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**TRUNK
FORTRAN PROGRAM FOR THE BUCKLING
OF TRUNCATED SPHERES**

F. R. CZERNIEJEWSKI

FEBRUARY, 1967

BATTELLE-NORTHWEST

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TRUNK
FORTRAN PROGRAM FOR THE BUCKLING OF TRUNCATED SPHERES

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TABLE OF CONTENTS

INTRODUCTION.	1
THEORY	2
The Lattice	2
Partial Difference Formulation	5
Boundary Conditions	10
Bare Truncated Sphere	13
Reflected Truncated Spheres	15
Solution for the Buckling	16
RESULTS	18
Internal Workings	20
Output	24
APPENDIXES	
A. Analytical Solution of Helmholtz Equation for a Hemisphere	25
B. Top Surface Boundary Conditions	29
C. An Analytic Expression for B^2 vs. h for Bare Spheres	37
D. Reflected Sphere Albedo Calculations	38
E. Illustrative Solution of $\frac{A}{\lambda} X = \frac{B}{\lambda} X$ by Residual Method	40
F. Note on Curvature of Fluxes Near Corners	42
REFERENCES	44
DISTRIBUTION.	45

LIST OF FIGURES

1	Two Dimensional Lattice for Truncated Sphere	4
2	Nine Point Pattern for Difference Equations	6
3	An Area Adjacent to $\theta = 0, \pi$ Axis	11
4	Lattice Near Top Surface Boundary	14
5	Reflected Truncated Sphere Geometry "Infinite" Water Reflector	16
6	The Physical and Extrapolated Truncated Bare Sphere	20
7	Buckling Versus Height Above Hemisphere for for Unit Radius	21
8	Normalized Radial Flux Plot for $H = 0.5$	22
9	Normalized Angular Flux Plot for $H = 0.5$	23
10	Hemisphere Showing Coordinates on Extrapolated Boundaries	25
11	Lattice Points at the Top Surface	29
12	Special Case	33
13	Approximate Juncture of Curved and Top Extrapolated Boundaries	34

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F. R. Czerniejewski

INTRODUCTION

Ordinarily critical mass or buckling experiments are done in cylindrical or slab geometry. The reason for this, of course, is that only one dimension is varied in the approach to critical, and the Helmholtz equation ($\nabla^2 \phi + B^2 \phi = 0$) is separable in these geometries; hence, the geometric buckling is well known. Thus, the approximate critical size of other geometric shapes may be found by direct buckling conversion. For example, given the critical height of some fissile solution in a given cylinder, a reasonable approximation may be made for the size of a critical sphere with identical solution by simply converting the known cylindrical buckling to spherical geometry.

Recently, however, the Critical Mass Unit of Battelle-Northwest has been doing plutonium critical mass experiments in spherical geometry. Thus, the problem of finding the buckling for partially filled spheres arose. For these experiments to yield information about the critical size of a given plutonium solution in spherical (or any other) geometry, it is necessary to know the geometric buckling at critical, which may occur anywhere between a hemisphere and a full sphere in these experiments.

Since the Helmholtz equation is not separable except for the sphere and hemisphere, some numerical method must be adopted to find the lowest eigenvalue buckling, B^2 . Of the several possible methods (Green's functions, "cylindrical stacking," exact bispherical coordinates,⁽¹⁾ etc.) the most straightforward was chosen; namely, partial differences in spherical coordinates. Although considerable algebra was necessary for the analysis using this scheme, the comparison of the approximations with analytic functions is on much firmer

ground in spherical coordinates. Mistakes or poor approximations come to light much more quickly when the observer is well versed in using the coordinate system. Furthermore, valuable conclusions on the relative merits of various methods in two dimensional geometry have been obtained. This experience may prove invaluable for future efforts in multi-dimensional analysis of any form of the neutron transport equation.

The approximations and methods described below have been incorporated into a computer code called TRUNK which finds the buckling for a given truncated sphere and converts it to an equivalent full sphere.

THEORY

The separability of the Helmholtz equation

$$\nabla^2 \phi + B^2 \phi = 0$$

is well established for the regular geometries: rectangular, cylindrical, and spherical. Less prominent is that for the hemispherical case, which is reproduced in Appendix A. For the unreflected case, separability leads to the fundamental (or lowest) eigenvalue buckling immediately by application of a $\phi = 0$ condition at the extrapolated boundaries.

Table I presents the unit extrapolated dimension bucklings for the regular geometries.

The Lattice

Considering that there is symmetry in the azimuthal angle and also about the $\phi = 0, \pi$ axis, Figure 1 is adequate to form a two-dimensional lattice of points for the truncated sphere. Note that radial and angular cuts are placed to meet exactly on the top surface.

TABLE 1. Bucklings for Regular Geometry, Unit Extrapolated Dimensions

I.	Rectangle	
	Each Dimension	π
II.	Cylinder, Upright	
	Axial	π
	Radial	2.4048
III.	Cylinder, Horizontal and Half Filled	
	Axial	π
	Radial	3.8317
IV.	Sphere	
	Full	π
	One-half Full	4.4934

Now, to facilitate the partial difference formulation with the uneven lattice, it is appropriate to average the Helmholtz equation over the volume of a small region, such as the shaded area in Figure 1. For volume v_1 , then,

$$\frac{\int_{v_1} \nabla^2 \phi \, dv + B^2 \int_{v_1} \phi \, dv}{\int_{v_1} dv} = 0 \quad (1)$$

$$\text{or } \overline{\nabla^2 \phi} + B^2 \overline{\phi} = 0. \quad (2)$$

If an equation like (2) can be written for every internal point in the lattice and if boundary conditions can be written for each outside point, then n equations would result with $n + 1$ unknowns; namely, an eigenvalue problem.

FIGURE 1. Two Dimensional Lattice for Truncated Sphere

Partial Difference Formulation

Ordinarily the partial difference formulation for the Laplacian in two dimensions uses five points in a star pattern, such as the circled points in Figure 1. Usually no averaging is necessary as the second central difference in each dimension approximates the Laplacian, and the flux is taken as the center point in the star.

The star pattern was tested for the truncated sphere problem using a $\phi = 0$ (extrapolated) boundary condition. The results are disastrous, however. In the limit as $h \rightarrow 0$ (hemisphere), it was found that B approached 3.8, while the analytic value is known to be 4.49. At the full sphere limit $h \rightarrow 1$, $B \rightarrow \pi$, the analytic value. This result could be expected. If the flux had actually been separable as this star pattern assumes, there would be no need to use numerical methods, because two relatively simple second order differential equations would have resulted which could have solved for B^2 analytically. From the numerical analysis point of view, the 5-point star pattern approximation for the flux neglects the cross product effects of $R(r)$ and $U(\theta)$, which become increasingly important as the system deviates further from the case of a sphere, since $\frac{\partial \phi}{\partial \theta} = 0$ for all points in the sphere. Furthermore we might conclude that $\frac{dB^2}{dh} \approx 0$ near the full sphere, whereas there is a relatively high slope of the curve near the hemisphere.

