A COMPARISON OF VARIOUS NUMERICAL SOLUTIONS
TO THE NONLINEAR VLASOV EQUATION

by

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ABSTRACT

Several methods for solving the nonlinear Vlasov equation are reviewed and compared. The numerical solutions of the Vlasov and Poisson set of equations for two physically significant problems are presented and a comparison is made with the results of particle simulations.

The problems considered are one-dimensional, with periodic boundary conditions and involve (1) Landau damping in a Maxwellian plasma, and (2) the two-beam instability with equal electron beams.

The numerical solutions of the Vlasov equation are based on the Fourier-Hermite and the Power Transform methods. A review of the Fourier-Hermite method is given with emphasis on the truncation difficulties. The Power Transform method is discussed and several truncation techniques are presented. It is found that an extrapolation procedure can be used which allows a truncation of the infinite matrix without causing numerical instabilities in the nonlinear system. Close quantitative agreement between the results of the two methods is found.

The particle simulations are based on the particle-in-cell method. Length weighting and the "quiet start" technique are used to suppress the noise and accurately control the initial conditions.

A small external electric field is utilized to initially perturb the system. Due to the relatively low noise level, Landau damping can be observed for a period of time. A decrease in the amplitude of the trapping oscillations is observed for the two-beam instability.

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I. INTRODUCTION

It has become increasingly apparent lately that computer simulation plays an indispensable role in the understanding of plasma phenomena. Although the dynamical equations of plasma physics are essentially Maxwell's equations and Newton's equations of motion, they are highly nonlinear and pose an impossible task for satisfactory analytical analysis; however, the gross macroscopic behavior of the plasma is quite simple to analyze through numerical simulation (see, e.g., Methods in Computational Physics, Volume 9).

Two fundamentally different formulations have emerged in the numerical simulation of plasmas. One method consists of numerically solving the Vlasov-Poisson set of equations while the other computes the dynamics of a large number of charged particles as they move in their self-consistent electric fields.

If a direct numerical simulation of the Vlasov equation is attempted, the independent variable v must be transformed in order to avoid a breakdown of the solution (Kellogg, 1965). The Fourier-Fourier Transform method (Knorr, 1963) incorporates a Fourier transformation of both phase space variables; whereas in the case of the Fourier-Hermite transformation method (Armstrong, 1966, 1967), only the spatial variable is Fourier transformed and the velocity variable is represented by a series using Hermite polynomials. The major

difficulty with the Fourier-Hermite method is the apparent numerical instability that arises when the infinite Hermite series is truncated. A large number of coefficients must be kept in order to compute for times of interest.

A new transformation scheme, the Power Transform method, has recently appeared (Joyce, Knorr and Meier, 1971). One of the purposes of this work is to apply the Power Transform method to two physically significant problems and compare the results with those obtained by the Fourier-Hermite method. Extremely close agreement is found. We will primarily be concerned with truncation procedures and the stability of each method. We find that the Power Transform method is amenable to an extrapolation procedure which yields an appreciable savings in computer time.

The second purpose of this thesis is to examine the particle simulation procedure. The particle simulation method represents the plasma by a large number of discrete charged masses which are followed by solving their equations of motion. The electric field is calculated from the particle positions at each time step and the system is allowed to evolve self-consistently. The "quiet-start" (Byers and Grewal, 1970) initialization procedure is employed. What is new is the application of an external electric field to initially perturb the plasma. This procedure allows for a firmer control of the initial conditions. We will apply the particle code to the same initial conditions as those considered with the Power Transform method and compare the results.

The problems considered are one-dimensional with periodic boundary conditions. The plasma is considered to be a collisionless electron gas; i.e., only the motions of the electrons will be considered and the system is macroscopically neutral with a uniform immobile positive background. Only electrostatic forces between the charges are considered. Solutions for both stable and unstable initial conditions are presented.

A brief outline follows: Section II reviews the Fourier-Hermite method with particular emphasis on the truncation difficulties. The results presented are due to Armstrong (1966, 1967, 1970). In Section III we present the Power Transform method as applied to stable and unstable initial conditions and compare the results with those obtained with the Fourier-Hermite code. A review of the discrete particle code is given in Section IV with a discussion of the initialization procedures. A comparison of the two methods is made for the stable and unstable cases.

II. REVIEW OF THE FOURIER-HERMITE TRANSFORM METHOD

A. Reduction of Equations to Dimensionless Form and Boundary Conditions

A collisionless one-dimensional plasma is described by the Vlasov equation

$$\frac{\partial f(x, v, t)}{\partial t} + v \frac{\partial f}{\partial x} - \frac{e}{m} E(x, t) \frac{\partial f}{\partial v} = 0, \quad (2.1)$$

where $f(x, v, t)$ denotes the one-dimensional distribution function.

The self-consistent electric field is given by Poisson's equation

$$\frac{\partial E(x, t)}{\partial x} = 4\pi n_0 e \left[1 - \int_{-\infty}^{+\infty} f(x, v, t) dv \right], \quad (2.2)$$

where n_0 is the density of the uniform, immobile ion background.

By making time measurements in units of ω_p^{-1} , where ω_p is the plasma frequency $\left(\left(\frac{4\pi n_0 e^2}{m} \right)^{\frac{1}{2}} \right)$, length measurements in terms of the Debye length (L_D) and velocities in terms of the thermal velocity ($V_{Th} = \omega_p L_D$) the above can be written in terms of dimensionless units:

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - E \frac{\partial f}{\partial v} = 0 \quad (2.3)$$

$$\frac{\partial E}{\partial x} = 1 - \int_{-\infty}^{+\infty} f(x, v, t) dv. \quad (2.4)$$

Dimensionless units will be used throughout this article.

The unbounded plasma can be depicted through a finite representation by considering the plasma to be confined between rigid, perfectly reflecting boundaries at $x = 0$ and $x = L$. For the reflection condition to hold for all time, the following equations must be satisfied:

$$\begin{aligned} f(0, v, t) &= f(0, -v, t) \\ f(L, v, t) &= f(L, -v, t) \\ E(0, t) &= E(L, t) = 0. \end{aligned} \quad (2.5)$$

Montgomery and Gorman (1962) have shown that the dynamics of a plasma confined in this manner are the same as for the unbounded plasma. If the initial distribution obeys the relation $f(-x, v, 0) = f(x, -v, 0)$ in the interval $0 > x > -L$ and is periodic in x with period $2L$, it follows that f is well-defined over the whole of the interval and can be represented by a Fourier series with wavelengths proportional to $2L$. See Figure 1.

Expanding $f(x, v, t)$ and $E(x, t)$ in a Fourier series of period $2L$ we obtain

$$f(x, v, t) = \sum_{n=-\infty}^{+\infty} f_n(v, t) e^{ink_0 x} \quad (2.6)$$

$$E(x,t) = \sum_{n=-\infty}^{+\infty} E_n(t) e^{ink_0 x} \quad (2.7)$$

where

$$f_n(v,t) = \frac{1}{2\pi} \int_{-L}^{+L} e^{-ink_0 x} f(x,v,t) dx \quad (2.8)$$

$$E_n(t) = \frac{1}{2\pi} \int_{-L}^{+L} e^{-ink_0 x} E(x,t) dx. \quad (2.9)$$

Substituting the above expressions into the Vlasov and Poisson equations yields the following set of Fourier transformed equations:

$$\frac{\partial f_n(v,t)}{\partial t} + ink_0 v f_n - \sum_{\substack{q=-\infty \\ q \neq n}}^{+\infty} E_{q-n} \frac{\partial f_q}{\partial v} = 0. \quad (2.10)$$

$$ink_0 E_n(t) = \delta_{n,0} - \int_{-\infty}^{+\infty} f_n(v,t) dv \quad (2.11)$$

with $n = 0, \pm 1, \pm 2, \dots$

Montgomery and Gorman (1962) have shown that conditions 2.5 are satisfied for all time by those $f_n(v,t)$'s which satisfy the following symmetry conditions

$$f_{-n}(v,t) = f_n(-v,t) = f_n^*(v,t) \quad (2.12)$$

and thus

$$E_{-n}(t) = -E_n(t) = E_n^*(t) .$$

Using these conditions in Equations (2.10) and (2.11) we obtain the same result for either $\pm n$, thus we need only compute for $n \geq 0$.

Considering the plasma to be confined between perfectly reflecting boundaries is equivalent to requiring periodic boundary conditions, as long as the distribution function is symmetric.

