REFORMULATION OF N/D METHOD AND ITS APPLICATION TO πN PHASE SHIFT ANALYSIS

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REFORMULATION OF N/D METHOD AND ITS
APPLICATION TO π N PHASE SHIFT ANALYSIS *

Chih Kwan Chen

Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720

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ABSTRACT

The N/D method is reformulated rigorously in the first part of this article, and a specific class of models, in which the high- and intermediate energy behavior of the input potential can be expressed by Regge parameters, is then investigated. A smoothing procedure of partial-wave phase shifts is discussed in the second part of this article, based on a circular application of the N/D method formulated in the first part. In this smoothing procedure CDD pole parameters are handled unambiguously, and the output solution is guaranteed to be free from causality violating singularities. The application of this smoothing procedure to π N partial-wave phase shift analysis is discussed, and an example of smoothing of a π N P_{33} -wave is given.

**PART I. THE GENERALIZED PARTIAL-WAVE DISPERSION
RELATION AND THE H/D METHOD***

1. INTRODUCTION

The N/D method¹⁻³ linearizes the nonlinear integral equation derived from unitarity and the partial-wave dispersion relation with a finite number of subtractions. Given contributions from subtractions and unphysical cuts (the "potential") as well as the inelastic factor and CDD pole parameters (if necessary), the N/D integral equations allow calculation of the elastic phase shift. Nevertheless, N/D methods have difficulties which, if not resolved, destroy their usefulness. In this article we resolve some of the difficulties and partially resolve the remaining ones.

The first question about the N/D method is its validity. The usual N/D method is based on the partial-wave dispersion relation with a finite number of subtractions (throughout this article we refer to the latter as "the ordinary partial-wave dispersion relation"), which is satisfied if the Mandelstam representation⁴ with a finite number of subtractions can be written down. It has become realized recently that both the Mandelstam representation and the ordinary partial-wave dispersion relation with a finite number of subtractions are violated in models with indefinitely rising Regge trajectories, so the validity of the N/D method becomes questionable. We will show in Sec. 2 and Sec. 3 of this article that the unitarity bound⁵ in the physical region guarantees that a generalized partial-wave dispersion relation can always be formulated, and the N/D integral equations can then be derived from the generalized partial-wave dispersion relation.

A second question about the N/D method is how to determine the CDD pole parameters. It is known⁶⁻⁹ that the origin of CDD poles¹⁰ in a one-channel N/D calculation can be interpreted as the effect of inelastic channels. We thus can expect to avoid the appearance of CDD poles by considering a multichannel formulation, if the potentials for each channel are given. Thus the question of CDD pole parameters is replaced by the question of obtaining potentials from reasonable physical models for each channel. In this article we will not consider CDD poles, but restrict our attention to a method for obtaining the input potentials.

A physical crucial question for the N/D method, evident from the previous paragraph, is how the input potential is to be obtained. The behavior of the input potential in the low-energy region can be estimated in a variety of ways (we will discuss these briefly in Sec. 2). Our major concern is how to obtain the high-energy behavior of the potential, because the solution of N/D equations depends critically thereon even if the low-energy behavior is given. It is this mathematical property of the N/D integral equation together with the difficulty (discussed in some detail in Sec. 4) of obtaining the high-energy behavior of the input potential that in previous N/D methods¹¹ has led to arbitrary cutoff parameters or to arbitrary assumptions about the properties of distant singularities (the high-energy behavior of the input potential is closely related to distant singularities in crossed channels). We shall resolve this difficulty for a certain class of models by expressing the high-

energy behavior of the input potential in terms of Regge parameters. The class of model, though not completely general, is large enough to contain the Veneziano model.¹²⁻¹⁵ These matters are discussed in Secs. 4 through 6.

A "restricted" partial-wave dispersion relation is defined in Sec. 4, and in Sec. 5 constraints on Regge parameters are inferred from assumptions used to formulate this "restricted" relation. It turns out that some of our assumptions force Regge residues to have nonsense-wrong-signature zeros and also force the intercept of the leading Regge trajectory to be less than one. In Sec. 6 a definition is given of "partial-wave duality" for models which satisfy the restricted partial-wave dispersion relation, and it is shown¹⁶ that the high-energy tail of the input potential can be expressed in terms of Regge parameters if partial-wave duality is imposed. Thus for models obeying partial-wave duality the N/D integral equations connect low-energy parameters (from the solution of the N/D equations and the input low-energy behavior of the potential) and high-energy parameters (from the input high-energy tail of the potentials). We note that this connection between low- and high-energy parameters in the N/D method is based explicitly on the nonlinear content of unitarity, whereas the role of unitarity in the finite energy sum rule¹⁷ is not explicit.

We only consider spinless, equal mass particles, the generalization to include spins and unequal mass particles being straightforward, though complicated.

