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Phase-Space Integrals for Beta-Decay and Nuclear Matrix Elements[†]

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Abstract: The use of phase-space integrals to determine nuclear matrix elements is discussed. The procedure for determining matrix elements is illustrated by analyzing the allowed and superallowed beta-decays of isotopes with mass numbers in the range five to ten, utilizing the recent summary of the properties of nuclei in this region compiled by Ajzenberg-Selove and Lauritsen. Some f -values for other beta-decays of interest are also given.

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1. Introduction

Phase-space integrals for nuclear beta-decay, which are frequently called "f-values," are useful for two reasons. The phase-space integrals for a given transition can often be used in conjunction with the measured half-life, $\tau_{1/2}$, and certain selection rules to determine uniquely the experimental nuclear matrix elements pertinent to the decay. The matrix elements thus obtained can then be compared with the predictions of theoretical models. Since the advent of the conserved vector current theory, several f-values have been determined very accurately ($\sim \frac{1}{2}\%$) in order to test the prediction that pure Fermi transitions ($0^+ \rightarrow 0^+$) have matrix elements that depend only on the isotopic spin quantum numbers of the initial and final nuclear states. One can also use the order of magnitude of the transition matrix elements, or equivalently the product $f\tau_{1/2}$, to help establish the spins and parities of the nuclear states that are involved. Transition matrix elements that pertain to different spin-parity changes, i.e., different degrees of forbiddenness, have values that characteristically differ by one or more orders of magnitude.

In this note, we are primarily concerned with showing how one can conveniently use phase-space integrals in conjunction with lifetime measurements and selection rules to determine nuclear matrix elements. We illustrate the procedure by analyzing the allowed and superallowed decays of isotopes with mass numbers in the range five to ten, utilizing the recent summary of the properties of nuclei in this region that has been compiled by Ajzenberg-Selove and Lauritsen ¹⁾. A future compilation by these authors will incorporate the results of a similar analysis for radioactive nuclei with mass numbers in the range ten to twenty. As a check on our numerical work, we have also calculated phase-space integrals for a number of standard decays that have previously been accurately studied.

2. Definition and Selection Rules

The dimensionless phase-space factor, f , is defined for $\beta^\pm$ -emissions and for electron capture decays by the relation ²⁾:

$$f\tau_{\frac{1}{2}} = 2\pi^3 \ln 2 (\hbar/m_e c)^{+7} m_e^2 c^3 G^{-2} \xi^{-1}, \quad (1)$$

where m_e , G , and ξ are, respectively, the electron's mass, the vector coupling constant of nuclear beta-decay, and a sum of squares of nuclear matrix elements. We first give explicit formulae for f and then discuss the matrix elements that occur in ξ .

2.1. Electron and Positron Emissions

For allowed beta-decays involving either electron or positron emission, one can show that ²⁾:

$$f = f(\pm Z, W_0) \quad (2a)$$

$$= m_e^{-5} c^{-9} \int_{m_e c^2}^{W_0} dW F(\pm Z, W) pW (W - W_0)^{+2}, \quad (2b)$$

where W_0 is the difference between initial and final nuclear energies (i.e., masses times c^{+2}) and p, W are the emitted electron's momentum and energy respectively. The quantity $F(\pm Z, W)$ is just the ratio of the relativistic electron density at the nucleus to the density at infinity (hence $F = 1$ for $Z = 0$). Of course, f is proportional to the total phase-space available to the electron and neutrino in the final state. In our calculations, we have approximately taken account of the small effect due to the finite size of the nucleus by evaluating at the nuclear radius R the expression for F appropriate to a point nucleus. One thus

obtains ²⁾:

$$F(\pm Z, W) = 2(1 + \gamma_0)(2pR/\hbar)^{-2(1 - \gamma_0)} e^{+\pi v} |\Gamma(\gamma_0 + i v)|^{+2} / [\Gamma(2\gamma_0 + 1)]^{+2} \quad (3a)$$

where

$$\gamma_0 \equiv \left[1 - (\alpha Z)^2 \right]^{\frac{1}{2}}, \quad \alpha \equiv e^2/\hbar c, \quad \text{and} \quad v = \pm \alpha Z W / c p. \quad (3b)$$

The plus sign is to be used for electron emission and the minus sign for positron emission; the quantity Z is the charge of the final nucleus.