B. Expansion in Hermite Polynomials

Following the method originated by Armstrong (1966, 1967), we expand the velocity variable in the Hermite series:

$$f_n(v, t) = e^{-v^2/2} \sum_{m=0}^{\infty} Z_{n,m}(t) h_m(v) , \quad (2.13)$$

where

$$h_m(v) = \frac{(-1)^m e^{v^2/2}}{[m! \sqrt{2\pi}]^{1/2}} \frac{d^m e^{-v^2/2}}{dv^m} \quad (2.14)$$

is the orthonormal Hermite polynomial. The Hermite expansion had been used previous to Armstrong's work in analytical calculations by Weissglas (1962) and Engelmann, et al. (1963)

The expression for the distribution function becomes

$$f(x, v, t) = \sum_{n=-\infty}^{+\infty} \sum_{m=0}^{\infty} Z_{n,m}(t) e^{ink_0 x} h_m(v) e^{-v^2/2} . \quad (2.15)$$

Substituting this series into the Vlasov equation and collecting the coefficients for each mode and Hermite function, we obtain the following system of differential equations:

$$\begin{aligned} \dot{Z}_{n,m}(t) + nk_o \left[\sqrt{m} Z_{n,m-1} + \sqrt{m+1} Z_{n,m+1} \right] \\ + \frac{i\sqrt{m}}{k_o} (2\pi)^{\frac{1}{4}} \sum_{\substack{q=-\infty \\ q \neq n}}^{+\infty} \frac{Z_{q,m-1}}{n-q} Z_{n-q,o} = 0 \end{aligned} \quad (2.16)$$

and Poisson's equation takes the form

$$ik_o n E_n(t) = - (2\pi)^{\frac{1}{4}} Z_{n,o}(t) \quad (2.17)$$

with $-\infty < n < \infty$, $m \geq 0$.

We now have a matrix representation of the distribution function of the form

$$f \rightarrow \begin{bmatrix} Z_{00} & Z_{10} & Z_{20} & \cdot & \cdot & \cdot \\ Z_{01} & Z_{11} & Z_{21} & \cdot & \cdot & \cdot \\ Z_{02} & Z_{12} & Z_{22} & \cdot & \cdot & \cdot \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \end{bmatrix}.$$

The electric field is recovered from the first row of the matrix since

$$E_n(t) = \frac{i(2\pi)^{\frac{1}{4}}}{n k_o} Z_{n,o}(t) \quad (2.17)$$

In this representation, a Maxwellian distribution appears as $(2\pi)^{\frac{1}{4}}$ in the upper left corner of the matrix and zeros everywhere else. With $f(v,0) \sim v^n e^{-v^2/2}$ where n is of low order, the nonzero $Z_{n,m}(0)$'s can be confined to a small rectangular region in the upper left corner.

As the time increases, the region of nonzero matrix elements spreads out in both the Fourier and Hermite indices. This spreading creates a severe problem since the matrix cannot be allowed to expand indefinitely but must be truncated at some time. The difficulties that arise from the truncation mechanism are discussed in the next section.

C. Truncation Difficulties

We now have a set of first-order, nonlinear, coupled differential equations which lend themselves to numerical integration. Unfortunately, the system of Equations (2.16) and (2.17) is infinite and must somehow be truncated in both the Fourier and Hermite index before numerical integration can be done. One of the problems we are considering is a comparison of the various truncation procedures and the difficulties which beset each method.

One need only require that $f(x,v,t)$ and $E(x,t)$ be expandable in a strongly convergent Fourier series to remove the difficulty of a truncation in n . This requirement limits the class of initial conditions to those which are nearly homogeneous, but it is appropriate for the problems considered here. The more serious difficulty arises in the truncation of the Hermite index.

The term arising from $v \frac{\partial f}{\partial x}$ couples each Hermite index to the one above and below and in order to compute $Z_{n,m}$ for the next time step, $Z_{n,m+1}$ must be known. One cannot simply ignore the contribution of $Z_{n,M+1}$, where M is the boundary of the matrix, to the $Z_{n,M}$, because $Z_{n,M-1}$ and $Z_{n,M+1}$ are large and of opposite sign nearly cancelling one another out.

Although it is mainly the coefficients in the upper left corner of the matrix which are of interest, the solution spreads out into the rest of the matrix in the m -direction and the coefficients at the boundary grow large by the time $t = \frac{\sqrt{M}}{n k_0}$ (Armstrong, 1966). By choosing M large and setting $Z_{n,m} = 0$ for $m > M$, one can compute for times $t \leq \frac{\sqrt{M}}{n k_0}$ before the error propagates back from the boundary of the matrix and becomes significant. This instability is apparently due to the attempt of representing a continuous eigen-spectrum by a discrete finite spectrum (Joyce, Knorr and Meier, 1971). By following this truncation procedure, however, one is forced to choose $M \sim 0$ (1000) in order to compute for any significant length of time. This results in the consumption of a great deal of computer time; thus it is economically desirable to seek another truncation procedure.

Armstrong (1966, 1967) has been able to compute for an extra $M \Delta t$ length of time by solving for one less coefficient at each time step beginning at $t = \frac{\sqrt{M}}{n k_0}$; this permits one to use a few less coefficients and still compute for times of interest.

With the Fourier-Hermite code in its present form, the most significant truncation procedure is the use of a "collision term" in the right side of Equations (2.16). A "collision term" would serve to smooth out the velocity derivatives and thus prevent $Z_{n,M}$ from ever growing large.

An appropriate collision term is the simplified Fokker-Planck term (Lenard and Bernstein, 1959; Taylor and Comisar, 1963; Chandrasekhar, 1943) of the form

$$\left(\frac{\partial f}{\partial t}\right)_c = v_c \left\{ \frac{\partial(fv)}{\partial v} + \frac{\partial^2 f}{\partial v^2} \right\} \quad (2.18)$$

which assumes the form

$$(\dot{Z}_{n,m}(t))_c = -v_c m Z_{n,m}(t) \quad (2.19)$$

This collision term selectively suppresses the higher Hermite coefficients. Grant and Feix (1967) have shown that if $M v_c \sim 0(1)$, the introduction of this term avoids the breakdown of the representation at $t = \frac{\sqrt{M}}{n k_0}$. Collision effects, however, do become dominant after a time $t \sim M$, since this term drives f to a Maxwellian in a time $t \sim \frac{1}{v_c}$. The time for which one can study approximately collisionless phenomena is thus extended from $\frac{\sqrt{M}}{n k_0}$ to M with a significant savings in computer time.

The basic problem with each of the aforementioned truncation procedures is the large number of coefficients which must be kept in order to compute for times of interest. With the Fourier-Hermite code in its present form there is as yet no satisfactory extrapolation scheme to accurately estimate the $Z_{n,m+1}$ coefficient from the $Z_{n,m}$'s. We will return to this problem later; we now want to discuss the results of these truncation procedures.

D. Numerical Results

The two cases considered are the stable (Landau damped) and unstable (two-beam instability) initial conditions. For the stable case the initial condition is taken to be a simple Maxwellian distribution

$$f_o(x, v, 0) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2} (1 + \epsilon \cos kx). \quad (2.20)$$

See Figure 2. In terms of the expansion coefficients $Z_{n,m}$, the initial conditions become

$$Z_{0,0}(0) = \frac{1}{(2\pi)^{\frac{1}{4}}} \quad (2.21)$$

and

$$Z_{1,0}(0) = \frac{\epsilon/2}{(2\pi)^{\frac{1}{4}}} \quad (2.22)$$

and all of the remaining $Z_{n,m}(0) = 0$. The Fourier components of the electric field are given by

$$E_n(t) = \frac{i(2\pi)^{\frac{1}{4}}}{n k_0} Z_{n,0}(t) \quad (2.23)$$

A typical result of Armstrong's is shown in Figure 3. For this case we chose $k = 0.5$ and $\epsilon = 0.04$. The typical Landau-damped behavior is evident and follows the linear prediction until $t = 26 \omega_p^{-1}$ when the damping decrement falls below the linear value. The addition of a "collision term" to the Landau-damped case serves to increase the damping decrement and thus cannot be used as a satisfactory truncation method for this problem.

As predicted by the resonant particle interaction picture, the number of particles below the phase velocity in $f_0(v,t)$ decreases and the number above the phase velocity increases. This effect is illustrated in Figure 4.

For the unstable case the initial condition is taken to be a strongly unstable counterstreaming plasma of the form

$$f_0(x,v,0) = \frac{1}{\sqrt{2\pi}} v^2 e^{-v^2/2} (1 + \epsilon \cos k_0 x) \quad (2.24)$$

which has growing waves if $|k_0| < 1$ (Grant and Feix, 1967). See Figure 5. As an illustration we chose $k = 0.5$ and $\epsilon = 0.01$. The

numerical results show that nonlinear effects cause the growth of the field to cease after a small fraction ($\sim \frac{1}{10}$) of the stream energy has been converted to electric field energy, and that the system approaches an inhomogeneous equilibrium in the long time regime (Armstrong, et al., 1970). See Figure 6. Figure 7 shows the "filling in" of the "hole" of the distribution function. In order to slow down the increase in Hermite polynomials, Armstrong introduces the collision term on the right-hand side of Equations (2.16). A small collision frequency produces a slight, but noticeable, change in the electric field. See Figure 8.