2. THE GENERALIZED PARTIAL-WAVE DISPERSION RELATION

It is shown in this section that a generalized partial-wave dispersion relation can be defined for any partial-wave amplitude even though the ordinary partial-wave dispersion relation with a finite number of subtractions is violated in models with indefinitely rising trajectories, such as the Veneziano model.¹²⁻¹⁵

A partial-wave amplitude $A_\ell(s)$ is defined as

$$A_\ell(s) = \frac{S_\ell(s) - 1}{2ip(s)} = \frac{1}{32\pi} \int_{-1}^1 dz_s P_\ell(z_s) A(s, t, u)$$

where

$$\rho(s) = [(s - 4\mu^2)/s]^{\frac{1}{2}}$$

and S_ℓ is the partial wave S-matrix element, $A(s, t, u)$ is the scattering amplitude and z_s is the cosine of the s-channel scattering angle defined as

$$z_s = 1 + \frac{2t}{s - 4\mu^2} = -1 - \frac{2u}{s - 4\mu^2}$$

The unitarity bound requires that $|S_\ell| \leq 1$, so⁵

$$\lim_{s \rightarrow \infty} \frac{A_\ell(s)}{s^\epsilon} = 0 \quad \text{for some positive } \epsilon. \quad (2.1)$$

For simplicity we only consider the case

$$\lim_{s \rightarrow \infty} s^\epsilon A_\ell(s) = 0 \quad \text{for some positive } \epsilon. \quad (2.2)$$

in this and the next section; the generalization to partial-wave amplitudes which satisfy Eq. (2.1) but not (2.2), is straightforward.

The constraint (2.2) allows us to define a function $V_\ell(s)$ as

$$V_\ell(s) = A_\ell(s) - \frac{1}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\text{Im } A_\ell(s')}{s' - s} - \sum_1^N \frac{g_1}{s - s_1}$$

N being a finite integer, (2.3)

where the s_1 's ($0 < s_1 < 4\mu^2$) are the positions of the bound-state poles contained in the partial-wave amplitude $A_\ell(s)$, and the g_1 's are the pole residues:

$$g_1 = \lim_{s \rightarrow s_1} (s - s_1) A_\ell(s).$$

The potential function $V_\ell(s)$ is free from poles and also does not contain the s -channel normal thresholds, since $V_\ell(s)$ is real in the region $s \geq 4\mu^2$. Transposing the terms of (2.3),

$$A_\ell(s) = V_\ell(s) + \frac{1}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\text{Im } A_\ell(s')}{s' - s} + \sum_1^N \frac{g_1}{s - s_1}, \quad (2.4)$$

we call Eq. (2.4) the generalized partial-wave dispersion relation. We note that Eq. (2.4) may contain one subtraction if the asymptotic behavior of Eq. (2.1) is used.

The behavior of the potential $V_\ell(s)$ in the low-energy s -channel physical region may be estimated from the assumption

channels. The validity of such methods can be checked by inserting the experimental phase shift for $A_i(s)$ in both sides of Eq. (2.4). The above procedure cannot be distinguished from that used in the analysis¹⁸ of ordinary partial-wave dispersion relation. Therefore these previous analyses can be carried over directly to the generalized partial-wave dispersion relation.

We will see in the next section that the N/D integral equations can also be derived from the generalized partial-wave dispersion relation.

3. DERIVATION OF THE N/D INTEGRAL EQUATIONS

We now discuss the possibility of decomposing a partial-wave amplitude into the N/D form. For simplicity as discussed in Sec. 2, we only consider the case

$$\lim_{s \rightarrow +\infty} s^\epsilon A_\ell(s) = 0 \quad \text{for some positive } \epsilon, \quad (3.1)$$

the generalization being straightforward. The generalized partial-wave dispersion relation is

$$A_\ell(s) = \sum_1^N \frac{g_1}{s - s_1} + V_\ell(s) + \frac{1}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\text{Im } A_\ell(s')}{s' - s} \quad (3.2)$$

where

$$4\mu^2 > s_1 > 0 \quad (i = 1, \dots, N).$$

The amplitude $A_\ell(s)$ can be parametrized as

$$A_\ell(s) = |A_\ell(s)| \cdot e^{-i\theta_\ell(s)}$$

where $\theta_\ell(s)$ is real. We are aiming at the construction of the N/D integral equation from the R-function method.¹ If we want to employ the Frye-Warlock¹⁹ method, we need to consider the phase of $S_\ell(s)$ instead of that of $A_\ell(s)$. In the remaining discussion of this article we consider only the R-function method.

