factor	Cubic	Corrected Cubic ^a	Improvement Factor
4	.078	.041	2	.050	.039	1.3
8	.069	.019	3	.029	.011	2.6
16	.040	.008	5	.016	.002	8.0
32	.024	.004	6	.0082	.0006	13.7

^a with $x \log x$ add on term in the first interval.

TABLE V.

Average Errors in Spline Approximations of the Transient Flux $Q(x) = \bar{\phi}(x) - x$,
 $0 \leq x \leq 10$, in the Milne Problem for Conservative Scattering (Huang Functional with
 Uniform Mesh)

Mesh	Linear	Corrected Linear ^a	Improvement Factor	Cubic	Corrected Cubic ^a	Improvement Factor
4	.025	.010	2.5	.012	.009	1.3
8	.015	.003	5.0	.0044	.0013	3.4
16	.0067	.0011	6.0	.0018	.0002	9.0
32	.0027	.0005	5.4	.00079	.00005	15.8

^a with $x \log x$ add on term in the first interval

TABLE VI. Eigenvalue Problem

Half Thickness	$c(\bar{\Phi}=1)$	$c(\bar{\Phi}=1-\text{sing.term})$	$c(\bar{\Phi}=1-ax^2)$	$c=(\bar{\Phi}-ax^2-\text{sing.term})$	$c(\text{EXACT})$
.2500	2.256	2.235093	2.235055	2.235015	2.235004
.5000	1.640	1.615575	1.615410	1.615388	1.615379
.7366	1.428	1.400361	1.400024	1.400013	1.400000
.7500	1.419	1.392083	1.391736	1.391724	1.391707
1.0000	1.307	1.277712	1.277117	1.277112	1.277100
1.2893	1.232	1.200996	1.200026	1.200024	1.200000
1.5000	1.196	1.164444	1.163137	1.163136	1.163124
2.0000	1.142	1.110791	1.108475	1.108475	1.108462
2.5000	1.111	1.081179	1.077583	1.077583	1.077569
5.6655	1.046	1.032847	1.020032	1.020031	1.020000

TABLE VII. Extended Eigenvalue Problem

Half Thickness	$C_F(\Phi = 1)$	$C_F(\Phi = 1 - \text{sing. term})$	$C_F(\Phi = 1 - ax^2)$	$C_F(\Phi = 1 - ax^2 - \text{sing. term})$	α_0^2 ($C_M = \alpha_0^2 C_F$)
					.2
.25	2.073	2.068	2.05531	2.05526	
.50	1.578	1.568	1.55634	1.55624	
.75	1.387	1.374	1.36229	1.36220	
1.0	1.287	1.271	1.25967	1.25959	
1.5	1.186	1.166	1.15522	1.15518	
2.0	1.136	1.115	1.10415	1.10412	
2.5	1.107	1.087	1.07495	1.07493	
5.6655	1.045	1.035	1.01968	1.01968	
					.4
.25	1.839	1.834*	1.8374	1.8339	
.50	1.490	1.484	1.4809	1.4793	
.75	1.339	1.330	1.3241	1.3231	
1.0	1.256	1.245	1.2368	1.2362	
1.5	1.169	1.156	1.1448	1.1444	
2.0	1.124	1.110	1.0984	1.0982	
2.5	1.098	1.084	1.0714	1.0713	
5.6655	1.042	1.035	1.0192	1.0192	
					.6
.25	1.613	1.605	1.6094	1.5995	
.50	1.390	1.383*	1.3891	1.3834	
.75	1.280	1.273*	1.2752	1.2715	
1.0	1.215	1.207	1.2069	1.2044	
1.5	1.145	1.135	1.1306	1.1293	
2.0	1.108	1.098	1.0905	1.0897	
2.5	1.086	1.076	1.0666	1.0661	
5.6655	1.037	1.032	1.0185	1.0185	
					.8
.25	1.415	1.404	1.3994	1.3847	
.50	1.287	1.279*	1.2849	1.2730	
.75	1.214	1.207*	1.2143	1.2056	
1.0	1.169	1.161*	1.1676	1.1612	
1.5	1.116	1.108*	1.1111	1.1074	
2.0	1.088	1.080	1.0793	1.0770	
2.5	1.071	1.062	1.0595	1.0580	
5.6655	1.031	1.027	1.0175	1.0173	