As noted previously, the collision term cannot be used in the Landau-damped case as it adds to the damping. In the unstable case, the collision term competes with the waves in interacting with the distribution function; thus, the usefulness of the collision term becomes doubtful if the shape of $f_0(v)$ is of importance.

We have cited only two of the many experiments performed by Armstrong but they will suffice as a basis for comparison with other truncation methods.

III. THE POWER TRANSFORM METHOD

A. Introduction

As we noted, the major difficulty with the Fourier-Hermite transform method is the large number of coefficients that must be kept in order to compute for times of interest. Although the collision term does speed the calculation by allowing one to use less coefficients, it cannot be used for the stable case and its usefulness for the unstable case is somewhat doubtful if the distribution function is to be examined.

Recent work by Joyce, Knorr and Meier (1971) has provided a method of truncating the infinite matrix without causing numerical instabilities. We now want to analyze this method and compare the results with the previous work.

B. The Power Transform Method

Recalling the Fourier transformed Vlasov and Poisson equations

$$\frac{\partial f_n(v,t)}{\partial t} + i k_0 n v f_n(v,t) - \sum_{q=-\infty}^{+\infty} E_{n-q} \frac{\partial f_q}{\partial v} = 0 \quad (3.1)$$

$$i k_0 n E_n(t) = - \int f_n(v,t) dv, \quad (3.2)$$

we let

$$F_n(y, t) = \int_{-\infty}^{+\infty} f_n(v, t) e^{i v y} dv \quad (3.3)$$

where

$$F_{-n}(y, t) = F_n^*(-y, t). \quad (3.4)$$

The above then take the form

$$\frac{\partial F_n(y, t)}{\partial t} + k_0 n \frac{\partial F_n(y, t)}{\partial y} + \sum_{q=-\infty}^{+\infty} i E_q y F_{n-q}(y, t) = 0 \quad (3.5)$$

$$i E_q = \frac{-1}{k_0 q} F_q(0, t). \quad (3.6)$$

Writing $F_n(y, t)$ as an expansion in powers of y , we obtain

$$F_n(y, t) = \sum_{v=0}^{\infty} a_{n, v}(t) g_v y^v e^{-y^2/2}, \quad (3.7)$$

where

$$g_v = 2^{v/2} \frac{\Gamma\left(\frac{v+2}{2}\right)}{\Gamma(v+1)}. \quad (3.8)$$

Substituting this expansion in Equation (3.5), we obtain

$$\begin{aligned} \dot{a}_{n,v}(t) + k_o n \left[(v+1) \frac{g_{v+1}}{g_v} a_{n,v+1}(t) - \frac{g_{v-1}}{g_v} a_{n,v-1} \right] \\ + i \sum_{q=-\infty}^{+\infty} E_q(t) a_{n-q,v-1} \frac{g_{v-1}}{g_v} = 0 \end{aligned}$$

Noting that

$$\frac{g_{v+1}}{g_v} = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{2} \Gamma\left(\frac{v+2}{2}\right)} \equiv \gamma_v,$$

then

$$\frac{g_{v-1}}{g_v} = v \gamma_v.$$

Using these expressions, the above becomes

$$\begin{aligned} \dot{a}_{n,v}(t) + k_o n \gamma_v \left[(v+1) a_{n,v+1}(t) - v a_{n,v-1} \right] \\ + i v \gamma_v \sum_{q=-\infty}^{+\infty} E_q(t) a_{n-q,v-1}(t) = 0 \end{aligned} \quad (3.9)$$

with

$$i E_q(t) = - \frac{a_{q,0}(t)}{k_o q}. \quad (3.10)$$

This set of equations is termed the Power Transform method (Joyce, Knorr and Meier, 1971). Equation (3.9) was integrated forward in time on the IBM 360/65 computer at the University of Iowa by using a modified Adams-Bashford predictor-corrector algorithm.

Note that for symmetric initial conditions, as used in this work, the $a_{n,v}$ are all real, whereas the $Z_{n,m}$ are alternately real and imaginary. Joyce, et al. (1971) have shown that the $a_{n,v}$ are equal to the $Z_{n,m}$ except for a complex factor.

C. Truncation Difficulties

When the system of Equations (3.9) is numerically integrated, we are faced with the same problem of truncating the v -index as with the Hermite index for the F-H method.

If we set $a_{n,v} \equiv 0$ for $v > N$ ($N \equiv v_{\max}$), we are again forced to choose N very large to obtain correct macroscopic quantities for a reasonable length of time. This, of course, would provide no advantage over the F-H code. The present form of the linear Vlasov equation (Equation 3.9), however, lends itself to a polynomial extrapolation scheme whereby we can accurately estimate the $N + 1$ coefficient.

There is no reason to assume any regularity between the $a_{n,v}$; however, when a version of Equation (3.9) corresponding to the linearized Vlasov equation is numerically integrated, the $a_{n,v}$ ($v = 0, 1, 2, \dots$) form a very regular pattern for large n for the linear system. Figure 9. The $N + 1$ coefficient can

therefore be determined by a polynomial extrapolation and thus close the system. With a fourth order polynomial extrapolation scheme, Joyce, et al. (1971) have shown that the linear Vlasov system can be reasonably represented by as few as 10 coefficients. We will discuss the effect the number of coefficients has on the solution for the electric field in Section III-E.

For unstable initial conditions (the two-beam instability) we find that the coefficients no longer lie on a smooth curve but are somewhat scattered. Even with this scattering, we find reasonable agreement between the lower order coefficients for N as small as 20. In Figures 10 and 11 we have plotted the coefficients for $N = 80$, 40 and 20 for times just before the electric field reaches saturation ($t = 19 \omega_p^{-1}$) and after saturation ($t = 28 \omega_p^{-1}$).

We find that by closing the system with the polynomial extrapolation procedure, the inaccuracies in describing the electric field are very slight even with $N \sim 0$ (20). If one wishes to examine the interaction of the waves with the distribution function, however, it is advisable to incorporate a larger number of coefficients. We will elaborate further upon this point in Section III-E and F.

D. Use of a Damping Term

If we modify the system of Equations (3.9) in such a way that the amplitudes of the $a_{n,v}$ with v close to N can never become large, then a truncation has the same effect as a reasonable guess of $a_{n,N+1}$. By adding a damping term, similar to that used with the

F-H code, to the right side of Equation (3.9), we find that the coefficients with small v are not affected by the damping of the $a_{n,v}$ which have large v . The damping term we have chosen is

$$\left(\frac{\delta f}{\delta t} \right)_c = \left[\eta_c \left(\frac{\partial(v)}{\partial v} + \frac{\partial^2}{\partial v^2} \right) \right]^l f \quad (3.11)$$

which assumes the form

$$\left(\frac{\delta a_{n,v}}{\delta t} \right)_c = - (\eta_c v)^l a_{n,v}(t) \quad (3.12)$$

With a collision term of this form, the coefficients can be written as

$$a_{n,v} = C e^{-(\eta_c v)^l t},$$

with $\eta_c \ll 1$. The coefficients with small v are only slightly affected by the collision term since $v \eta_c \ll 1$, thus $(v \eta_c)^l \ll (v \eta_c)$; however, the coefficients with v large are heavily damped since $v \eta_c \gg 1$ and $(v \eta_c)^l \gg v \eta_c$. This collision term thus selectively suppresses only the large v -derivatives of the distribution function.

We find that for $(\eta_c v) \sim 0(1)$, the instability that appears in the Landau-damped case is suppressed without adding noticeably

to the damping decrement; and we obtain excellent results for N as small as 80. For the two-stream instability problem, it is necessary to keep $(\eta_c N) < 1$ so that the macroscopic quantities are not significantly affected.

We will now discuss the results of these cutoff procedures in more detail with specific examples for both the stable and unstable initial conditions. In Section III-E we discuss the stable initial condition (the simple Maxwellian distribution); and in Section III-F we present the results for the unstable case (the two-beam instability).

E. Numerical Results for Stable Initial Conditions

In all of the cases to be considered in this section, the form of the initial distribution is the same as the stable case for the F-H code, that is

$$f_o(v,0) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}.$$

(Figure 2.) In the presentation of numerical data, the $|E_n(t)|$ and not the $a_{n,v}(t)$ will be used. We found early in the calculations that $|E_2(t)|$ was at least an order of magnitude smaller than $|E_1(t)|$; thus only the results with $n = 1$ will be presented.

We will consider wave numbers of $k = 0.4, 0.5$ and 0.9 with a perturbation $\epsilon = 0.035$. For each case we will first present the

results of the extrapolation scheme. We find that we can easily avoid the difficulty of a cutoff by accurately estimating the $N + 1$ coefficient and we are able to describe the system for a reasonable length of time with as few as 10 coefficients. We will then discuss the results obtained by using the collision term to avoid an early breakdown of the system.