The zeros of $A_\ell(s)$ require attention. Certain of the zeros of $A_\ell(s)$ may correspond to CDD poles,¹⁰ and certain may not. On the other hand, CDD poles can be interpreted either as poles of the D function or zeros of the N function

(the reverse of this statement is not true). If we have a D function with poles, we can always introduce a new set of D and N functions, by multiplying the original D and N functions by an appropriate polynomial, such that the new D function is free from poles but has different asymptotic behavior, while the new N function contains additional zeros. In this section we concentrate on the D function which is free from all poles, and whose zeros are in one-to-one correspondence with the bound state poles of $A_\ell(s)$. The CDD poles (if any) are represented by zeros of the N function, the asymptotic behavior of the D function having a close connection thereto. We defer further discussion of the CDD ambiguity to the end of the section. We first want to show that if a partial-wave amplitude $A_\ell(s)$ is given, we can construct a D function, under certain assumptions, such that the D function is power-bounded, has the phase of $A_\ell(s)$ in the region $s \geq 4\mu^2$, is real in the region $s < 4\mu^2$, and its zeros are in one-to-one correspondence with the bound-state poles of $A_\ell(s)$. Assuming that

$$\lim_{s \rightarrow \infty} |O_\ell(s)| < \infty,$$

we define a function $D_\ell(s)$ as

$$D_\ell(s) = D_\ell(0) \cdot \left\{ \prod_{i=1}^N \frac{s_i - s}{s_i} \right\} \exp \left\{ \frac{s}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{O_\ell(s')}{s'(s' - s)} \right\}, \quad (3.3)$$

where the s_i 's are the positions of poles of $A_\ell(s)$. This function $D_\ell(s)$ may have some poles at $z = \bar{s}_i$ ($i=1, \dots$) when

$$\lim_{\epsilon \rightarrow 0} (\Theta_\ell(\bar{s}_i + \epsilon) - \Theta_\ell(\bar{s}_i - \epsilon)) = -\pi.$$

We assume that $\Theta_\ell(s)$ in the asymptotic energy region is determined by Regge phase. The Regge phase derived from Regge poles is continuous. Thus we may assume that the number of possible poles contained in Eq. (3.3) is finite. If such poles appear in Eq. (3.3), we multiply $D_\ell(s)$ by an adequate polynomial to remove these poles.

We consider this new function as the function $D_\ell(s)$. The function $D_\ell(s)$ has the phase $\theta_\ell(s_\pm)$ in the region $s_\pm \geq k_\mu^2$, is real in the region $s < k_\mu^2$, and its zeros are in one-to-one correspondence with poles of $A_\ell(s)$. The function $D_\ell(s)$ is power-bounded, so it satisfies a dispersion relation with a finite number of subtractions. If $\theta_\ell(s)$ is unbounded as $s \rightarrow +\infty$, the existence of a function $D_\ell(s)$, power-bounded, with the phase $\theta_\ell(s_\pm)$ in the region $s_\pm \geq k_\mu^2$ and real in the region $s < k_\mu^2$, is an open question. We note that this problem concerning the existence of $D_\ell(s)$ is not specially associated with the generalized partial-wave dispersion relation but already occurs for the N/D method based on the ordinary partial-wave dispersion relation. We also note that an unbounded $\theta_\ell(s)$ implies that the ratio $\text{Re } A_\ell(s)/\text{Im } A_\ell(s)$ oscillates as $s \rightarrow +\infty$. Here we assume that either $\theta_\ell(s)$ is bounded or that, if unbounded, a function $D_\ell(s)$, with the properties enumerated above, nevertheless exists. A function $N_\ell(s)$ is next defined as

$$N_\ell(s) = A_\ell(s) \cdot D_\ell(s), \quad (3.4)$$

so we have [note that $\text{Im } A_\ell(s) = \text{Im } V_\ell(s)$ for $s < s_L$]

$$\text{Im } N_\ell(s) = \begin{cases} 0 & \text{for } s \geq k_\mu^2 \\ D_\ell(s) \text{Im } V_\ell(s) & \text{for } s < s_L, \end{cases} \quad (3.5)$$

where s_L is the position of the nearest left-hand singularity of $A_\ell(s)$.

We consider below for simplicity the case that $D_\ell(s)$ satisfies a dispersion relation with one subtraction, but no poles. The N function may contain zeros. This case corresponds to the absence of CDD poles, since it can be shown¹⁶ that the solution is completely determined by giving the potential, the inelastic function (the R -function), and (if required) the subtraction constant in the generalized partial-wave dispersion relation. The generalization to include CDD poles is straightforward and is discussed briefly at the end of this section.