A different scheme was chosen which allows for the non-separability of the differential equation and thus greatly improves accuracy. This method incorporates (1) products of $R(r)$ and $U(\theta)$ terms, (2) a more accurate computation of derivatives to replace the central difference method which is only accurate to about $\pm 5\%$, (3) a better approximation

of the flux in the neighborhood of a point (r, θ) , and (4) average quantities, which smear deviations of the approximations from the actual flux.

Figure 2 is a section of the lattice of Figure 1. All nine⁽³⁾ of the circled points are used to fit the flux in the region around point 5.

FIGURE 2. Nine Point Pattern for Difference Equations

Since a Lagrange fit⁽²⁾ passes through all lattice points exactly, define

$$L_{x_i} = \frac{\prod_{j \neq i}^3 (x - x_j)}{\prod_{j \neq i}^3 (x_i - x_j)} = \text{Lagrange coefficient for (3) point } i \text{ in the } x \text{ dimension,}$$

and write

$$\phi \approx \sum_{k=1}^9 L_{r_i} L_{\theta_j} \phi_k, \quad (4)$$

where r_i and θ_j refer to the r and θ values at point k . For example, the $k = 4$ component of the sum (4) is

$$\frac{(r-r_2)(r-r_3)(\theta-\theta_1)(\theta-\theta_3)}{(r_1-r_2)(r_1-r_3)(\theta_2-\theta_1)(\theta_2-\theta_3)} \phi_4. \quad (5)$$

Then, the average flux $\bar{\phi}$ in the region between r_1 and r_3 , and θ_1 and θ_3 is

$$\bar{\phi} = \frac{\int_{\Psi=0}^{2\pi} \int_{r_1}^{r_3} \int_{\theta_1}^{\theta_3} \phi \, dv}{\int_{\Psi=0}^{2\pi} \int_{r_1}^{r_3} \int_{\theta_1}^{\theta_3} dv}. \quad (6)$$

For the 3 dimensions of the sphere (θ, Ψ, r)

$$dv = (r d\theta)(r \sin \theta d\Psi)(dr) = r^2 \sin \theta dr d\theta d\Psi. \quad (7)$$

Integration on Ψ will cancel, of course, leaving

$$\bar{\phi} = \frac{1}{V} \sum_{k=1}^9 \phi_k \int_{r_1}^{r_3} r^2 L_{r_i} dr \int_{\theta_1}^{\theta_3} \sin \theta L_{\theta_j} d\theta, \quad (8)$$

$$\text{where } V = \frac{1}{3} (r_3^3 - r_1^3) (\cos \theta_1 - \cos \theta_3). \quad (9)$$

$$\text{thus } \bar{\phi} = \sum_{k=1}^9 C_k \phi_k, \quad (10)$$

$$\text{where } C_k = \frac{1}{V} \int_{r_1}^{r_3} r^2 L_{r_i} dr \int_{\theta_1}^{\theta_3} \sin \theta L_{\theta_j} d\theta \quad (11)$$

$$\text{and } \sum_{k=1}^9 C_k = 1,$$

as a consequence of the averaging process.

Now, the average Laplacian is

$$\begin{aligned} \overline{\nabla^2 \phi} &= \frac{1}{V} \left\{ \int_{\theta_1}^{\theta_3} \int_{r_1}^{r_3} \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) r^2 \sin \theta dr d\theta \right. \\ &\quad \left. + \int_{r_1}^{r_3} \int_{\theta_1}^{\theta_3} \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \phi}{\partial r} \right) r^2 \sin \theta d\theta dr \right\} \\ \overline{\nabla^2 \phi} &= \frac{1}{V} \left\{ \int_{\theta_1}^{\theta_3} \left(r^2 \frac{\partial \phi}{\partial r} \right)_{r_1}^{r_3} \sin \theta d\theta \right. \\ &\quad \left. + \int_{r_1}^{r_3} \left(\sin \theta \frac{\partial \phi}{\partial r} \right)_{\theta_1}^{\theta_3} dr \right\} \quad (12) \end{aligned}$$

before using a Lagrange fit in the alternate dimension. A flux fit (Equation 4) has already been obtained; hence, it would seem reasonable to use it to compute derivatives. However, the derivatives so obtained are the same as the central differences used in the star pattern formulation.

If all points along a particular r or θ cut are used in fitting the flux, a much better approximation for the derivative results. That is, use

$$\phi(x, y_i) = \sum_{k=1}^N L_{x_k} \phi_k, \quad (13)$$

where, analogous to Equation (3) for N points on the cut,

$$L_{x_k} = \frac{\prod_{j \neq k}^N (x - x_i)}{\prod_{j \neq k}^N (x_i - x_j)} \quad (14)$$

The derivative of this function may be taken analytically and evaluated at $x = x_\ell$, the coordinate of the point at which the derivative is desired.

It can be seen from Equation (14) that for each point except $x = x_\ell$, this evaluation will yield only one nonzero term. For example, the k^{th} component of the derivative is

$$D_k = \frac{\prod_{j \neq k, \ell}^N (x_\ell - x_j)}{\prod_{j \neq k}^N (x_k - x_j)} \quad (15)$$

It can also be shown by considering a function $\phi(x) = \text{const.}$, that the component at the point of the derivative is

$$D_{\ell} = - \sum_{k \neq \ell}^N D_k. \quad (16)$$

Hence the N^{th} order difference approximation for the first derivative $\frac{\partial \phi}{\partial x}$ evaluated at x_{ℓ} is

$$\frac{\partial \phi}{\partial x} (x_{\ell}, y_i) = \sum_{k=1}^N D_k \phi_k. \quad (17)$$

The general rule that the more points used in a difference calculation, the better the accuracy has been applied. About 8 points are used for the derivative rather than only 3. After the derivatives are found they may be fit in the other dimension with a 3 point Lagrange function and integrated according to Equation (12). This, of course, includes an approximation for the cross terms which were discarded before.

Boundary Conditions

The boundary conditions are numerous and varied, and several different kinds of approximations are made. The following discussion is applicable to either bare or reflected systems.

Boundary conditions may be used judiciously to eliminate the fluxes at lattice points along the $\theta = 0, \pi$ axis, including the point $r = 0$.