For Case I, we have chosen $k = 0.5$ and $\epsilon = 0.035$. Keeping only 80 coefficients, we then closed the set of equations by using a fourth order polynomial extrapolation scheme. This allows us to accurately estimate the $N + 1$ coefficient. No breakdown of the system appears for the total time computed ($50 \omega_p^{-1}$). As shown in Figure 12 the electric field undergoes linear decay with a damping decrement of $\gamma_1 = 0.158$ until $t = 28 \omega_p^{-1}$ when the decrement falls below the linear value. The damping decrement for $t < 28 \omega_p^{-1}$ agrees exactly with the linear theoretical calculations done by Gary (1966). The results also agree favorably with the work done by Armstrong (1966, 1967) where the decrease in the damping decrement appears at almost the same time. Compare Figure 3, which has slightly different initial conditions.

As noted previously, if we truncate the system with a small number of coefficients, we can suppress the large v -derivatives by adding a damping term of the form $-(\eta_c v)^\ell a_{n,v}(t)$ to the right side of Equations (3.9). With 80 coefficients and $\eta_c = 0.016$, $\ell = 4$, we find that the breakdown of the system is completely suppressed;

however, the field now undergoes strictly linear decay for the total time of computation. This is shown in Figure 13. As the results indicate, this damping term selectively suppresses only the higher order coefficients without adding significantly to the damping decrement for $t < 30 \omega_p^{-1}$. We do note that for $t > 30 \omega_p^{-1}$, the collision term adds to the damping decrement obtained with the extrapolated code.

The breakdown of the system manifests itself through a large jump in the size of the electric field. With $k = 0.5$ and $\epsilon = 0.035$ and truncating the matrix at 80 coefficients, we find that $|E_1(t)|$ undergoes linear decay until $t \approx 32 \omega_p^{-1}$ when breakdown becomes apparent (Figure 14). We note that the sharp growth in the electric field appears at $t = \frac{2\sqrt{N}}{n k_0}$. This corresponds to the predicted time at which the v th coefficient attains its maximum value, $t = \frac{\sqrt{v}}{n k_0}$ (Armstrong, 1966).

By choosing N larger, the instability appears at later times; and with $N \sim 0$ (250), the solution is in exact agreement with the extrapolated solution with only 80 coefficients.

For Case II we have $k = 0.4$ and $\epsilon = 0.035$. Using the fourth order extrapolation scheme with $N = 80$, we find that the electric field undergoes linear decay with a damping decrement $\gamma_1 = 0.072$ until $t = 34 \omega_p^{-1}$, when the decrement decreases. This is shown in Figure 15. We are able to obtain exactly the same result from the truncated code ($N = 80$) with a collision frequency of

$\eta_c = 0.01$ and $\ell = 4$. The collision frequency does not change the damping rate for this case.

In Figure 16 we show the results of the interaction of the wave with the distribution function for both the damped truncated and extrapolated code for $k = 0.4$. Both results are for the time $t = 50 \omega_p^{-1}$. The typical decrease of the number of particles below the phase velocity and the increase in the number above the phase velocity is apparent. The linear phase velocity is indicated on the velocity axis. We find that the ripple in $f_0(v)$ is slightly greater for the truncated code. This discrepancy is probably due to the increase in the size of the coefficients at the boundary of the matrix.

As a further example of the extrapolation procedure, we show the results of a run with $k = 0.9$ and $\epsilon = 0.035$ in Figure 17 (Case III). The electric field decays away rapidly with a damping decrement $\gamma_1 = 0.68$, in agreement with the linear theoretical result (Gary, 1966). The field undergoes a slight deviation at $t = 18 \omega_p^{-1}$, after which it continues to damp away quite rapidly with a decrement slightly less than -0.64 . The results we obtained with a large wave number agree quite well with Armstrong's even to the extent of the slight deviation that appears at $t \sim 0 (15 \omega_p^{-1})$ (Armstrong, 1966).

We now want to investigate the extrapolated Power Transform method further by examining the change in $|E_1(t)|$ as we reduce the number of coefficients. We chose $k = 0.5$ and $\epsilon = 0.035$. We find

that the results are in exact agreement for a reasonable length of time with N as small as 10. We have plotted the envelope of $|E_1(t)|$ for $N = 80, 40, 20$ and 10 in Figure 18. Only with $N = 10$ is there any change in $|E_1(t)|$ before $t = 30 \omega_p^{-1}$.

We note that as the number of coefficients is decreased, the damping rate approaches its linear value. With only 20 coefficients, the field undergoes strict Landau damping with $\gamma_1 = -0.158$ for the total time computed. We recall that a similar increase in the damping rate was found for very large collision frequencies. (Recall Figure 13 and see Armstrong, 1966.)

As the solutions for stable initial conditions quite clearly show, we are able to circumvent the difficulty of a cutoff mechanism by using an extrapolation scheme with the Power Transform method. This scheme is capable of estimating the $N + 1$ coefficient to such a degree that relatively few coefficients are needed in order to obtain excellent results.

We now want to apply the Power Transform method to the solution of unstable initial conditions. We find that the results are again in excellent agreement with the Fourier-Hermite Transform method.

F. Numerical Results for Unstable Initial Conditions

We will now investigate the results of the three cutoff procedures on the unstable initial condition, i.e.,

$$f_o(v, o) = \frac{1}{2\pi} v^2 e^{-v^2/2} .$$

The numerical solutions will again be presented in terms of the electric field and not the coefficients.

We found early in this work that we could not obtain reasonable results by just truncating the matrix at $N = 80$. The coefficients grew too large, resulting in early failure of the program. By using a small damping term ($\eta_c \sim 0.0062$), however, we are able to obtain somewhat reasonable results for times of interest. We had no difficulties with the extrapolated code.

For Case IV, we have chosen $k = 0.5$ and $\epsilon = 0.02$. The solution of $E_1(t)$ for the fourth order polynomial extrapolation scheme with $N = 80$ is presented in Figure 19. $E_1(t)$ was observed to grow exponentially with a growth rate of $\gamma_1 = 0.245$. The growth ceased at $t = 20.5 \omega_p^{-1}$, after which the field undergoes a long period of fluctuation in magnitude. Very small fluctuations appear in the electric field between $t = 34$ and $t = 44$. These ripples can be removed by adding a very small damping term ($\eta_c = 0.0062$) to the extrapolated code. This is also shown in Figure 19.

It is thought that this roughness in $E_1(t)$ is due more to the fact that we are solving the $n = 1$ solution only and not to inaccuracies in the extrapolation procedure. We find that these ripples do not appear for $n \geq 2$. It is to be noted that the

growth rate for $k = 0.5$ ($\gamma_1 = 0.245$) is in exact agreement with Armstrong's results (Armstrong, 1966).

The solution for the damped truncated code with $N = 80$, $\eta_c = 0.0062$ and $\ell = 4$ is very similar to the extrapolated result. The only disparity is in the size of the oscillations that appear between $t = 34$ and $44 \omega_p^{-1}$ (see Figure 20). This is probably due to truncating the matrix at such a low number.

We find that if we use a larger perturbation, $\epsilon = 0.04$, Case V, and truncate the matrix at $N = 80$, we cannot suppress these ripples unless the damping term is as large as $\eta_c \sim 0.01$.

With the same wave number and perturbation ($k = 0.5$, $\epsilon = 0.04$), we calculated the solution of $E_1(t)$ by using the fourth order extrapolation scheme. We find that the extrapolated solution is in very close agreement with the truncated solution with $\eta_c = 0.01$. This is shown in Figure 21 where we have superposed the two solutions. We also show in this figure the early regrowth of $E_1(t)$ that comes about if we set $\eta_c > .01$. At no time did the damping term change the solution of the electric field before a time $t \sim 0 (25) \omega_p^{-1}$. This is in contrast to Armstrong's collision term which affects the magnitude of the field as early as $t \sim 0 (8) \omega_p^{-1}$. See Figure 8.

A slightly larger discrepancy appears between the extrapolated and truncated code in the solution for the distribution function. The filling in of the "hole" near the minimum ($v = 0$) of $f_0(v, t)$ occurs for both solutions; however, variations begin to

appear at times $t \sim 0 (30) \omega_p^{-1}$. In Figures 22 and 23 we have superposed the two solutions of $f_0(v, t)$ for the times $t = 37$ and $t = 47 \omega_p^{-1}$ respectively. The difference between the two solutions is again probably due to the suppression of the higher order coefficients by the damping term.

We now want to discuss the results of the extrapolated scheme for various initial perturbations.