The unitarity relation can be written as

$$\text{Im} \left\{ \frac{1}{A_\ell(s)} \right\} = -\rho(s) \cdot R_\ell(s), \quad (3.6)$$

where

$$\rho(s) = \frac{1}{2}[(s - 4\mu^2)/s]^{\frac{1}{2}}$$

and

$$R_\ell(s) = \frac{\sigma_\ell^{\text{total}}(s)}{\sigma_\ell^{\text{elastic}}(s)} = \frac{\text{Im } A_\ell(s)}{\rho(s) \cdot |A_\ell(s)|^2}.$$

From Eqs. (3.4), (3.5), and (3.6), we have the relation

$$\text{Im } D_\ell(s) = \begin{cases} -\rho(s) \cdot R_\ell(s) \cdot N_\ell(s) & \text{for } s \geq 4\mu^2 \\ 0 & \text{for } s < 4\mu^2. \end{cases} \quad (3.7)$$

The dispersion relation for $D_\ell(s)$ (only the case with one subtraction is considered here) can be written with the help of Eq. (3.7) as

$$D_\ell(s) = D_\ell(0) - \frac{s}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\rho(s') R_\ell(s') N_\ell(s')}{s'(s' - s)}. \quad (3.8)$$

A function $C_\ell(s)$ may be defined as

$$C_\ell(s) = N_\ell(s) - V_\ell(s) \cdot D_\ell(s).$$

From Eqs. (3.5) and (3.7), we see that

$$\operatorname{Im} C_\ell(s) = \begin{cases} \rho(s) \cdot R_\ell(s) \cdot V_\ell(s) \cdot N_\ell(s) & \text{for } s \geq 4\mu^2 \\ 0 & \text{for } s < 4\mu^2. \end{cases}$$

As mentioned in Sec. 2 and at the beginning of this section, we only consider, for simplicity, the case that the asymptotic condition of Eq. (3.1) is satisfied. Therefore

$$\lim_{s \rightarrow +\infty} C_\ell(s) = 0,$$

since

$$D_\ell(s) \xrightarrow{s \rightarrow +\infty} \text{const.}$$

$$N_\ell(s), \text{ and } V_\ell(s) \xrightarrow{s \rightarrow +\infty} 0.$$

At the other asymptotic direction on the s -plane, the function $A_\ell(s)$ is not necessarily power-bounded, but the function $C_\ell(s)$ always vanishes at such limits. To see this explicitly, we write

$$s = |s| e^{i\theta},$$

and suppose at some limit $|s| \rightarrow +\infty$ with $\theta = \Psi / 2\pi$ ($n = 0, 1, 2, \dots$), we have the relation

$$\lim_{\substack{|s| \rightarrow +\infty \\ \theta = \Psi / 2\pi \\ (n=0,1,2,\dots)}} |s|^n / A_\ell(|s|e^{i\theta}) = 0 \quad \text{for any positive integer } n.$$

This expression just implies that we are assuming $A_\ell(s)$ is not power-bounded at the direction $\theta = \Psi / 2\pi$, $|s| \rightarrow +\infty$. From the definition of $v_\ell(s)$

$$v_\ell(s) = A_\ell(s) - \frac{1}{\pi} \int_{\mu^2}^{\infty} ds' \frac{\text{Im } A_\ell(s')}{s' - s} - \sum_1^N \frac{g_1}{s - s_1}$$

and the constant asymptotic behavior of $D_\ell(s)$ (we are considering this case as mentioned before), we have the relation

$$\lim_{\substack{|s| \rightarrow +\infty \\ \theta = \Psi / 2\pi}} C_\ell(|s|e^{i\theta})$$

$$= \lim_{\substack{|s| \rightarrow +\infty \\ \theta = \Psi / 2\pi}} (N_\ell(|s|e^{i\theta}) - v_\ell(|s|e^{i\theta}) \cdot D_\ell(|s|e^{i\theta}))$$

$$= \lim_{\substack{|s| \rightarrow +\infty \\ \theta = \Psi / 2\pi}} (D_\ell(|s|e^{i\theta}) \cdot [A_\ell(|s|e^{i\theta}) - v_\ell(|s|e^{i\theta})])$$

$$= \left\{ \lim_{\substack{|s| \rightarrow +\infty \\ \theta = \Psi / 2\pi}} D_\ell(|s|e^{i\theta}) \right\} \cdot \left\{ \lim_{\substack{|s| \rightarrow +\infty \\ \theta = \Psi / 2\pi}} \frac{1}{\pi} \int_{\mu^2}^{\infty} ds' \frac{\text{Im } A_\ell(s')}{s' - s} + \sum_1^N \frac{g_1}{s - s_1} \right\} = 0.$$

The above discussion implies that $C_\ell(s)$ satisfies a dispersion relation with no subtraction, i.e.

$$C_\ell(s) = N_\ell(s) - V_\ell(s) \cdot D_\ell(s) \\ = \frac{1}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\rho(s') R_\ell(s') V_\ell(s') N_\ell(s')}{s' - s}.$$

Substituting Eq. (3.8) into this relation, we have

$$N_\ell(s) = D_\ell(0) \cdot V_\ell(s) + \int_{4\mu^2}^{\infty} ds' K_\ell(s; s') N_\ell(s') \quad (3.9)$$

where

$$K_\ell(s; s') = \frac{1}{\pi} \rho(s') \cdot R_\ell(s') \frac{s' V_\ell(s') - s V_\ell(s)}{s' - s}.$$

As mentioned before, this $N_\ell(s)$ may contain some zeros, but these zeros have nothing to do with CDD poles, since the ratio $N_\ell(s)/D_\ell(s)$ is determined completely¹⁶ by giving $V_\ell(s)$ and $R_\ell(s)$. Equations (3.8) and (3.9) are the N/D integral equations and their form is the same as the N/D integral equation^{2,3} associated with an ordinary partial-wave dispersion relation.