For an area adjacent to the $\theta = 0, \pi$ axis (Figure 3) it will be required in the computation of $\nabla^2 \phi$ and $\bar{\phi}$ to find such quantities as

$$(a) \quad \left. \frac{\partial \phi}{\partial \theta} \right|_{\text{pt.1}}$$

(b) $\frac{\partial \phi}{\partial r} \Big|_{\text{pt. 2}}$

(c) component of pt. 1 in $\bar{\phi}$

(d) component of pt. 1 in $\frac{\partial \phi}{\partial \theta} \Big|_{\text{pt. 3}}$

FIGURE 3. An Area Adjacent to $\theta = 0, \pi$ Axis (Note point numbers.)

The value of (a) is zero by symmetry, so no problem arises. The component (d) may be eliminated by decreasing the number of points along the angular cut to exclude the point at 1. There is negligible effect on the accuracy of derivatives approximated by 8 points or by 7 points along a cut, for instance.

Both (b) and (c) can be handled by the same trick. In each case there is a function whose derivative with respect to θ is zero at $\theta = 0$. That is, for (b)

$$\frac{\partial}{\partial \theta} \left(\frac{\partial \phi}{\partial r} \right) \Big|_{\theta=0} = 0 \quad (18)$$

and for (c)

$$\left. \frac{\partial \phi}{\partial \theta} \right|_{\theta=0} = 0 \quad (19)$$

If a 3 point (quadratic) fit is written and this condition applied, the flux value at point 1 can be eliminated. Using an arbitrary function $y(\theta)$ with lattice points (Figure 3) of θ_0 , θ_1 , and θ_2 and function values of y_1 , y_2 , y_3

$$y = \frac{(\theta - \theta_1)(\theta - \theta_2)}{(\theta_0 - \theta_1)(\theta_0 - \theta_2)} y_1 + \frac{(\theta - \theta_0)(\theta - \theta_2)}{(\theta_1 - \theta_0)(\theta_1 - \theta_2)} y_2 + \frac{(\theta - \theta_0)(\theta - \theta_1)}{(\theta_2 - \theta_0)(\theta_2 - \theta_1)} y_3$$

or, since $\theta_0 = 0$

$$y = \frac{(\theta - \theta_1)(\theta - \theta_2)}{\theta_1 \theta_2} y_1 + \frac{\theta(\theta - \theta_2)}{\theta_1(\theta_1 - \theta_2)} y_2 + \frac{\theta(\theta - \theta_1)}{\theta_2(\theta_2 - \theta_1)} y_3 \quad (20)$$

Then

$$\frac{\partial y}{\partial \theta} = \frac{[(\theta - \theta_1) + (\theta - \theta_2)]}{\theta_1 \theta_2} y_1 + \frac{[\theta + (\theta - \theta_2)]}{\theta_1(\theta_1 - \theta_2)} y_2 + \frac{[\theta + (\theta - \theta_1)]}{\theta_2(\theta_2 - \theta_1)} y_3$$

Evaluating at $\theta = 0$

$$0 = -\frac{\theta_1 + \theta_2}{\theta_1 \theta_2} y_1 - \frac{\theta_2}{\theta_1(\theta_1 - \theta_1)} y_2 - \frac{\theta_1}{\theta_2(\theta_2 - \theta_1)} y_3$$

which gives

$$y_1 = \frac{\theta_2^2}{\theta_2^2 - \theta_1^2} y_2 - \frac{\theta_1^2}{\theta_2^2 - \theta_1^2} y_3 \quad (21)$$

Substitution of (21) into (20) gives the required function $y(\theta)$ with no y_1 component, namely

$$y(\theta) = \left[\frac{\theta(\theta - \theta_2)}{\theta_1(\theta_1 - \theta_2)} + \left(\frac{\theta_2^2}{\theta_2^2 - \theta_1^2} \right) \frac{(\theta - \theta_1)(\theta - \theta_2)}{\theta_1} \right] y_2$$

$$+ \left[\frac{\theta(\theta - \theta_1)}{\theta_2(\theta_2 - \theta_1)} - \left(\frac{\theta_1^2}{\theta_2^2 - \theta_1^2} \right) \frac{(\theta - \theta_1)(\theta - \theta_2)}{\theta_2} \right] y_3$$

Components at $r = 0$ may be eliminated by similar methods. Both the flux and radial component of the Laplacian are multiplied by $r^2 \rightarrow 0$ and are thus eliminated. Obviously $\frac{\partial \phi}{\partial \theta} \big|_{r=0} = 0$ and the point at $r = 0$ may be excluded in calculating $\frac{\partial \phi}{\partial r}$ at other points in the lattice.

The boundary conditions around the outside surface are different depending on whether the truncated sphere is bare or reflected.

Bare Truncated Sphere

Two methods for the external boundary conditions have been examined. The most obvious is to set the flux to zero all around the outside. That is, to define the extrapolated boundary of the system. A zero flux around the curved section of the boundary is no problem. But along the top surface it is found that the 9-point pattern used above to average the flux and the Laplacian does not fit within the system. That is, one of the 9 points lies out in the void as in Figure 4.

FIGURE 4. Lattice Near Top Surface Boundary (Point Number 9 lies outside the system.)

A rigorous method may be applied here, which although being very tedious algebraically, does give the correct answer. It consists of analytically substituting the $\phi = 0$ boundary condition and eliminating the unwanted point before carrying out the averages. It is briefly sketched in Appendix B.

With the $\phi = 0$ outer boundary condition, the buckling becomes independent of the extrapolation length, i.e., independent of material cross sections. Hence, a universal curve of buckling versus normalized height may be calculated for the bare system which is applicable to all materials. See Figures 7, 8, and 9. One must still know an extrapolation length to really apply this curve, however, since it is the height of the physical system which is actually measured experimentally. An analytic expression which fits this curve well was proposed by W. A. Reardon, unpublished. See Appendix C.

A second method is available for bare systems which also allows simple extension to reflected systems. The boundary condition can be written $\underline{J}^- = 0$, i.e., no reentrant neutrons. At each point on the boundary the condition

$$\phi + 2.13 D \frac{\partial \phi}{\partial r} = 0 \quad (22)$$

can be applied, where D is the diffusion coefficient (either thermal or energy averaged) and 2.13 is a transport theory corrected constant.

Just inside the top surface the problem of an extraneous point will crop up again. A simple approximation can be made to get rid of it. See Appendix B.

Reflected Truncated Spheres

Most of the Critical Mass Unit experiments were carried out with the geometry shown in Figure 5.

The simplest approximation that can be made is to set up the boundary conditions around the outside as

$$\underline{J}^- = \alpha \underline{J}^+, \quad (23)$$

where α is the albedo. The calculation of the side and top albedoes is shown in Appendix D.