The above expression for F should be corrected for the effect of screening by atomic electrons. This can be accomplished approximately ³⁾ by replacing $F(\pm Z, W)$ with

$$F(\pm Z, W \mp V_0) \left[\frac{(W \mp V_0)^2 - m_e^2 c^4}{W^2 - m_e^2 c^4} \right]^{\frac{1}{2}} \left(\frac{W \mp V_0}{W} \right), \quad (4)$$

where V_0 is the shift in the potential energy at the nucleus caused by the screening. Durand ⁴⁾ has recently shown that:

$$V_0 \approx 1.45 \alpha^2 |Z|^{+4/3} m_e c^2. \quad (5)$$

2.2. Electron Capture

The appropriate phase-space factor for electron capture can be obtained from eq. (1) and the theory of allowed beta-decay. One finds:

$$f_{e.c.}(Z, W_0) = \pi^2 (\hbar/m_e c)^{+3} m_e^{-2} c^{-4} \sum_n a_{ns} |\psi_{ns}(0)|^{+2} q_{ns}^{+2}, \quad (6)$$

where a_{ns} is the number of s -electrons in the initial atom with principal quantum number n , $\psi_{ns}(0)$ is the value of the electron's wave function evaluated at the nucleus, and q_{ns} is the energy of a neutrino emitted when

an ns-electron is captured by the nucleus,

$$q_{ns} = W_0 + m_e c^2 - b(n,s) , \quad (7)$$

where $b(n,s)$ is the positive binding energy of an ns-electron in the initial atom. Accurate numerical expressions for $\psi_{ns}(0)$ are available for almost all cases of interest from detailed self-consistent field calculations.

2.3. Matrix Elements and Selection Rules

The quantity ξ that appears in eq. (1) is defined by the relation

$$\xi = \langle 1 \rangle^{+2} + (C_A/C_V)^{+2} \langle \sigma \rangle^{+2} , \quad (8)$$

with

$$\langle 1 \rangle = \langle I_f(M_f) \left| \sum_{i=1}^A \tau_{\pm}(i) \right| I_i(M_i) \rangle \quad (9a)$$

and

$$\langle \sigma \rangle = \langle I_f(M_f) \left| \sum_{i=1}^A \sigma_k(i) \tau_{\pm}(i) \right| I_i(M_i) \rangle / \langle I_f(M_f) 1(k) \left| I_i(M_i) \right\rangle . \quad (9b)$$

Here I_i, M_i (I_f, M_f) are the total angular momentum and its projection along a definite axis of the nucleus in its initial (final) state; $\tau_{\pm}$ and σ_k are ⁵⁾, respectively, the familiar isotopic spin raising and lowering operators and the Pauli spin matrices combined to form a spherical tensor of rank one. The ratio $(C_A/C_V)^{+2}$ is approximately:

$$(C_A/C_V)^{+2} = 1.40 \pm 0.05, \quad (10)$$

as we shall see later.

The matrix element $\langle 1 \rangle$ vanishes unless $\Delta I = 0^+$, $\Delta T = 0$, and $\Delta T_z = \pm 1$,

where the superscript plus indicates no parity change and T is the total isotopic spin. Similarly, $\langle \sigma \rangle$ vanishes unless $\Delta I = 0^+$ or $\pm 1^+$ and $\Delta T = 0, \pm 1$, with, of course, $\Delta T_z = \pm 1$.

The conserved vector current theory ⁶⁾ implies that for transitions among members of an isotopic spin multiplet, i.e., superallowed decays:

$$\langle 1 \rangle^2 = \left[T \mp T_z(i) \right] \left[T \pm T_z(i) + 1 \right] , \quad (11a)$$

where the upper (lower) sign is to be used if the z-component of the isotopic spin increases (decreases) in the transition. Here $T_z(i)$ is the z-component of the isotopic spin in the initial state. Equation (11a) can also be written in the following more symmetric form:

$$\langle 1 \rangle^2 = \left[T \mp T_z(i) \right] \left[T \pm T_z(f) \right] . \quad (11b)$$

Equations (11a) and (11b) are of course valid for either convention [τ_z (neutron) equals $+\frac{1}{2}$ or $-\frac{1}{2}$] regarding the sign of T_z .