We have demonstrated the derivation of the N/D integral equation for the simplest case, i.e., the case that $D_\ell(s)$ satisfies a dispersion relation with one subtraction and no poles. For the case that $D_\ell(s)$ does not have poles but $n+1$ subtractions are required to write down a dispersion relation

for $D_2(s)$, we can go through a similar argument as in the case with one subtraction and derive the N/D integral equations. In this case the dispersion relation for $C_2(s)$ requires n subtractions, and we have $2n+1$ arbitrary parameters (i.e., $2n+1$ subtraction constants) in the N/D integral equations. Only $2n$ of $2n+1$ parameters are nontrivial, since one parameter will set the scale. This case corresponds to the existence of n CDD poles.

4. THE RESTRICTED PARTIAL-WAVE DISPERSION RELATION

The N/D integral equations are derived in the previous section, but such equations are not useful unless we find reasonable ways to approximate the input potential. It is a mathematical property of the N/D integral equations that both the low-energy and high-energy behavior of the input potential is important in determining the structure of the output solution at low energy. We should not confuse this statement with the principle of nearby singularity dominance in the analysis of ordinary or generalized partial-wave dispersion relations. In such analysis the required inputs are the low-energy behavior of the potential $V_\ell(s)$ and the behavior of $\text{Im } A_\ell(s)$ in the region $s \geq 4\mu^2$, and the output is the behavior of $\text{Re } A_\ell(s)$ in the low-energy region. On the other hand, the outputs of the N/D integral equations are both $\text{Im } A_\ell(s)$ and $\text{Re } A_\ell(s)$ in the low-energy region, and we are saying that the required inputs are the behavior of the potential $V_\ell(s)$ in the low-, intermediate-, and high-energy regions, as well as the inelastic function $R_\ell(s)$ (and CDD pole parameters if necessary). If the behavior of the potential $V_\ell(s)$ in the intermediate- and high-energy region can be expressed in terms of high-energy phenomenological parameters (e.g., Regge parameters) without referring to the knowledge about the low-energy phase shift, the N/D method will provide some connection between the low- and high-energy phenomenological parameters, which cannot be obtained from the above mentioned analysis of the generalized

partial-wave dispersion relation. One may think that the high-energy behavior of the potential $V_f(s)$ can be expressed in terms of Regge parameters from the relation

$$V_f(s) \sim \operatorname{Re} A_f(s) - \frac{1}{\pi} P \int_C^\infty ds' \frac{\operatorname{Im} A_f(s')}{s' - s}$$

for $s \gg C \gg 4\mu^2$ (4.1)

by inserting the partial-wave projected Regge asymptotic behavior for $\operatorname{Re} A_f(s)$ and $\operatorname{Im} A_f(s')$. But such high-energy behavior of $V_f(s)$ cannot be extrapolated down to the intermediate-energy region, since at $s = C$ $V_f(s)$ of Eq. (4.1) has a spurious singularity which should not exist in the potential $V_f(s)$ from its definition Eq. (2.3). The intermediate-energy behavior of $V_f(s)$, as can be seen from the more general expression, is

$$V_f(s) = \operatorname{Re} A_f(s) - \frac{1}{\pi} P \int_{4\mu^2}^\infty ds' \frac{\operatorname{Im} A_f(s')}{s' - s}, \quad (4.2)$$

which is sensitive to the input behavior of $\operatorname{Im} A_f(s')$ in the low-energy region. Therefore the N/D method gives no useful information beyond the generalized partial-wave dispersion relation, if Eqs. (4.1) and (4.2) are used to estimate the behavior of $V_f(s)$ in the high- and intermediate-energy region. In the remaining part of this article, we shall mainly be concerned with the possibility of expressing $V_f(s)$ in the high- and intermediate-energy region in terms of high-energy phenomenological parameters alone.

The low-energy behavior of the potential $V_f(s)$ is dominated by nearby singularities in crossed channels. On the other hand, the intermediate- and high-energy behavior of the potential is controlled by both distant and nearby singularities of crossed channels, and only if the distant singularities are very weak, is the nearby contribution dominant. In previous N/D calculations,¹¹ the difficulty of estimating the contribution of distant singularities is handled by neglecting them completely (e.g., in models that approximate the potential by several nearby poles), by introducing some arbitrary regulating functions to approximate the discontinuity across the left-hand cut of the partial-wave amplitude $A_f(s)$, or by inserting some arbitrary cutoff parameters into the calculation. All such approaches, of course, lack a profound physical basis. Starting in this section we will investigate the possibilities of expressing the intermediate and high-energy behavior of $V_f(s)$ in terms of high-energy Regge parameters, and as will be shown in Sec. 6, we succeed for a certain class of model, which contains the Veneziano model^{12-15,21} with all trajectory intercepts less than one. In this section we begin by formulating "the restricted partial-wave dispersion relation," which applies to a certain class of model (such as that of Veneziano).