For Case VI we have $k = 0.5$, $\epsilon = 0.2$ and $n = 2$. The solution is shown in Figure 24. $E_1(t)$ undergoes exponential growth after $t = 10 \omega_p^{-1}$ with a growth rate of $\gamma_1 = 0.245$. The field saturates at $t = 22 \omega_p^{-1}$ after which it undergoes a smooth fluctuation in magnitude until computation was stopped at $t = 50$. $E_2(t)$ grows at a rate $\gamma_2 \sim 2\gamma_1$ until it reaches a maximum at $t = 18.75 \omega_p^{-1}$, after which it undergoes a period of fluctuations in magnitude with changes in sign. These results agree very favorably with Armstrong's results as can be noted by comparing Figures 24 and 6.

By retaining the first two modes, we have a smoother regrowth of $E_1(t)$ and the strictly exponential growth of the first mode ceases at an earlier time than for $n = 1$, although it reaches its saturation point at a slightly later time. These differences are due to the k wave feeding energy into the $2k$ wave through the $E_1 \frac{\partial f_1}{\partial v}$ term.

A more striking difference arises between the solutions of $f_0(v, t)$ when we include the first harmonic in the calculation.

With $n = 2$, a "ripple" appears in the distribution function at a phase velocity of $v \sim 1.0$. This result does not appear for $n = 1$. Also, the "ripple" appears only at times when $E_1(t)$ is growing and $E_2(t)$ is decaying, or vice versa. We have plotted $f_0(v, 20)$ in Figure 25 for the cases $n = 1$ and $n = 2$. For $t = 25$, the "ripple" is not as prominent but is still obvious as shown in Figure 26.

As another illustration of the excellent results obtained with the extrapolated Power Transform method, we plot the solution of E_1 and E_2 for $k = 0.5$ and $\epsilon = 0.03$ (Case VII) in Figure 27. Again, the growth rate, the maximum size of E_1 , the time of saturation and the period of the large amplitude oscillation are in almost exact agreement with Armstrong's results.

The "ripple" also appears in the distribution function for this perturbation. We have plotted $f_0(v, t)$ in time increments of $10 \omega_p^{-1}$ in Figures 28 and 29. The "ripple" becomes apparent at $t = 40$, when E_1 is growing and E_2 is decaying. It is proposed that the interaction of the two modes that arises from the $E_1 \frac{\partial f_1}{\partial v}$ term gives rise to a wave that interacts with the distribution function in such a manner as to produce this aberration.

As the numerical results clearly indicate, the extrapolated Power Transform method is equally applicable to the solution of unstable initial conditions.

As we stated earlier (Section III-C), there is some scattering among the higher order coefficients for the unstable case (recall

Figures 10 and 11). We want to investigate the solutions of the electric field and the distribution function as we reduce the number of coefficients in the extrapolated scheme. We obtain excellent results for the total time computed with N as small as 20.

The solution of E_1 is shown in Figure 30 for $k = 0.5$ with $N = 80, 40$ and 20. The solutions with $N = 80$ and 40 are remarkably close, and with $N = 20$ there is only a slight shift in the regrowth of the field. We note that the error introduced by using a small number of coefficients is similar to the error resulting from suppressing the higher order coefficients by a large collision term. Recall Figure 21.

The variations between the solutions for f_0 are shown in Figure 31, where we have plotted $f_0(v, 37.5)$ for $N = 80, 40$ and 20. There is a very slight discrepancy between the $N = 80$ and $N = 40$ run. A larger aberration appears with $N = 20$, indicating that if the distribution is to be rigorously examined one should incorporate at least 40 coefficients in the calculation scheme. The electric field, however, is adequately represented by as few as 20 coefficients.

G. Summary of Results

The incorporation of a fourth order polynomial extrapolation scheme into the Power Transform method has provided us with a rapid and efficient means of solving the nonlinear Vlasov equation for two important initial conditions. We find that we are able to avoid

the difficulty of a cutoff. The numerical results indicate that this system is somewhat more stable numerically than the F-H code; and the inclusion of a damping term does not influence the solution to as large a degree as with the F-H code.

We find that the extrapolation procedure accurately estimates the $N + 1$ coefficient to such an extent that less than 80 coefficients are needed to describe the system for times $t \sim 0 (50 \omega_p^{-1})$. We note that for the unstable case, if we use only 20 coefficients the regrowth of E_1 is shifted by only $\Delta t = 5 \omega_p^{-1}$. See Figure 30. Comparing this with Armstrong's results (Figure 8), we see that with a collision frequency as small as $\eta_c = 0.001$, variations appear in the electric field as early as $t = 8 \omega_p^{-1}$ and the regrowth is shifted by a $\Delta t = 4 \omega_p^{-1}$. Incorporating very few coefficients in the solution has not affected the limiting amplitude of the field nor has it changed the asymptotic behavior.

As an example of the savings in computer time: The F-H code requires approximately one-half hour compute time for unstable initial conditions; whereas, the extrapolated Power Transform method takes two minutes with 80 coefficients and only 17 seconds for $N = 20$.

We now want to discuss a third computational technique as it is applied to plasma physics: the Particle Code. We will apply the particle code to the same two initial conditions and compare the solutions with the previous results.

IV. THE PARTICLE SIMULATION METHOD

A. Introduction

The third computational technique we want to examine is the particle simulation method. The computations are performed using the particle-in-cell, or PIC model. This model is widely used in computational plasma physics (Morse and Nielson, 1969; Morse, 1970; Denavit and Kruer, 1971).

In the collisionless particle code, particles are assigned velocities and positions which vary in time and they move as if they were real charged particles in the plasma. The self-consistent electric field which accelerates these particles is computed and stored on a spatial Eulerian grid. There is no velocity grid with this code.

As with the Fourier-Hermite and the Power Transform methods, it is the aim of the particle simulation procedure to formulate solutions of the system of equations

$$\frac{\partial f(x, v, t)}{\partial t} + v \frac{\partial f}{\partial x} - \frac{e v}{m} E(x, t) \frac{\partial f}{\partial v} = 0, \quad (4.1)$$

$$\frac{\partial E}{\partial x} = +4\pi n_0 e \left(1 - \frac{n(x)}{n_0} \right). \quad (4.2)$$

We recall that the major difficulty with the transform methods is the development of steep gradients in the distribution function which requires a large amount of smoothing; i.e., an increasing number of matrix elements are required in order to compute for a significant length of time. We have shown that the Power Transform method effectively solves this problem by an extrapolation scheme which accurately predicts the next matrix element. This allows us to use a relatively small number of coefficients and still obtain excellent results.

The particle simulation method circumvents this problem entirely by integrating the characteristic equations of Equation (4.1):

$$\frac{dv}{dt} = -\frac{e}{m} E \quad (4.3)$$

and

$$\frac{dx}{dt} = v. \quad (4.4)$$

The particles are then promoted along these characteristics and the distribution function is simply the density of the particles in phase space.

The ions are treated as a uniform positive background and only the electrons are represented as simulation particles. As we previously mentioned, the self-consistent electric field is obtained by means of a spatial Eulerian grid which is superposed on the particle positions. All particles can then be thought of as lying in some grid interval, and the particle's position is defined by the number of the particular cell it is in and its location in that cell.

The first step in determining the electric field is to calculate the charge density of the particles on the grid. The particle positions are used to calculate a "grid" number density, ρ , by the method of length weighting. A particle which is located in the Nth cell a distance x from the Nth grid point is distributed on the Nth and the $(N + 1)$ th grid points by the following scheme:

$$\rho_N = (\Delta x - x) \rho_0 \quad (4.5)$$

and

$$\rho_{N+1} = x \rho_0, \quad (4.6)$$

where $0 \leq x \leq \Delta x$. Δx is defined as the cell size and is generally taken to be 1. ρ_0 is the charge density of an individual particle and it assumes the value 1 in our set of dimensionless units. We note that this scheme conserves charge and the smearing out of the

charges tends to reduce any fluctuations that may appear as a result of poor statistics.

Each particle is treated in this manner resulting in a number density on the grid points which can be used to calculate the electric field from Poisson's equation. If the number of grid points is a power of two, fast Fourier transform techniques can be employed to solve for the electric field in the Fourier transform space, and the field can then be transformed back to the spatial grid. Because the electric field is known only at the grid points, an interpolation scheme must be used to calculate the electric field at each particle's position. If we again consider the particle lying a distance x from the N th grid point, the interpolation formula is simply

$$E(x) = x E_N + (\Delta x - x) E_{N+1} , \quad (4.7)$$

where E_N and E_{N+1} are the values of the electric field at the N th and $(N + 1)$ th grid points respectively.

This interpolation eliminates the self-interactions of the particles and modifies the electric fields somewhat, as it tends to smooth out the interactions of close encounters. Two particles moving through each other experience a smooth electric field instead of a discontinuous field which is the exact solution of Poisson's equation. The effect of this smoothing has been discussed by Langdon and Birdsall (1970).