3. Numerical Results

The expression for $f(\pm Z, W_0)$ that is obtained when eqs. (1)-(5) are combined has been programmed for numerical calculations on an IBM 7094. For most of our computations we have used a nuclear radius R equal to $1.3 \times 10^{-13} A^{1/3}$ cm and a screening potential given by eq. (5). For all cases with A less than ten that we have investigated, a change in the value of R by $\pm 30\%$ changes the resultant f -value by less than or of the order of a tenth of a percent. The results for higher A are, of course, more sensitive to the choice of R ; for example, a 30% change in the nuclear radius results in a 1% change in the f -value for V^{46} and Co^{54} . As one might guess

from eq. (4), the inclusion of electron screening results in a fractional change in the calculated f-values of the order of magnitude of $\pm 2V_0/(W_0 - m_e c^2)$, where the minus (plus) sign obtains for electrons (positrons). Equation (5) gives much too large a value for V_0 in the special case of the decay of H^3 ; fortunately, one can find V_0 exactly in this case with the aid of the known atomic wave function [$V_0(H^3) = e^2/a_0 \approx 27$ eV]. In addition to the effects we have included, there are ⁷⁾ some other small corrections due to radiative effects, second forbidden contributions, overlap and exchange effects involving the atomic wave functions, and detailed averaging of the emitted electron's wave function over the nuclear surface. These corrections are small (typically < 1%) and can be ignored in all applications except those experiments designed to test the conserved vector current theory.

Some f-values calculated in the manner described above are given in column three of table 1; the cases shown were chosen because accurate hand calculations of the f-values for these decays already exist in the literature ⁷⁻¹¹⁾. For almost all cases, our results agree with the published values (without radiative corrections) to within a tenth of one percent. The only exceptions are the decay of B^{12} , N^{12} , and V^{46} . For both the B^{12} and N^{12} decays, our results appear to exceed the values of Fisher ¹⁰⁾ by one percent; our results for these decays are in agreement with the values obtained by Kavanagh ¹¹⁾. Our f-value for V^{46} is three percent smaller than the f-value quoted by Jänecke ⁸⁾.

In order to determine the uncertainty in f that results from an experimental uncertainty in the energy difference W_0 , it is convenient to introduce the quantity

$$\Delta \equiv f^{-1} \left(\frac{df}{dW_0} \right)_{W_0 = W_0(\text{experimental})} \quad (12)$$

For a finite but small uncertainty, δW_0 , in the experimental energy, the fractional uncertainty in f is then equal to $\Delta \times \delta W_0$. The calculated values of Δ are given in column four of table 1 in units of MeV^{-1} .

The f -value for the decay of tritium is extremely sensitive to the energy, W_0 , that is assumed. A change of 0.7 keV in the assumed value of W_0 produces a ten percent change in the calculated f -value. Two apparently accurate experiments ^{9,12)} have, in fact, yielded values of W_0 that differ by 0.7 keV, indicating that theorists should be cautious in making inferences that depend upon knowing the f -value of H^3 to much better than 10%.

4. Nuclear Matrix Elements

Equation (1) can be used to determine the nuclear matrix elements pertaining to a given transition if the f and $\tau_{\frac{1}{2}}$ values (i.e., energy and lifetime) of the transition are known. For this purpose, it is convenient to eliminate the constants occurring in eq. (1) in favor of the $f\tau_{\frac{1}{2}}$ -value of some standard transition whose matrix elements are well known from theoretical considerations and whose f -value and lifetime have been accurately measured. We adopt the $0^{14} \rightarrow \text{N}^{14*}$ decay ($\langle 1 \rangle^2 = 2$, $\langle \sigma \rangle^2 = 0$), which has been extensively studied both theoretically and experimentally. One can then rewrite eq. (1) as follows:

$$\langle 1 \rangle^2 + (C_A/C_V)^2 \langle \sigma \rangle^2 = 2(f\tau_{\frac{1}{2}})_{0^{14}} / (f\tau_{\frac{1}{2}}) \quad , \quad (13)$$

where ¹³⁾ $(f\tau_{\frac{1}{2}})_{0^{14}} = (3.10 \pm 0.03) \times 10^{+3}$ secs.