A partial-wave amplitude $A_f(s)$ is said to satisfy "the restricted partial-wave dispersion relation" if the following four properties are satisfied:

- (1) A scattering amplitude $A(s, t, u)$ is a linear combination of three functions $A_{st}(s, t)$, $A_{tu}(t, u)$, and $A_{us}(u, s)$,

each having special properties. Let (x,y,z) be an arbitrary cyclic permutation of (s,t,u) . The function $A_{xy}(x,y)$ contains the x and y channel normal thresholds and poles, but does not contain those for the z channel. Although the function $A_{xy}(x,y)$ is allowed to contain spurious singularities which are not contained in the scattering amplitude $A(s,t,u)$ [which will be cancelled in the linear combination that constructs $A(s,t,u)$], we assume that such spurious singularities are absent from A_{xy} in the region $x \leq 0$ with y fixed and $y \leq 0$ with x fixed.

(2) The function A_{xy} has Regge asymptotic behavior in the limit x (or y) $\rightarrow \pm\infty$ with y (or x) fixed, but it damps out faster than any power at the limit x (or y) $\rightarrow \pm\infty$ with z fixed.

We need to consider certain implications of assumption (1) before going on to state assumptions (3) and (4). The function $A_{st}(s,t)$ can be written in terms of s and z_s , where z_s is the cosine of the scattering angle of the s channel:

$$z_s = 1 + \frac{2t}{s - 4\mu^2} = -1 - \frac{2u}{s - 4\mu^2}.$$

In the s -channel physical region, i.e., $s \geq 4\mu^2$, we look at the singularities of $A_{st}(s,z_s)$ in the complex z_s -plane with s fixed. From assumption (1) all the singularities of A_{st} with s fixed are in the region $t > 0$, so the singularities of A_{st} in the z_s plane are in the region $z_s > 1$. We thus can draw a Lehman ellipse²² with foci at $z_s = \pm 1$ for A_{st} , implying that A_{st} has a partial-wave expansion in the s -channel physical region. The functions A_{tu} and A_{us} also have their own partial-wave expansions in the s -channel physical

region, by a similar argument, and of course the corresponding result is true in the t - and u -channel physical regions as well.

We denote the partial-wave amplitudes of A_{ℓ}^{st} , A_{ℓ}^{tu} , and A_{ℓ}^{us} in the s -channel physical region as $A_{\ell}^{st}(s)$, $A_{\ell}^{tu}(s)$, and $A_{\ell}^{us}(s)$ respectively. Assumptions (3) and (4) will now be formulated in the s channel, but they, of course, are also true in the t and u channels.

(3) The partial-wave amplitudes $A_{\ell}^{st}(s)$ and A_{ℓ}^{us} are power-bounded in the s plane, although such is not required for the amplitude $A_{\ell}^{tu}(s)$.

(4) The following asymptotic condition holds:

$$\lim_{s \rightarrow +\infty} \frac{A_{\ell}^{xy}(s)}{A_{\ell}(s)} < +\infty.$$

We note that models which satisfy the ordinary partial-wave dispersion relation are not necessarily special cases of the restricted partial-wave dispersion relation, since assumption (1) is not required for the ordinary partial-wave dispersion relation [assumption (2) is rather independent of the partial-wave dispersion relations]. On the other hand, assumption (3) is more general than the ordinary partial-wave dispersion relation in some sense, since $A_{\ell}(s)$ is not required to be power-bounded in the complex s plane.

The partial-wave amplitude $A_{\ell}(s)$ satisfies the unitarity bound, as discussed in Sec. 2,

$$\lim_{s \rightarrow +\infty} \frac{A_{\ell}(s)}{s^{\epsilon}}.$$

From assumptions (3) and (4) the amplitudes $A_f^{st}(s)$ and $A_f^{us}(s)$ satisfy dispersion relations with at most one subtraction. We only consider the simplest case in this section, i.e.,

$$\lim_{s \rightarrow \infty} A_f(s) \cdot s^\epsilon = 0 \quad \text{for some positive } \epsilon,$$

but the generalization is obvious. The dispersion relations for $A_f^{st}(s)$ and $A_f^{us}(s)$ are then

$$A_f^{st}(s) = \frac{1}{\pi} \int_{\text{L.H.C.}} ds' \frac{\text{Im } A_f^{st}(s')}{s' - s} + \frac{1}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\text{Im } A_f^{st}(s')}{s' - s} \quad (4.2)$$

$$A_f^{us}(s) = \frac{1}{\pi} \int_{\text{L.H.C.}} ds' \frac{\text{Im } A_f^{us}(s')}{s' - s} + \frac{1}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\text{Im } A_f^{us}(s')}{s' - s}.$$