The electric field is determined at all particle positions and is then used to advance the particle positions and velocities by the following scheme:

$$v^{n+\frac{1}{2}} = v^{n-\frac{1}{2}} + E^n \Delta t, \quad (4.8)$$

$$x^{n+1} = x^n + v^{n+\frac{1}{2}} \Delta t, \quad (4.9)$$

where n indicates the time in units of Δt .

We note from this iterative scheme that the order of operations is time centered and thus leaves errors of $O(\Delta t^3)$. Our numerical results indicate that a Δt of $1/8 \omega_p^{-1}$ is generally adequate; there is no significant difference in the results if we choose a smaller time step.

The cycle is thus complete: The particle positions yield the charge density, which determines the electric field through Poisson's equation. The self-consistent field then advances the particle positions and velocities according to Equations (4.8) and (4.9). By repeating the cycle, the system is advanced in time.

The system we are considering is one dimensional in both the spatial and velocity variables. The system is made infinite by imposing periodic boundary conditions so that if a particle goes beyond the periodicity interval, its x -coordinate is masked and it appears at the opposite end of the interval with the correct velocity and position.

In Sections IV-C and IV-D we present the results obtained by applying the particle code to the stable Maxwellian distribution problem and to the two-beam instability. In the next section we discuss the noise suppression techniques and the initialization procedure.

B. Noise Suppression and Initialization Procedures

The major difficulty with the particle simulation method is the large thermal fluctuation (noise) level that results from the finite number density of the particles. If the particles are initially distributed uniformly in x and randomly in v according to the initial distribution function, the most unstable mode can be observed to grow out of the noise level after a period of time (Morse and Nielson, 1969). There is essentially no control over the initial conditions with this procedure, other than choosing a different set of random velocities; and because of the very high noise level $\left(O \left(\frac{1}{n L_D} \right) \right)$, it can only be used to examine strongly unstable initial conditions. A very large number of particles ($N \sim O(40 \times 10^3)$) is required in order to keep the ratio of the thermal fluctuation energy to the field energy in the physical problem of interest very small.

In order to reduce the necessity for an extremely large number of particles, examine stable initial conditions, obtain better statistics, save computer time and gain a firm control over the initial

conditions, we employed the "quiet start" loading scheme (Byers and Grewal, 1970; Denavit and Kruer, 1971) and initially perturbed the system with a small external electric field.

The "quiet start" technique refers to beginning the calculation with ordered initial conditions. A set of N discrete velocities is generated by using the probability function

$$p(v_o) = \int_{-\infty}^{v_o} f(v) dv, \quad (4.10)$$

where $f(v)$ is the desired distribution function.

The function $p(v_o)$ takes on values between 0.5 and 1 as v_o goes from zero to infinity. The method of obtaining a velocity distribution for $2N + 1$ particles is to divide $p(v_o)$ into $N + 1$ equal intervals, and to solve Equation (4.10) for v_o for each value of p . These values of v_o plus the negatives yield $2N + 1$ particle velocities distributed according to $f(v)$.

For example, if the particles are distributed according to a Maxwellian distribution, the equation we need to solve is

$$M(v_o) \equiv p(v_o) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{v_o} e^{-v^2/2} dv = 0 \quad (4.11)$$

for $2p = 1, 1 + \frac{1}{N}, 1 + \frac{2}{N}, \dots, 2$. If N is fairly large, the solution of this equation can be approximated by one iteration of Newton's method of successive approximation. Noting that for

$p = 0.5$ $v_0 = 0$, we can find the n th value of v_0 from the $(n - 1)$ th value from the equation

$$v_n = v_{n-1} + \exp\left(\frac{v_{n-1}^2}{2}\right) \left[\sqrt{2\pi} p_n(v_n) - \sqrt{\pi/2} \left(1 + \operatorname{erf}\left(\frac{v_{n-1}}{\sqrt{2}}\right)\right) \right]. \quad (4.12)$$

This approximation works quite well except for very large values of velocity. However, since there are very few particles with large velocities, we do not consider particles with $v \geq 3.5$, as a matter of convenience.

A similar algorithm can be used for the two-beam instability.

In this case we set

$$S(v_0) \equiv p(v_0) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{v_0} v^2 e^{-v^2/2} dv = 0, \quad (4.13)$$

and obtain from Newton's method

$$v_n = \frac{v_{n-1}^2 + 1}{v_{n-1}} + \frac{\exp\left(\frac{v_{n-1}^2}{2}\right)}{v_{n-1}^2} \left[\sqrt{2\pi} p_n(v_n) - \sqrt{\pi/2} \left(1 + \operatorname{erf}\left(\frac{v_{n-1}}{\sqrt{2}}\right)\right) \right]. \quad (4.14)$$

We note that for this problem, no particle can have an initial velocity of zero. We are able to calculate as many as 4×10^3 velocity values with this simple method.

This set of velocities is then divided into j subsets by choosing every j th value of the original set. The subsets are loaded at different grid points, repeating the same loading every j th cell. The repetition length should be much smaller than the important wavelengths. Equal numbers of particles are loaded in each cell. This scheme essentially eliminates the natural fluctuations that arise from a finite set of particles near an equilibrium state, and allows us to start the computations with specified initial conditions.

The system is initially perturbed by applying an external electric field. A small external field is "turned on" and the particles are backed up one time step. The particles, in their attempt to shield the external field, produce a density perturbation when the field is "turned off". The system is then allowed to propagate forward in time. This permits close control of the initial conditions and we can then "match" initial conditions with the transform codes.

The "quiet start" technique in one dimension is essentially a multibeam representation of $f(v)$. This can give rise to multibeam instabilities which become apparent in times on the order of $1/(k\Delta v)$, where Δv is the typical beam separation (Byers, 1970). It is our feeling that in our work this problem is insignificant, since this instability should not appear until times on the order of several hundred ω_p^{-1} in our system.

Another difficulty that must be considered is the nonphysical interaction between the particles and the grid. This grid noise can become so large as to mask the physical result. Langdon (1970) has shown that the interaction force between particles may have a non-physical dependence on the particle's position in the grid. The consequence of this grid force is that perturbations at wavenumber k are coupled to perturbations at wavenumbers differing from k by integer multiples of $\frac{2\pi}{\Delta x}$, where Δx is the grid separation. With our simulation, however, we keep the Debye length much larger than the cell size (Δx); and this procedure effectively suppresses any noise arising from particles interacting with the grid according to Langdon (1970).

We present the particle code solutions to the Landau-damped problem in the next section and to the two-beam instability in Section IV-D.

C. Numerical Results for Stable Initial Conditions

The problems of Landau damping solved in Section II-D by the Fourier-Hermite method and in Section III-E by the Power Transform method are reconsidered in the present section using the particle-in-cell method.

By incorporating our length weighted scheme and the "quiet start" technique in our calculations, we are able to reduce the natural fluctuations in the electric field that arise from the

discrete number density by several orders of magnitude. However, these fluctuations soon reach a level, albeit small, such that it becomes impossible to observe Landau damping after a period of time; i.e., the electric field soon damps into the noise level.

The ratio of the energy density of the electrostatic noise to the thermal energy density is of the order of the plasma parameter $\left(\frac{1}{n L_D}\right)$. $n L_D$ is the number of particles per Debye length. Eldridge and Feix (1962) have shown from a linearized theory that the thermodynamic fluctuations are given by

$$\frac{\langle E^2 \rangle}{nkT} = \frac{2\pi}{n L_D} \quad (4.15)$$

In our system this ratio yields the values 1.24×10^{-2} , 9.86×10^{-3} and 8.24×10^{-3} for wavenumbers $k = 0.4$, 0.5 and 0.6 respectively, with the following conditions:

1. L (periodicity length) = $32 \Delta x$
2. $\Delta x = 1$
3. $\Delta t = 1/8 \omega_p^{-1}$
4. N (total number of particles) = 8000.

We are able to reduce these ratios by several orders of magnitude by using the "quiet start" technique. We find that the noise level that our system finally attains is dependent upon only the number of particles. It does not depend upon the time step nor the length of the system. We will now discuss the results of the Landau damped problem for wavenumbers $k = 0.6$, 0.5 and 0.4 .

For Case I, we have $k = 0.6$, $L = 32 \Delta x$, $\Delta t = 1/8 \omega_p^{-1}$ and $N = 4 \times 10^3$ particles. We find that the electric field Landau damps with the correct linear damping rate ($\gamma_1 = -0.265$) until it reaches the noise level. The results are shown in Figure 32, where we have also plotted the noise level of the system. We attained the noise level by running the system without any initial perturbation. The level of the fluctuations as predicted by Equation (4.15) is denoted on the electric field axis.

If we double the number of particles ($N = 8 \times 10^3$) but hold the other parameters fixed (Case II), we are able to obtain smooth Landau damping for a slightly longer period of time since the noise level is lower (Figure 33). The damping decrement is in agreement with linear predictions.