The value of $(C_A/C_V)^2$ can be determined by inserting in eq. (13) the theoretical values ($\langle 1 \rangle^2 = 1$, $\langle \sigma \rangle^2 = 3$) of the matrix elements for the neutron decay, the measured lifetime of the neutron ¹⁴⁾, and the value of f_{neutron}

given in table 1. One thus finds the result, $(C_A/C_V)^2 = 1.40 \pm 0.05$, given in eq. (10), where the main uncertainty in $(C_A/C_V)^2$ arises from the uncertainty in the lifetime of the neutron.

Equations (10) and (13) can be combined to yield a convenient equation for the nuclear matrix elements of an allowed or superallowed transition in terms of the experimentally determined $f\tau_{1/2}$ -value. One finds:

$$\langle 1 \rangle^2 + (1.40 \pm 0.05) \langle \sigma \rangle^2 = (6.20 \pm 0.06) \times 10^{+3} / (f\tau_{1/2}). \quad (14)$$

The energies and lifetimes given in ref. (1) have been used to compute the most probable $f\tau_{1/2}$ -values given in column five of table 2 for all allowed and superallowed decays of isotopes with mass numbers in the range five to ten. For Be^7 , which decays by electron capture, the values of $|\psi_{ns}(0)|^2$ used in ref. (15) were adopted. Errors in f and $\tau_{1/2}$ are not explicitly indicated in table 2 since these may be readily inferred from the stated values of Δ and the indicated errors in the data ¹⁾. Moreover, the errors in f and $\tau_{1/2}$ are not as important for theoretical interpretations as the overall uncertainty in the nuclear matrix elements.

The nuclear matrix elements given in the last column of table 2 were calculated with the aid of eq. (14), the selection rules given in section 2.3, and the computed $f\tau_{1/2}$ -values. For all except the $\text{Li}^9 \rightarrow \text{Be}^9$ and $\text{C}^{10} \rightarrow \text{B}^{10**}$ decays, the largest uncertainty in the derived matrix elements is due to the uncertainty in $(C_A/C_V)^2$, i.e., in the lifetime of the neutron. For both the $\text{Li}^9 \rightarrow \text{Be}^9$ and $\text{C}^{10} \rightarrow \text{B}^{10**}$ decays, the largest uncertainty in the matrix elements is due to the lack of a precise measurement of the relevant branching ratios.

Some observations, not original, can be made on the basis of the matrix elements given in table 2. Notice first that matrix elements for superallowed

decays are typically of the order of unity and matrix elements for allowed decays are typically of the order of one-tenth, although considerable dispersion from these values occurs. The fact that $\langle \sigma \rangle^2$ for He^6 is less than the theoretical value of six which obtains for a $^1S_0 \rightarrow ^3S_1$ transition suggests that Li^6 is not a particularly pure 3S_1 -configuration.

The most interesting entry in table 2 occurs in the last row, the decay of C^{10} to B^{10**} . The value of $\langle 1 \rangle^2$ predicted for this decay by the conserved vector current theory [see eq. (11)] is two. The experimental value of $\langle 1 \rangle^2$ differs from the predicted value by 0.8, which is twice the quoted sum of errors. If one accepts the prediction of the conserved vector current theory and the estimate of the energy uncertainty (13 keV) given in ref. (1), then one can show that the branching ratio for the decay $\text{C}^{10} \rightarrow \text{B}^{10**}$ should be $(2.2 \pm 0.15)\%$ instead of the quoted value of $(1.65 \pm 0.2)\%$. It would be useful to remeasure the branching ratio as well as the transition energy in order to remove the above mentioned discrepancy¹⁵⁾. In order to test the conserved vector current theory significantly with this decay, one would require a knowledge of the branching ratio and total lifetime accurate to about 0.5% and a knowledge of the transition energy accurate to about 1 keV. Since C^{10} is the lightest available isotope participating in a superallowed $0^+ \rightarrow 0^+$ transition, an experiment which determined the branching ratio, total lifetime, and transition energy to the stated accuracy would also be useful in searching for a dependence upon mass number of such suggested interactions as a non-isotopic spin conserving part of the nuclear Hamiltonian.