The amplitude $A_f(s)$ can be written from assumption (1) as

$$A_f(s) = A_f^{st}(s) + A_f^{tu}(s) + A_f^{us}(s).$$

Since the function $A_f^{tu}(s)$ does not contain any singularities in the region $s \geq 4\mu^2$ and also does not contain any s -channel bound state poles [from assumption (1)], the potential $V_f(s)$ of the generalized partial-wave dispersion relation is

$$V_f(s) = A_f^{tu}(s) + \frac{1}{\pi} \int_{\text{L.H.C.}} ds' \frac{\text{Im } A_f^{st}(s') + \text{Im } A_f^{us}(s')}{s' - s} \quad (4.3)$$

It has been shown in the Veneziano model¹²⁻¹⁵ that the functions $A_s^{st}(s)$ and $A_s^{us}(s)$ satisfy²³ the dispersion relations of Eq.(4.2), but the function $A_s^{tu}(s)$ is not power-bounded at the limit $s \rightarrow \infty$.²⁴ Therefore the Veneziano model satisfies the restricted partial-wave dispersion relation, but violates the ordinary partial-wave dispersion relation. We note that if the Pomeranchuk intercept of Wong's version of the Veneziano model²¹ is less than one, it satisfies the restricted partial-wave dispersion relation, but Wong's version of the Veneziano model with Pomeranchuk intercept one violates our condition. We also note that the Virasoro model²⁵ violates the restricted partial-wave dispersion relation. We will see these points clearly in the next section.

5. THE CONSTRAINTS ON REGGE PARAMETERS FROM THE

RESTRICTED PARTIAL-WAVE DISPERSION RELATION

Assumption (1) of Sec. 4 is a strong assumption. When combined with assumption (2) of Sec. 4, there is implied a set of constraints on Regge parameters. We consider only Regge poles, although the analysis can be generalized to Regge cuts.

The Regge asymptotic behavior of the scattering amplitude may be parameterized as

$$A(s, t, u) \xrightarrow[s \rightarrow \infty]{x \text{ fixed}} \sum_i \frac{-\beta_i(x)}{\Gamma[\alpha_i(x) + 1] \cdot \sin \pi \alpha_i(x)} \cdot \left\{ \tau_i \left(x \frac{s}{s_1} \right)^{\alpha_i(x)} + \left(-\sigma_x \frac{s}{s_1} \right)^{\alpha_i(x)} \right\} \quad (5.1)$$

where $\sigma_t = +1$, $\sigma_u = -1$, the subscript 1 denotes the different Regge poles, and $\alpha_i(x)$ and $\beta_i(x)$ are the Regge trajectory and residue functions respectively, both assumed real in the region $t \leq 4u^2$. The symbol $\tau_i = \pm 1$ is the signature of the i th Regge trajectory and s_1 is a real constant. Assumption (1) of Sec. 4 says that we can decompose $A(s, t, u)$ into a linear combination of A_{st} , A_{tu} , and A_{us} as

$$A(s, t, u) = A_{st}(s, t) + A_{tu}(t, u) + A_{us}(u, s).$$

Assumption (2) means that

$$A_{sx} \xrightarrow[s \rightarrow \infty]{x \text{ fixed}} \sum_i \frac{-\beta_i(x)}{\Gamma[\alpha_i(x) + 1] \cdot \sin \pi \alpha_i(x)} \cdot \left(-\sigma_x \frac{s}{s_1} \right)^{\alpha_i(x)} \quad (5.2)$$

$$A_{tu} \xrightarrow[s \rightarrow +\infty]{x \text{ fixed}} \sum_1 \frac{-\beta_1(x)}{\Gamma[\alpha_1(x) + 1] \cdot \sin \pi \alpha_1(x)} \cdot \left(\sigma_x \frac{s}{s_1} \right)^{\alpha_1(x)}, \quad (5.3)$$

where $\sigma_t = +1$ and $\sigma_u = -1$. The requirement [from assumption (1)] about the absence of the spurious singularities of A_{sx} and A_{tu} in the region $x \leq 0$ with s or y fixed (if $x = t$, $y = u$, if $x = u$, $y = t$) forces us to choose $\alpha_1(x)$ and $\beta_1(x)$ in Eqs. (5.2) and (5.3) such that

$$\alpha_1(0) < 1 \quad \text{if } \beta_1(0) \neq 0 \quad (5.4)$$

and

$$\lim_{\alpha_1(x) \rightarrow 0} \left| \frac{\beta_1(x)}{\sin \pi \alpha_1(x)} \right| < +\infty \quad (x \leq 0) \quad (5.5)$$

$$\lim_{\alpha_1(x) \rightarrow -N} \left| \frac{\beta_1(x)}{\Gamma[\alpha_1(x) + 1] \cdot \sin \pi \alpha_1(x)} \right| < +\infty$$

for $N = 1, 2, 3, \dots$, $x \leq 0$.