This condition for points along the fuel solution boundary can be written as

$$\phi(1 - \alpha) + 2.13 D(1 + \alpha) \frac{\partial \phi}{\partial x} = 0 \quad (24)$$

where $\frac{\partial \phi}{\partial x}$ represents the derivative normal to the surface of interest. Around the sides this derivative is simply $\frac{\partial \phi}{\partial r}$. But along the top surface $\frac{\partial \phi}{\partial x}$ must be broken into the normal components in r and θ coordinates, namely

$$\frac{\partial \phi}{\partial x} = \frac{\sin \theta}{r} \frac{\partial \phi}{\partial \theta} - \cos \theta \frac{\partial \phi}{\partial r} \quad (25)$$

Recall that $\theta > \pi/2$, $\cos \theta < 0$.

FIGURE 5. Reflected Truncated Sphere Geometry "Infinite"
Water Reflector

Solution for the Buckling

Once the boundary conditions are incorporated, an eigenvalue matrix is obtained. Using the notation of double underline for matrices and single underline for vectors, the equation has the form

$$\underline{A} \underline{\phi} = - B^2 \underline{C} \underline{\phi} \quad (26)$$

Equation (26) is not the ordinary kind of eigenvalue matrix equation because of the presence of matrix $\underline{C}$. Furthermore, the lowest eigenvalue is desired, whereas often the largest is sought.

Equation (26) can be solved by inverting $\underline{A}$ and setting into standard form as follows:

$$\underline{A} \underline{\phi} = - B^2 \underline{C} \underline{\phi}$$

$$\underline{\phi} = - B^2 \underline{A}^{-1} \underline{C} \underline{\phi}$$

$$(-\underline{A}^{-1}\underline{C}) \underline{\phi} = \frac{1}{2} \frac{\underline{\phi}}{B}$$

$$\text{or } \underline{G} \underline{\phi} = \lambda \underline{\phi} , \quad (27)$$

where the largest positive λ is sought. Although this would be a straightforward solution, numerical difficulties arise in the inversion of $\underline{A}$ which is about 150×150 . Only 3 or 4 figure accuracy can be gained by present computer techniques using single precision arithmetic on such a matrix. Furthermore, the largest $\lambda = \frac{1}{2} \frac{1}{B}$ would be (for the bare case) between $1/\pi^2$ and $1/4.49^2$ or about 0.1 and 0.05. Thus further loss of accuracy results making the answers meaningless. An alternate method has been devised which a) is usually faster than inverting a large matrix, b) gives approximate results while converging to warn of trouble, and c) can give accuracy of 5 or 6 figures in B^2 if necessary.

This method briefly involves a) assuming a realistic value of B^2 (since the limits for the hemisphere and sphere are known), b) holding one value of the flux vector constant, i.e., normalizing, c) solving the resulting set of $n-1$ simultaneous equations for the other $n-1$ fluxes, d) finding the deviation from zero of the n^{th} equation using the n fluxes

and the assumed buckling, and c) repeating the process for a new buckling which will give a deviation closer to zero in the n^{th} equation. Since there is usually rapid convergence for B^2 and several simultaneous sets can be solved in the same time as one inversion, a computer time saving can be realized. Furthermore, each iteration may be checked to assure that the calculation is proceeding correctly in the early stages. Since there is no partial solution or simple approximation for a 150×150 set, this method also aids in debugging the large computer program. A sample hand calculation of this type is given in Appendix E.

RESULTS

Accuracy of the approximations is the primary criterion to assess the value of the methods and the meaningfulness of the results. The necessary approximations given above were used to calculate the bare hemisphere and sphere using the $\phi = 0$ boundary condition. Both are degenerate cases, since the 9-point lattice fits exactly for the hemisphere and there is no angular derivative in the sphere.

TABLE II.

Case	Analytic B^2 (Unit Extrapolated Radius)	Calculated B^2
Sphere	$\pi^2 = 9.86960$	9.86947
Hemisphere	$(4.4934)^2 = 20.19073$	20.18827

Table II indicates that the approximations are accurate to about four figures for these cases

For checking the accuracy of the approximations for the top surface of a general truncated sphere (Appendix B), it was useful to find a function, representing a flux, which has the proper boundary conditions. This may be done for a limited portion of the region near the surface by assuming such a function in 2 rectangular coordinates, then transforming to spherical coordinates. This analytic function may

then be used to find the flux at the mesh points of interest. Furthermore, this function may be analytically differentiated and integrated over the volume of interest to give exact averages of $\nabla^2 \phi$ and ϕ . The 9-point flux approximation may be tested against these analytic averages, since the mesh point fluxes are known. Use of this analytic function also aids in programming. The results of such analysis show that the approximations are good to about $\pm 0.05\%$ of the analytic averages for $\nabla^2 \phi$ and ϕ . Therefore, it can be assumed that the eigenvalues $B^2(h)$ are no more than 0.5% from the true value for one group theory, other things being consistent.

It must be pointed out that there is a slight, but significant difference in the physical system and the model used here. The model here uses extrapolated dimensions, while experimental measurements are made on physical dimensions. An extrapolation length is often available from theory or experiment and should be used to correct the results to a physically obtainable system. The slight distortion of the sphere in changing from an extrapolated to a physical system is not considered important. See Figure 6.

The resulting graph of B^2 versus h for unit extrapolated radius is shown in Figure 7. It appears to have the correct limiting behavior for spheres and hemispheres.

Radial and angular flux plots for the case of a 3/4 full ($h = 1/2$) sphere are given in Figures 8 and 9. A note on the strange flux shape on the radial cuts near the corner of the truncated sphere (ϕ_7, ϕ_8, ϕ_9 Figure 8) is given in Appendix F.

FIGURE 6. The Physical and Extrapolated Truncated Bare Sphere

TRUNK - Code

Input

<u>Format</u>	<u>Meaning</u>
I 10	0 bare, 1 reflected by "infinite" water
E 10.6	HR, actual height above bottom
E 10.6	RR, actual fuel radius
E 10.6	D, Diffusion coefficient in same units as HR and RR; or (extrapolation)/2.13
E 10.6	GMLPU, Pu concentration in gms/l. (Used for reflected cases only.)

Internal Workings

The TRUNK code uses the above methods plus several others to give maximum numerical stability, flexibility and simplicity for the user's convenience.

Neg. 0651202-2

FIGURE 7. Buckling Versus Height Above Hemisphere for Unit Radius

Neg. 0651202-1

FIGURE 8. Normalized Radial Flux Plot for H = 0.5

Neg. 0651202-3

FIGURE 9. Normalized Angular Flux Plot for $H = 0.5$

Besides the above methods, such conveniences as internal setting of the lattice and automatic flux plotting and full-sphere conversion are also included.

Output

TRUNK prints the input data supplied. If the buckling converges with all positive fluxes or if it does not converge after 8 iterations, the final iteration flux and buckling are tabulated. If the run is successful, a full equivalent sphere calculation is made and the flux of the truncated sphere is plotted for both radial and angular dimensions. If the case is unsuccessful, no conversion or plot is made, but the source of the trouble is pinpointed as well as possible.