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Table 1
Some Calculated f-values

Isotope	W_0 (MeV)	f	Δ (MeV ⁻¹)
1_0n	1.293	1.6896	5.0
3_1H	0.52895 (0.52961)	2.576×10^{-6} (2.892×10^{-6})	$1.9 \times 10^{+2}$ ($1.8 \times 10^{+2}$)
${}^{12}_5B$	13.880	$5.6459 \times 10^{+5}$	3.6×10^{-1}
${}^{12}_7N$	16.941 (16.838)	$1.1622 \times 10^{+6}$ ($1.1273 \times 10^{+6}$)	3.0×10^{-1} (3.0×10^{-1})
${}^{14}_8O$	2.3236	$4.3039 \times 10^{+1}$	2.4
${}^{26}_{13}Al^{26*}$	3.719	$4.737 \times 10^{+2}$	1.4
${}^{34}_{17}Cl$	4.971	$1.956 \times 10^{+3}$	1.0
${}^{46}_{23}V$	6.552	$7.125 \times 10^{+3}$	7.7×10^{-1}
${}^{54}_{27}Co$	7.7398	$1.538 \times 10^{+3}$	6.5×10^{-1}

Table 2

The $f\tau$ -values and nuclear matrix elements for allowed and superallowed beta-decays of isotopes with mass numbers in the range five to ten.

Isotope	W_0 (MeV)	f	Δ (MeV ⁻¹)	$f\tau_{\frac{1}{2}}$ (sec)	$\langle 1 \rangle^2$	$\langle \sigma \rangle^2$
He ⁶ → Li ⁶	4.0194 ± 0.004	1.00 × 10 ⁺³	1.3	8.02 × 10 ⁺²	"0"	5.6 ± 0.2
Li ⁸ → Be ^{8*}	13.613 ± 0.040	4.88 × 10 ⁺⁵	3.7 × 10 ⁻¹	4.14 × 10 ⁺⁵	"0"	(1.10 ± 0.05) × 10 ⁻²
Li ⁹ → Be ⁹	14.126 ± 0.040	5.87 × 10 ⁺⁵	3.5 × 10 ⁻¹	3.97 × 10 ⁺⁵	"0"	(11 ± 6) × 10 ⁻³
Li ⁹ → Be ^{9*}	11.696 ± 0.040	2.28 × 10 ⁺⁵	4.3 × 10 ⁻¹	5.13 × 10 ⁺⁴	"0"	(8.8 ± 0.4) × 10 ⁻²
Be ⁷ → Li ⁷	0.8616 ± 0.0003	3.86 × 10 ⁻⁴	2.3	2.00 × 10 ⁺³	"1"	1.49 ± 0.05
Be ⁷ → Li ^{7*}	0.3838 ± 0.0004	7.70 × 10 ⁻⁵	5.2	3.45 × 10 ⁺³	"0"	1.26 ± 0.06
B ⁸ → Be ^{8*}	14.568 ± 0.040	5.70 × 10 ⁺⁵	3.4 × 10 ⁻¹	4.41 × 10 ⁺⁵	"0"	(1.03 ± 0.05) × 10 ⁻²
C ¹⁰ → B ^{10*}	2.378 ± 0.013	5.13 × 10 ⁺¹	2.3	1.01 × 10 ⁺³	"0"	(4.5 ± 0.2)
C ¹⁰ → B ^{10**}	1.356 ± 0.013	1.90 ± 0.12	4.7	2.24 × 10 ⁺³	2.77 ±0.4	"0"

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