As a check on possible error arising from the size of the time step, we made runs with $\Delta t = \frac{1}{16} \omega_p^{-1}$ and there was no significant change in the damping of the electric field or in the noise level of the system.

We then doubled the length of the system ($L = 64 \Delta x$) to determine if the noise was possibly a result of the particles interacting with the grid. Again, there was no change in the electric field. As we mentioned in Section IV-B, it is our feeling that the grid instability is stabilized in our case as a result of the Debye length being larger than the cell size.

For Case III, we again doubled the number of particles ($N = 16 \times 10^3$), keeping $k = 0.6$ and $L = 32 \Delta x$. The large particle number lowers the noise level to $0 (7 \times 10^{-4})$ and the electric field Landau damps until it reaches the noise level (Figure 34). The damping decrement for this run, $\gamma_1 = -0.264$, is slightly below the value predicted by linear theory. In Figure 35 we have plotted $f_0(v, 12)$ which shows the typical ripple that appears near the phase velocity.

With $k = 0.5$, $N = 16 \times 10^3$ and $L = 32 \Delta x$ (Case IV), we once more obtain smooth Landau damping for several oscillations until the electric field reaches the noise level. This run is shown in Figure 36. The damping decrement is equal to -0.16 which is in good agreement with linear theory.

For Case V, we set $k = 0.4$ with $N = 8 \times 10^3$ and $L = 32 \Delta x$. The electric field again damps with the correct damping decrement ($\gamma_1 = -0.077$) until it reaches the noise level. The results are illustrated in Figure 37.

We have shown that our length weighting scheme and the "quiet start" loading lowers the inherent fluctuations in the electric field by several orders of magnitude from thermal levels. This permits us to examine Landau damping through several oscillations of the electric field. We find that the noise level of our system depends only upon the particle density. If, as we increase the

number of particles, the noise level continues to decrease at its present rate, we should be able to obtain Landau damping for a long enough time to observe the decrease in the damping rate with $N \sim 0 (12 \times 10^4)$ particles. However, a system of this size is not feasible at this time.

We note that without the "quiet start" procedure, this system would require on the order of 2×10^5 particles to reduce the noise level to the value we have attained with only 16×10^3 particles.

The relatively low noise level ceases to be such a problem when examining instabilities, since the electric field soon grows well above the fluctuation level. This problem is discussed in the next section.

D. Numerical Results for Unstable Initial Conditions

In this section we will apply the particle code method to the problem of the two-beam instability. As we noted earlier, the electric field for the unstable case soon grows to several orders of magnitude above the noise level and thus the fluctuations due to the finite particle density cease to be a problem. As a consequence, the numerical results are not too dependent upon the particle number. The results for $N = 2 \times 10^3$ and $N = 16 \times 10^3$ are essentially the same; however, the larger number density does give rise to a smoother electric field because of the better statistics. We will consider the two-beam instability for wavenumbers $k = 0.5$ and $k = 0.6$.

For Case I, we have $k = 0.5$, $N = 8 \times 10^3$ and $L = 32 \Delta x$. After $t \approx 10 \omega_p^{-1}$, the solution for $E_1(t)$ was observed to grow at a nearly exponential rate until $t = 20 \omega_p^{-1}$ after which the field undergoes a series of small oscillations. See Figure 38. The growth rate ($\gamma_1 \approx +0.24$) was in very close agreement with the result obtained by the Power Transform method.

The major discrepancy between this solution and that obtained with the Power Transform and F-H method is the marked decrease in the size of the fluctuations after saturation. Armstrong and Nielson (1971) have shown that in the F-H code as one increases the number of Fourier modes the amplitude oscillations become smaller. The larger matrix is necessary in order to better represent the trapped particles accurately. (Armstrong, private communication, 1971)

The particle code can better represent the trapped particles which soon obtain random phases after saturation. This random phasing results in a decrease in amplitude of the oscillation. In Figure 38, we also show the slight increase in the size of the fluctuation if we incorporate only 16 Fourier modes in the particle solution.

For ease of comparison between the two solutions we superposed the particle code solution and the corresponding Power Transform solution in Figure 39. The similarities and differences are apparent.

In Figure 40, we have plotted $f_0(v, t)$ for the times $t = 13.5 \omega_p^{-1}$ and $t = 19 \omega_p^{-1}$. We again obtain the typical

filling-in of the "hole". We also note that the "ripple" that appeared in $f_0(v)$ for the Power Transform solution, whenever we incorporated two Fourier modes, appears again for the particle code.

For Case II, we have $k = 0.6$, $N = 8 \times 10^3$ and $L = 32 \Delta x$. The solution for $E_1(t)$ is shown in Figure 41. The results are similar to the previous run with a growth rate in this case of $\gamma_1 = +0.22$, which is in good agreement with Armstrong's results (1966).

E. Summary of Results

It is apparent that the particle code method is more appropriate for solving instability problems than for examining Landau damping. The fluctuations that arise as a result of the discrete number density are too large to enable one to observe damping for a significant length of time.

By using a length weighted scheme to smear out the charge density and a "quiet start" loading scheme we can lower the noise level enough to observe Landau damping through several oscillations.

The particle code solution agrees with the Power Transform solution of the two-beam instability up to the point at which the electric field saturates. The small amplitude oscillations that appear in the particle code are a result of the random phasing of the particles trapped in the potential well.

Figure 1. Geometry for a plasma slab confined between perfectly reflecting boundaries.

"SLAB" GEOMETRY

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Figure 1

Figure 2. Plot of $f_o(v,o)$ for the Maxwellian distribution problem.

Figure 2

Figure 3. Fourier-Hermite code solution of the second order stable system for $k = 0.5$, $\epsilon = 0.04$ (from Armstrong, 1966).

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Figure 3

Figure 4. Plot of $f_o^{(2)}(v,t)$ and $f_o(v,t)$ for $k = 0.5$,
 $\epsilon = 0.04$, $t = 35$ (from Armstrong, 1966).

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Figure 4

Figure 5. Plot of $f_o(v,o)$ for the two-beam instability problem.

Figure 5

Figure 6. Fourier-Hermite code solution of the second order unstable system for $k = 0.5$, $\epsilon = 0.01$ (from Armstrong, 1966).

Figure 6

Figure 7. Plot of $f_o^{(2)}(v,t)$ and $f_o(v,t)$ for $k = 0.5$, $\epsilon = 0.01$,
 $t = 35$ (from Armstrong, 1966).

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Figure 7

Figure 8. Effect of a collision term on the electric field for the unstable case (from Armstrong, 1967).

Figure 8

Figure 9. Plot of the real amplitudes a_v for the linear case.
● represents results when maximum v was 10. x represents results when maximum v was 20, 50, 100 (from Joyce, Knorr and Meier, 1971).

Figure 9

Figure 10. Plot of the real amplitudes a_v for the nonlinear unstable case at $t = 19$ (before saturation) for maximum $v = 20, 40, 80$.

Figure 10

Figure 11. Plot of the real amplitudes a_v for the nonlinear unstable case at $t = 28$ (after saturation) for maximum $v = 20, 40, 80$.

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Figure 11.

Figure 12. Numerical integration of the first order system for $k = 0.5$, $\epsilon = 0.035$, $N = 80$, for stable initial conditions using the extrapolation scheme, Case I.

Figure 12

Figure 13. Effect of a collision term on the truncated solution,
 $N = 80$, $\eta_c = 0.016$, $\ell = 4$; stable initial conditions,
 $k = 0.5$, $\epsilon = 0.035$, Case I.

Figure 13

Figure 14. The truncated solution with no collision term, $N = 80$,
 $k = 0.5$, $\epsilon = 0.035$, Case I. Breakdown occurs at
 $t \approx 32 \omega_p^{-1}$.

Figure 14.

Figure 15. Numerical integration of the first order system for $k = 0.4$, $\epsilon = 0.035$, for stable initial conditions, using the extrapolation scheme with 80 coefficients, Case II.

Figure 15

Figure 16. Comparison of $f_0(v, t = 50)$ for the damped truncated and extrapolated code, $k = 0.4$, $\epsilon = 0.035$, Case II. Eighty coefficients used with $\eta_c = 0.016$, $\ell = 4$ for the damped truncated code.

Figure 16

Figure 17. Numerical integration of the first order system for $k = 0.9$, $\epsilon = 0.035$, for stable initial conditions, using the extrapolation scheme with 80 coefficients, Case III.

Figure 17

Figure 18. Plot of the envelope of $|E_1(t)|$ for maximum $\nu = 10, 20, 40, 80$, for the extrapolated scheme for $k = 0.5$, $\epsilon = 0.035$, for stable initial conditions, Case I.