APPENDIX A

ANALYTICAL SOLUTION OF HELMHOLTZ EQUATION
FOR A HEMISPHERE

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APPENDIX A

Analytical Solution of Helmholtz Equation for a Hemisphere

FIGURE 10. Hemisphere Showing Coordinates on Extrapolated Boundaries.

The Helmholtz equation is

$$\nabla^2 \phi + B^2 \phi = 0 \quad (28)$$

which for spherical geometry and symmetry in 1 coordinate is

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \phi}{\partial \theta} \right) + B^2 \phi = 0. \quad (29)$$

Assume separability

$$\phi(r, \theta) \equiv R(r) T(\theta). \quad (30)$$

Substituting (30) into (29)

$$\frac{1}{r^2} T \frac{\partial}{\partial r} \left(r^2 \frac{\partial R}{\partial r} \right) + \frac{R}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial T}{\partial \theta} \right) + B^2 RT = 0 \quad (31)$$

Multiplying through by $\frac{r^2}{TR}$ yields

$$\frac{1}{R} \frac{\partial}{\partial r} \left(r^2 \frac{\partial R}{\partial r} \right) + B^2 r^2 + \frac{1}{T \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial T}{\partial \theta} \right) = 0$$

Now, of course, the r and θ terms are separate and equal to a constant, say $\ell(\ell + 1)$

Rearranging the θ equation

$$\frac{d}{d\theta} \sin \theta \frac{dT}{d\theta} + \ell(\ell + 1) \sin \theta T = 0 \quad (33)$$

Define

$$x = \cos \theta, \quad (34)$$

Then

$$\begin{aligned} \sin \theta &= \sqrt{1 - x^2} \\ \frac{dx}{d\theta} &= -\sin \theta = -\sqrt{1 - x^2} \\ \frac{dT}{d\theta} &= \frac{dT}{dx} \frac{dx}{d\theta} = -\sqrt{1 - x^2} \frac{dT}{dx} \\ \frac{d}{d\theta} \sin \theta \frac{dT}{d\theta} &= -\frac{d}{d\theta} (1 - x^2) \frac{dT}{dx} = -(1 - x^2) \frac{d}{dx} \frac{dT}{d\theta} \\ &= -(1 - x^2) \frac{d}{dx} \left[-\sqrt{1 - x^2} \frac{dT}{dx} \right] \\ &= (1 - x^2)^{3/2} \frac{d^2 T}{dx^2} - 2x\sqrt{1 - x^2} \frac{dT}{dx} \end{aligned} \quad (35)$$

Substituting these relations into (33) yields

$$(1 - x^2) \frac{d^2 T}{dx^2} - 2x \frac{dT}{dx} + \ell(\ell + 1) T = 0 \quad (36)$$

which is the standard form of Legendre's equation having solutions

$$T_\ell(x) = P_\ell(x), \quad \ell = 0, 1, 2, \dots \quad (37)$$

The first nontrivial solution occurs for $\ell = 1$, that is,

$$T(x) = P_1(x) = x = \cos \theta. \quad (38)$$

Equation (38) is the first acceptable solution for the coordinate which meets the boundary condition of $T = 0$ at $\theta = \frac{\pi}{2}$ and $\frac{dT}{d\theta} = 0$ at $\theta = 0$. Substituting the value $\ell = 1$ into the equation for the r portion of the flux, (32), gives

$$\frac{d}{dr} r^2 \frac{\partial R}{\partial r} + (B^2 r^2 - 2) R = 0 \quad (39)$$

or

$$r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} + (B^2 r - 2) R = 0 \quad (40)$$

Equation (40) is a Bessel equation of the form

$$x^2 y'' + ax y' + (b^2 x^2 + c) y = 0, \quad (41)$$

where

$$x = r$$

$$y = R$$

$$a = 2$$

$$b = B$$

$$c = 2,$$

which has the solution

$$y = x^\alpha [c_1 J_m(bx) + c_2 Y_m(bx)], \quad (42)$$

where

$$\alpha = \frac{1-a}{2} = \frac{1-2}{2} = -1/2$$

$$m = \frac{\sqrt{(1-a)^2 - 4c}}{2} = \frac{\sqrt{1+8}}{2} = 3/2$$

and c_1 and c_2 are arbitrary constants.

Thus (40) has the solution

$$R = r^{-1/2} [c_1 J_{3/2}(Br) + c_2 Y_{3/2}(Br)]. \quad (43)$$

Applying boundary conditions, c_2 is forced to zero. The lowest eigenvalue B occurs for unit extrapolated radius when

$$J_{3/2}(B) = 0 \quad (44)$$

which is

$$B = 4.4934 \quad (45)$$

The flux solution is

$$\phi(r, \theta) = c_1 \sqrt{\frac{\cos \theta}{r}} J_{3/2}(4.4934 r) \quad (46)$$

Identities between trigonometric and Bessel functions may be used to obtain

$$\phi(r, \theta) = c_1 \cos \theta \left[\frac{\sin Br}{(Br)^2} - \frac{\cos Br}{Br} \right]. \quad (47)$$

It may be easily shown that B may also be obtained from the transcendental function

$$B = \tan B. \quad (48)$$

APPENDIX B

TOP SURFACE BOUNDARY CONDITIONS

APPENDIX B

Top Surface Boundary Conditions

FIGURE 11. Lattice Points at the Top Surface

It is apparent from Figure 11 that the 9-point flux pattern will not fit properly at the top surface. The boundary condition $\phi = 0$ may be applied along this surface in the bare case since it represents the extrapolated boundary. (Recall coordinate system for solution of hemisphere.) A nice trick would be to eliminate the unwanted point (Number 9 in Figure 11) by use of the boundary condition.

Ordinarily, the 9-point pattern flux was approximated by

$$\phi(r, \theta) = \sum_{k=1}^9 L_{r_i} L_{\theta_j} \phi_k . \quad (4)$$

For the top surface we substitute the condition

$$\phi = 0 \quad \text{for } r = h / \cos \theta \quad (49)$$

(See Figure 1), and solve for ϕ_9 ($\phi_1, \phi_2, \dots, \phi_8$). We need not carry the coefficients for ϕ_6 and ϕ_8 , however, since the flux is zero anyway. Substitution of this function in equation (4) gives

$$\phi(r, \theta) = \sum_{\substack{k=1 \\ k \neq 6}}^7 \phi_k L_{\theta_j} \left(L_{r_i} - L'_{r_i} \frac{L_{r_3}}{L'_{r_3}} \right), \quad (50)$$

where the L' indicates evaluation at $r = -h/\cos \theta$. L_{r_3} and L'_{r_3} are the Lagrange functions as Point 9, of course.