Figure 18

Figure 19. Comparison of the extrapolated and the extrapolated
with collisions solutions of the unstable case, $k = 0.5$,
 $\epsilon = 0.02$, $N = 80$, $\eta_c = 0.0062$, $\ell = 4$, Case IV.

Figure 19

Figure 20. The damped truncated results for Case IV, $N = 80$,
 $\eta_c = 0.0062$, $\ell = 4$, $k = 0.5$, $\epsilon = 0.02$, unstable
initial conditions.

Figure 20

Figure 21. Comparison of the extrapolated and damped truncated solutions for the unstable case, Case V, $N = 80$, $\eta_c = 0.01$ and 0.025 with $\ell = 4$, $k = 0.5$, $\epsilon = 0.04$.

Figure 21

Figure 22. Comparison of $f_0(v,t)$ for the extrapolated and damped truncated code at $t = 37 \omega_p^{-1}$. $N = 80$ and $\eta_c = 0.01$, $\ell = 4$, with $k = 0.5$, $\epsilon = 0.04$.

Figure 22

Figure 23. Comparison of $f_o(v,t)$ for the extrapolated and damped truncated code at $t = 47 \omega_p^{-1}$. $N = 80$ and $\eta_c = 0.01$, $l = 4$, with $k = 0.5$, $\epsilon = 0.04$.

Figure 23

Figure 24. Numerical integration of the second order system for unstable initial conditions, using the extrapolation scheme, with $k = 0.5$, $\epsilon = 0.02$, $N = 80$, Case VI.

Figure 24

Figure 25. Comparison of $f_o(v,t)$ for the first and second order system at $t = 20 \omega_p^{-1}$, for $k = 0.5$, $\epsilon = 0.02$.

Figure 25

Figure 26. Plot of $f_o(v,t)$ for the second order system, $t = 25 \omega_p^{-1}$,
Case VI, with $k = 0.5$, $\epsilon = 0.02$.

Figure 26

Figure 27. Numerical integration of the second order system for unstable initial conditions, using the extrapolation scheme, with $k = 0.5$, $\epsilon = 0.03$, $N = 80$, Case VII.

Figure 27

Figure 28. Plot of $f_o(v,t)$ for Case VII at $t = 10, 20, 30 \omega_p^{-1}$.
 $k = 0.5, \epsilon = 0.03$ with $n = 2$.

Figure 28

Figure 29. Plot of $f_o(v,t)$ for Case VII at $t = 40, 50 \omega_p^{-1}$.
 $k = 0.5, \epsilon = 0.03$ with $n = 2$.

Figure 29

Figure 30. Comparison of the extrapolated solutions for $|E_1(t)|$ for the two-beam instability, with v maximum = 20, 40, 80, with $k = 0.5$, $\epsilon = 0.02$, $n = 1$.

Figure 30

Figure 31. Comparison of $f_0(v, 37.5)$ for v maximum = 20, 40, 80,
with $k = 0.5$, $\epsilon = 0.02$, $n = 1$.

Figure 31

Figure 32. Particle simulation solution of $|E_1(t)|$ for the stable case, with $k = 0.6$, $N = 4 \times 10^3$, Case I.

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Figure 32

Figure 33. Particle simulation solution of $|E_1(t)|$ for the stable case with $k = 0.6$, $N = 8 \times 10^3$, Case II.

Figure 34. Particle simulation solution of $|E_1(t)|$ for the stable case with $k = 0.6$, $N = 16 \times 10^3$, Case III.

Figure 34

Figure 35. Plot of $f_o(v,t)$ for the particle code, $t = 12 \omega_p^{-1}$,
Case III, $k = 0.6$, $N = 16 \times 10^3$.

Figure 35

Figure 36. Particle simulation solution of $|E_1(t)|$ for stable initial conditions with $k = 0.5$, $N = 16 \times 10^3$, Case IV.

Figure 36

Figure 37. Particle simulation solution of $|E_1(t)|$ for stable initial conditions with $k = 0.4$, $N = 8 \times 10^3$, Case V.

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Figure 37

Figure 38. Particle simulation solution of $|E_1(t)|$ for the two-beam instability with $k = 0.5$, $N = 8 \times 10^3$, Case I, for $L = 32 \Delta x$ and $16 \Delta x$.

Figure 38

Figure 39. Comparison of the Power Transform and particle solutions for the unstable case. For the Power Transform, $k = 0.5$, $N = 80$, $n = 1$. For the particle solution, $k = 0.5$, $N = 8 \times 10^3$, $L = 32 \Delta x$.

Figure 39

Figure 40. Plot of $f_0(v,t)$ for unstable initial conditions from the particle code, for the times $t = 13.5, 19 \omega_p^{-1}$, Case I, $k = 0.5$, $N = 8 \times 10^3$.

Figure 40

Figure 41. Particle simulation solution of $|E_1(t)|$ for the two-beam instability for $k = 0.6$, $N = 8 \times 10^3$, Case II.

Figure 41

LIST OF REFERENCES

- Armstrong, T. P., Numerical studies of the nonlinear Vlasov equation, Ph.D. thesis, University of Iowa Res. Rept. 66-34, Iowa City, 1966.
- Armstrong, T. P., Numerical studies of the nonlinear Vlasov equation, Phys. Fluids, 10, 1269, 1967.
- Armstrong, T. P., and D. Montgomery, Asymptotic state of the two-stream instability, J. Plasma Phys., 1, 425, 1967.
- Armstrong, T. P., R. C. Harding, G. Knorr, and D. Montgomery, Solution of Vlasov's equation by transform methods, Methods in Computational Physics, Vol. 9, edited by B. Adler, S. Fernbach, and M. Rotenberg, p. 30, Academic Press, New York and London, 1970.
- Armstrong, T. P., and C. W. Nielson, Initial comparison of transform and particle-in-cell methods of collisionless plasma simulation, Phys. Fluids, 13, 1880, 1970.
- Byers, J. A., Noise suppression techniques in macroparticle methods of collisionless plasmas, Proceedings of the Fourth Conference on Numerical Simulation of Plasmas, p. 496, Washington, D. C., November 1970.
- Byers, J. A., and M. Grewal, Perpendicularly propagating plasma cyclotron instabilities simulated with a one-dimensional computer model, Phys. Fluids, 13, 1819, 1970.
- Chandrasekhar, S., Stochastic problems in physics and astronomy, Rev. Mod. Phys., 15, 1, 1943.
- Denavit, J., and W. L. Kruer, Comparison of numerical solutions of the Vlasov equation with particle simulation of collisionless plasmas, Phys. Fluids, 14, 1782, 1971.
- Eldridge, O. C., and M. Feix, One-dimensional plasma model at thermodynamic equilibrium, Phys. Fluids, 5, 1076, 1962.

LIST OF REFERENCES (continued)

- Engelmann, F., M. Feix, E. Minardi, and J. Oxenius, Non-linear effects from Vlasov's equation, Phys. Fluids, 6, 266-275, 1963.
- Gary, S. P., Third-order oscillations in a Maxwellian plasma, Phys. Fluids, 10, 570, 1967.
- Grant, F. C., and M. R. Feix, Fourier-Hermite solutions of the Vlasov equations in the linearized limit, Phys. Fluids, 10, 696, 1967.
- Joyce, G., G. Knorr, and H. Meier, Numerical integration methods of the Vlasov equation, J. Comp. Phys., 8, 1, 1971.
- Kellogg, P. J., Some properties of the two-stream instability at large amplitudes, Phys. Fluids, 8, 102, 1965.
- Knorr, G., Beiträge zur Lösung der nichtlinearen Vlasov-Gleichung, Thesis, Max Planck Institut für Physik und Astrophysik, München, 1963.
- Langdon, A. B., Effects of the spatial grid in simulation plasmas, J. Comp. Phys., 6, 247, 1970.
- Langdon, A. B., and C. K. Birdsall, Theory of plasma simulation using finite size particles, Phys. Fluids, 13, 2115, 1970.
- Lenard, A., and I. B. Bernstein, Plasma oscillations with diffusion in velocity space, Phys. Rev., 112, 1456, 1958.
- Montgomery, D., and D. Gorman, Boltzmann-Vlasov equation in a bounded region, Phys. Fluids, 5, 947, 1962.
- Morse, R. L., Multidimensional plasma simulation by the particle-in-cell method, Methods in Computational Physics, Vol. 9, edited by B. Adler, S. Fernbach and M. Rotenberg, p. 247, Academic Press, New York and London, 1970.
- Morse, R. L., and C. W. Neilson, Numerical simulation of warm two-beam plasma, Phys. Fluids, 12, 2418, 1969.

LIST OF REFERENCES (continued)

Taylor, E. C., and G. G. Comisar, Frequency spectrum of thermal fluctuations in plasmas, Phys. Rev., 132, 2379, 1963.

Weissglas, P., Longitudinal plasma oscillations, Plasma Phys., (J. Nucl. Energy: C), 4, 329, 1962.