Equation (50) would be feasible except for 2 minor difficulties. First, the quantity in brackets goes to zero at lattice Point Number 3, since it has the same radius at Point Number 9. This means that we are preserving the quadratic (i.e., 3 point) nature of the fitting process in 1 dimension, but since the flux at Point 10 has now been specified as zero, the fit will pass through it rather than Point 3. The loss of the coefficient for Point 3 results in lost accuracy since the flux is allowed merely to fall on the curve passing through Points 1, 2, and 10, and will not, in general, be the true value at that point. The solution to this difficulty is simply to use a 4-point or cubic fit along this cut, thus forcing the flux fit to pass through its actual value at Point 3.

Secondly, when this flux approximation is integrated to get the components of $\bar{\phi}$, some nonintegrable functions will show up because of the substitution of (49). To preserve accuracy, and eliminate any need for numerical integrations, the angular Lagrange interpolation function is formed on $\cos \theta$ rather than θ itself. Defining

$$\psi_j = \cos \theta_j, \quad (51)$$

the complete flux function is

$$\phi(r, \theta) = \phi_1 \left\{ \frac{(\psi - \psi_2)(\psi - \psi_3)(r - r_2)}{(\psi_1 - \psi_2)(\psi_1 - \psi_3)(r_1 - r_2)(r_1 - r_3)(r_1 - r_4)} \right\}$$

$$\begin{aligned}
& \left[\frac{(r-r_3)(r-r_4)}{h\psi_4} - \frac{(r-r_1)(\psi-\psi_1)(\psi-\psi_2)}{\psi\psi_1\psi_2(\psi-\psi_4)} \right] \Bigg\} \\
& + \phi_2 \left\{ \frac{(\psi_1-\psi_2)(\psi_1-\psi_3)(r_2-r_1)(r_2-r_3)(r_2-r_4)}{(\psi-\psi_2)(r-r_1)} \right. \\
& \left. \left[\frac{(r-r_3)(r-r_4)(\psi-\psi_3)}{h\psi_4} - \frac{(r-r_2)(\psi-\psi_2)(\psi-\psi_1)}{\psi\psi_1\psi_2} \right] \right\} \\
& + \phi_3 \left\{ \frac{(\psi-\psi_2)(\psi-\psi_3)(r-r_1)(r-r_2)}{(\psi_1-\psi_2)(\psi_1-\psi_3)(r_3-r_1)(r_3-r_2)(r_3-r_4)} \right. \\
& \left. \left[\frac{(r-r_4)}{h} - \frac{h(\psi-\psi_1)}{\psi\psi_1} \right] \right\} \\
& + \phi_4 \left\{ \frac{(\psi-\psi_1)(\psi-\psi_3)(r-r_2)}{(\psi_2-\psi_1)(\psi_2-\psi_3)(r_1-r_2)(r_1-r_3)} \right. \\
& \left. \left[\frac{(r-r_3)}{\psi_4} - \frac{\psi_4(r-r_1)(\psi-\psi_2)}{\psi_2(\psi-\psi_4)} \right] \right\} \\
& + \phi_5 \left\{ \frac{(\psi-\psi_1)(r-r_1)}{(\psi_2-\psi_1)(\psi_2-\psi_3)(r_2-r_1)(r_2-r_3)} \right. \\
& \left. \left[\frac{(r-r_3)(\psi-\psi_3)}{\psi_3(r-r_2)} - \frac{\psi_3(r-r_2)(\psi-\psi_2)}{\psi_2} \right] \right\} \\
& + \phi_7 \left\{ \frac{(\psi-\psi_1)(\psi-\psi_2)(r-r_2)}{(\psi_3-\psi_1)(\psi_3-\psi_2)(r_1-r_2)(r_1-r_3)} \right. \\
& \left. \left[\frac{(r-r_3)}{\psi_4} - \frac{\psi_4(r-r_1)(\psi-\psi_2)}{\psi(\psi-\psi_4)} \right] \right\} . \tag{52}
\end{aligned}$$

The flux fit (52) must now be integrated over the odd-shaped volume of Figure 11. That is,

$$\bar{\phi} = \frac{1}{V'} \left\{ \int_{r_1}^{r_3} \int_{\theta_1}^{\theta_2} \phi r^2 \sin \theta d\theta dr + \int_{\theta_2}^{\theta_3} \int_{r_1}^{-\frac{h}{\psi}} \phi r^2 \sin \theta dr d\theta \right\}, \quad (53)$$

where V' has a small correction to the volume of (9), namely,

$$V' = V - h \left[\frac{r_3^3}{3r_2} + \frac{r_2^2}{6} - \frac{r_3^2}{2} \right], \quad (54)$$

which has been derived by noting that some r_i and θ_j 's are related by virtue of the lattice points on the top surface.

After several attempts to get good approximations for the Laplacian at the top surface by methods similar to those in the Theory Section for interior points, these methods were abandoned for the following system. Since the flux approximation (52) already has the troublesome boundary condition included, it is possible to simply differentiate this function analytically to obtain the necessary components of the derivatives. Although this procedure is reasonably straightforward, it entails mountains of algebra inasmuch as these derivatives must be integrated over the odd-shaped volume to obtain $\nabla^2 \phi$. The resulting 35 pages of algebraic manipulations will not be reproduced here, as they add nothing to the grasp of the problem. A sample calculation might, however, give some insight to the numerical difficulties encountered in using high order polynomial fitting procedures. The flux fit, the $\phi = 0$ boundary condition, the derivatives, the volume differential and the integration of terms conspire to truncate the significant figures of the result so as to make it meaningless. For example, the terms in one section of a component integral (which may contain 10 or 12 such sections) might be these

$$4.21786438 - 1.69477351 + 3.16442986 - 2.16872719 \\ + 1.03247552 - 5.25964768 + 0.70837946 = 0.00000084. \quad (55)$$

The alternating signs arise from the product of several $(x - x_i)$ terms. Equation (55) is wholly typical of the great loss of significant figures in computing $\nabla^2 \phi$ and $\bar{\phi}$ near the top surface. It is so typical in fact, that the loss of significant figures was used as the primary criterion for debugging these 2 phases of the computer program. The computer can give satisfactory accuracy if double precision arithmetic is specified. Normally, the computer carries 8 significant figures, but under double precision logic it will carry 16 digits. The worst cases of (55) occur for the cubic fit of the first radial cut, when the loss of significant figures reaches 10. Use of double precision keeps the situation under control even at this extreme.

Two special cases occur which have not been mentioned. The first is illustrated in Figure 12.

FIGURE 12. Special Case. Circled Point Has Boundary Condition on $\frac{\partial \phi}{\partial r}$ and $\frac{\partial \phi}{\partial \theta}$.

In order to force the flux along the top surface to zero, there must be a relationship between $\frac{\partial \phi}{\partial r}$ and $\frac{\partial \phi}{\partial \theta}$. Namely, the

components of these derivatives along the top surface must be equal and opposite. That is,

$$\frac{\partial \phi}{\partial r} \cos \left(\theta - \frac{\pi}{2} \right) = \frac{1}{r} \frac{\partial \phi}{\partial \theta} \cos (\pi - \theta) \quad (56)$$

$$\frac{\partial \phi}{\partial \theta} = \frac{h \sin \theta}{\cos^2 \theta} \frac{\partial \phi}{\partial r} \text{ at } r = \frac{-h}{\cos \theta} . \quad (57)$$

Condition (57) is applied when $\frac{\partial \phi}{\partial r}$ is required for a region as shaded in Figure 12. The flux at the circled point is zero, of course.

The second special case is at the corner point where the curved and top extrapolated surfaces supposedly meet. There is no guarantee that such a point actually exists in a physical system, since extrapolated boundaries are mathematical tools rather than physical realities.

FIGURE 13. Approximate Juncture of Curved and Top Extrapolated Boundaries.

Little theoretical work has been done on the extrapolation length from curved surfaces, or at corners. Several text-books on reactor theory quote a limiting value of $4/3 \lambda_{TR}$.^(4,5,6) However, the result appears to be a mis-interpretation of a paper⁽⁷⁾ on the extrapolation length into a curved black absorber (control rod) from an infinite medium. Here we are interested in the extrapolation from a curved surface into an infinite black absorber (void), the inverse geometry of a control rod.

It appears intuitively that a standard calculation of neutron current at the boundary⁽⁸⁾ by integrating over the infinite half space on the medium side of the point would yield a slightly smaller extrapolation length. This integration would have some void sections for a sphere, whereas no void would be included for a slab surface. Figure 13 perhaps represents a better approximation for the extrapolated boundary than Figure 12. Further analysis would be required to clear up this difficulty, however.

With the boundary condition $\underline{J}^- = \alpha \underline{J}^+$ ($\alpha = 0$ for bare systems), some correction is necessary for the invalid lattice point (Point 9, Figure 11) in the calculation of the average $\nabla^2 \phi$ and ϕ for the shaded region. Since the integration is carried only to the physical boundary, the contribution of Point 9 to the averages is small. A reasonable and simple method for eliminating it altogether follows.

Assume a linear angular variation of the second radial derivative in the region of interest. The second derivative may be approximated by a fit of 3 points. The second derivative may be taken and will be seen to yield

$$\phi''(r_1 \psi) = 2 \left\{ \frac{\phi_1}{(r_1 - r_2)(r_1 - r_3)} + \frac{\phi_2}{(r_2 - r_1)(r_2 - r_3)} + \frac{\phi_3}{(r_3 - r_1)(r_3 - r_2)} \right\} \quad (58)$$

and similarly

$$\phi''(r_1 \psi_2) = 2 \left\{ \frac{\phi_4}{(r_1 - r_2)(r_1 - r_3)} + \frac{\phi_5}{(r_2 - r_1)(r_2 - r_3)} + \frac{\phi_6}{(r_3 - r_1)(r_3 - r_2)} \right\} \quad (59)$$

Extrapolating a linear fit of $\phi''(r_1\psi_1)$ and $\phi''(r_1\psi_2)$ to ψ_3 , and solving for $\phi_9(\phi_1, \phi_2 \dots \phi_8)$ one obtains

$$\begin{aligned} \phi_9 = & (r_3-r_1)(r_3-r_2) \frac{(\psi_2-\psi_3)\phi_1 + (\psi_3-\psi_2)\phi_4}{(\psi_2-\psi_1)(r_1-r_2)(r_1-r_3)} \\ & \frac{(\psi_2-\psi_3)\phi_2 + (\psi_3-\psi_2)\phi_5}{(\psi_2-\psi_1)(r_3-r_1)(r_3-r_2)} \frac{(\psi_2-\psi_3)\phi_3 + (\psi_3-\psi_2)\phi_6}{(\psi_2-\psi_1)(r_3-r_1)(r_3-r_2)} \\ & - \frac{\phi_7}{(r_1-r_2)(r_1-r_3)} - \frac{\phi_8}{(r_2-r_1)(r_2-r_3)} \end{aligned} \quad (60)$$

The same function is obtained by fitting quadratically in ψ and linearly in r , as a consequence of the condition that radii and angles meet at the top surface.

APPENDIX C

AN ANALYTIC EXPRESSION
FOR B^2 vs. h FOR BARE SPHERES

APPENDIX C

An Analytic Expression for B^2 vs. h for Bare Spheres

W. A. Reardon has proposed the following scheme for calculating the buckling of a truncated bare sphere.

The buckling is approximated by a linear interpolation between the hemisphere and the sphere based on the surface-to-volume ratio. This expression is accurate to about 1% with respect to the few truncated sphere measurements available and also the matrix theory given in this paper.

Starting with the analytic expressions for the buckling of a sphere B_S^2 and a hemisphere B_H^2 in terms of the value of each

$$B_S^3 = \frac{4/3 \pi^4}{V} = \frac{129.8878}{V_S} \quad (61)$$

$$\text{and } B_H^3 = \frac{(4.4934)^3 (2/3) \pi}{V_H} = \frac{189.5822}{V_H} \quad (62)$$

It is convenient to interpolate according to

$$B^3 = \frac{1}{V} A \frac{aS}{V} + C, \quad (63)$$

where S/V is the surface to volume ratio and a is the sphere radius. Carrying out the calculation of A and C , one finds

$$B^3 = \frac{40.0896 \frac{aS}{V} + 9.6099}{V} \quad (64)$$

The calculation of B^2 is facilitated by defining the height of the system in units of the radius, i.e., $0 \leq H \leq 2$.

Then one finds

$$\frac{aS}{V} = \frac{3}{H} \frac{(4-H)}{(3-H)} \quad (65)$$

the volume of a partially filled sphere in the same units is

$$V = \frac{\pi a^3}{3} H^2 (3-H) \quad (66)$$

APPENDIX D

REFLECTED SPHERE ALBEDO CALCULATIONS

APPENDIX D

Reflected Sphere Albedo Calculations

For applying the boundary condition $\underline{J}^- = \alpha \underline{J}^+$ to reflected truncated spheres (Figure 5) some analytic form for α , the albedo, of side and top fuel surfaces must be available.

The albedo around the curved portion is approximated by fitting a cubic equation to integrated energy albedoes for full 20 in. diameter spheres reflected by a 6 in. (effectively infinite) water blanket as calculated by B. H. Duane's Program S transport theory code. For best results, the curve is broken into two sections:

From $x = 0$ to 320 gm/l Pu

$$\alpha_S = 0.552246 - 1.01532(E-3)x + 3.06407(E-6)x^2 - 3.54085(E-9)x^3 \quad (67)$$

and for $320 < x \leq 1000$ gm/l Pu

$$\alpha_S = 0.459096 - 1.46619(E-4)x + 1.59576(E-7)x^2 - 6.05892(E-11)x^3. \quad (68)$$

It is assumed that these albedoes apply around the curved surface except at the point of junction with the top surface.

The top surface albedo is found by assuming an isotropic flux on the inside of the reflector above the void (Figure 5) and averaging the return probability to the fuel surface.

The infinitesimal probability that a neutron leaving the inside reflector surface from some area da , will directly enter an area da_2 on the fuel surface is

$$dp = \frac{(\hat{n}_1 \cdot \vec{\ell})(\hat{n}_2 \cdot \vec{\ell}) J_1 da_1 da_2}{\pi \ell^4} \quad (69)$$

where $\hat{n}_1$ and $\hat{n}_2$ are the unit normals to da_1 and da_2 , respectively, and, $\vec{l}$ is the vector between da_1 and da_2 . J_1 represents the inward neutron current at da_1 .

The average probability is then

$$\bar{p} = \frac{\int dp}{\int J_1 da_1} . \quad (70)$$

When the analytic integration of (69) is carried out over the fuel surface (da_2), a rather surprising result obtains. Namely, it is found that the probability is independent of position on the reflector, and is a simple function of the solution height. Hence, a knowledge of the flux shape on the inside of the reflector is unnecessary since the integrals over current just cancel in equation (70). One expects such a result for the hemispherical fuel configuration, by simple geometry, but good fortune indeed is it that such a form holds for any truncated sphere. The actual first flight probability function is

$$\bar{p} = \frac{H}{2R} , \quad (71)$$

with H measured from the bottom of the sphere. Hence the value $\bar{p} = 0.5$ for a hemisphere, as expected.

Taking multiple reflections into account and assuming the reflector coefficient to be the same as for the sides, one finds that the top surface albedo can be written

$$\alpha_T = \frac{\bar{p}\alpha_S}{1 - (1-\bar{p})\alpha_S} , \quad (72)$$

a simple result to what at first appears to be a difficult problem.

At the corner point (junction of curved and top surfaces) a simple average is taken between the $\underline{J}^- = \alpha \underline{J}^+$ boundary conditions for side and top.

APPENDIX E

ILLUSTRATIVE SOLUTION

OF $\underline{A X} = \lambda \underline{B X}$ BY RESIDUAL METHOD

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APPENDIX E

Illustrative Solution of $\underline{A} \underline{X} = \lambda \underline{B} \underline{X}$ by Residual Method

Given the matrix equation

$$\begin{pmatrix} 3 & 7 & 2 \\ 2 & 6 & 5 \\ 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \lambda \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 3 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

and that the lowest eigenvalue lies between

$$0.05 < \lambda < 0.2 ,$$

the solution may be obtained as follows:

Choose $X_3 = 1$. At $\lambda = 0.05$, the first 2 equations (n - 1, in general) may be written

$$3X_1 + 7X_2 + 2 = 0.05 (X_1 + 3X_2)$$

$$2X_1 + 6X_2 + 5 = 0.05 (X_2 + 3) ,$$

a simultaneous set of n - 1 equation, whose solution is

$$X_1 = 5.5347$$

$$X_2 = -2.67553 .$$

The residual of the nth equation (third equation here) is

$$\delta(\lambda) = X_2 + 4 - \lambda (2X_1 - X_2 + 1) ,$$

which for the above values of X_1 and X_2 yields

$$\delta(0.05) = 0.58722 .$$

Reworking the problem at $\lambda = 0.2$, we find

$$3X_1 + 7X_2 + 2 = 0.2 (X_1 + 3X_2)$$

$$2X_1 + 6X_2 + 5 = 0.2 (X_2 + 3) ,$$

and

$$X_1 = 4.8139$$

$$X_2 = -2.4186 .$$

The residual is

$$\delta(0.2) = -1.0279 .$$

Interpolation between $\delta(0.05)$ and $\delta(0.2)$ predicts $\delta(\lambda') = 0$ at

$$\lambda' = 0.1045 .$$

Reworking the problem at $\lambda' = 0.1045$ yields the solution

$$X_1 = 5.2862$$

$$X_2 = -2.5882$$

and

$$\delta(0.1045) = -0.068 .$$

Interpolation between $\delta(0.05)$ and $\delta(0.1045)$ predicts $\delta(\lambda'') = 0$ at

$$\lambda'' = 0.0988 .$$

This process is continued until the required eigenvalue, 0.09855 for this problem, has been converged upon. In the physical problem of the truncated sphere all fluxes (x_i 's) must be positive for an acceptable solution.

APPENDIX F

NOTE ON CURVATURE
OF FLUXES NEAR CORNERS

APPENDIX F

Note on Curvature of Fluxes Near Corners

One might well be intuitively disturbed by graphs of the radial flux for a bare system near the junction of the side and top surfaces. (See curves θ_7 , θ_8 , θ_9 on Figure 8.) Near the corner the second derivative changes sign obviously, and this appears to imply that neutrons are returning from the void, an impossible result.

However, it can be shown with a little mathematical work that this change in the second derivative is correct and probably will occur for all such corners.

The flux near the corner is very small. If J^- is viewed as being the number of neutrons crossing a unit area inward toward the center of the fuel, then near the corner

$$\frac{\partial J^-}{\partial r} \approx 0, \quad (73)$$

since the flux is very small.

Carrying out the differentiating

$$\begin{aligned} \frac{\partial}{\partial r} \left(\frac{\phi}{4} + \frac{D}{2} \frac{\partial \phi}{\partial r} \right) &\approx 0 \\ \frac{1}{4} \frac{\partial \phi}{\partial r} + \frac{D}{2} \frac{\partial^2 \phi}{\partial r^2} &\approx 0 \end{aligned} \quad (74)$$

Since $\frac{\partial \phi}{\partial r} < 0$, $\frac{\partial^2 \phi}{\partial r^2} > 0$, as Figure 5 indicates.

A less intuitive approach is to calculate the flux variation from the center of a two dimensional slab reactor with equal sides to its corner. The flux is the well known $\cos Bx$ function for each dimension. Then in two dimensions, the flux from the center diagonally to a corner follows a $\cos^2 Bx'$ form, which is very similar to the graphs in question. The x' above indicates that we are stretching the coordinates by $\sqrt{2}$ as we go along a diagonal.

A very similar curve results if one plots the flux for the hemisphere (Appendix A) from near the middle of the reactor to a corner.

This result appears to be general for corners. Recalling that the Laplacian has four terms for the two dimensions of interest in the truncated sphere, namely,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial^2}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial}{\partial \theta} \quad (75)$$

The positive second radial derivative simply says that the sum of the other three terms is a larger negative quantity, making the total ∇^2 negative, as expected.

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