For the leading trajectory $i = P$ it is obvious from the optical theorem and the experimental total cross section that

$$\beta_P(0) \neq 0$$

so the constraint of Eq. (5.4) requires that the intercept of the leading trajectory should be less than one. If the class of models of Sec. 4 contains Pommeranchuk poles, this implies that the Pommeranchuk intercept (of this class of models) must

be less than one. The constraints of Eq. (5.5) imply not only the existence of ghost killing factors at the right signature points

$$\tau_1 \alpha_1(x) = 0, -2, -4, \dots,$$

but also the existence of zeros in the residue

$\rho_1(x)/\Gamma[\alpha_1(x) + 1]$ at the nonsense wrong signature points, i.e., the points

$$\tau_1 \alpha_1(x) = -1, -3, -5, \dots$$

The argument of this section makes clear why Wong's version²¹ with unit Pomeranchuk intercept and Virasoro's model²⁵ do not satisfy the restricted partial-wave dispersion relation. The function A_{sx} of Wong's version with unit Pomeranchuk intercept has a spurious singularity (a pole) at $x = 0$, which partially violates assumption (1). Virasoro's model does not satisfy assumptions (1) through (4).

6. PARTIAL-WAVE DUALITY

We have introduced in Sec. 4 the restricted partial-wave dispersion relation, which is satisfied by the Veneziano model including Wong's version with the Pomanchuk intercept less than one, but even within a model obeying the restricted partial-wave dispersion relation the N/D method still cannot be made useful. In this section we will show that if we add an additional assumption (the assumption of "partial-wave duality") to assumptions (1) through (4) of Sec. 4, we can express the high-energy behavior of the input potential in terms of Regge parameters which can be obtained, in principle, from phenomenological fitting of the high-energy scattering data, and this high-energy behavior of $V_L(s)$ can be extrapolated down to the intermediate energy region without introducing spurious singularities. The Veneziano model with all the intercepts of Regge trajectories less than one satisfy the additional requirement.

We again only consider for simplicity the case

$$\lim_{s \rightarrow \infty} s^\epsilon A_L(s) = 0 \quad \text{for some positive } \epsilon,$$

a condition that holds when all Regge trajectories have non-vanishing slopes and intercepts less than one. Statements (1) through (4) of Sec. 4 are assumed. The partial-wave amplitude $A_L^{sx}(s)$ ($x = t, u$) satisfies a dispersion relation as shown in Eq. (4.1)

$$A_f^{sx}(s)$$

$$= \frac{1}{\pi} \int_{\text{L.H.C.}} ds' \frac{\text{Im } A_f^{sx}(s')}{s' - s} + \frac{1}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\text{Im } A_f^{sx}(s')}{s' - s} \quad (x = t, u) . \quad (6.1)$$

We now introduce the additional assumption

(5) that the following asymptotic relations hold,

$$\lim_{s \rightarrow +\infty} \frac{\int_{\text{L.H.C.}} ds' \frac{\text{Im } A_f^{sx}(s')}{s' - s}}{\int_{4\mu^2}^{\infty} ds' \frac{\text{Im } A_f^{sx}(s')}{s' - s}} = 0 \quad (x = t, u) \quad (6.2)$$

and

$$\lim_{s \rightarrow +\infty} \left| \frac{A_f^{sx}(s)}{A_f^{tu}(s)} \right| < +\infty \quad (x = t, u) . \quad (6.3)$$

The relation (6.3) holds if the Regge trajectories have nonvanishing slopes, since then both $A_f^{sx}(s)$ and $A_f^{tu}(s)$ have the asymptotic behavior $1/s^{1-b} \ln s$ (b is the intercept of the leading Regge trajectory), which are obtained by partial-wave projecting the Regge asymptotic behavior of A_{sx} and A_{tu} respectively. We call the asymptotic behavior of $A_f^{xy}(s)$ at the limit $s \rightarrow +\infty$ the partial-wave Regge asymptotic behavior, since it is caused by the Regge asymptotic behavior of A_{xy} . Equation (6.2) implies

$$\lim_{s \rightarrow +\infty} A_\ell^{sx}(s) = \lim_{s \rightarrow +\infty} \frac{1}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\text{Im } A_\ell^{sx}(s')}{s' - s}.$$

This means that the partial-wave Regge asymptotic behavior of $A_\ell^{sx}(s)$ is dual to the "pure" s-channel singularities, i.e., the right-hand singularities of $A_\ell^{sx}(s)$. In this sense we call assumption (5) the assumption of partial-wave duality.

The potential $V_\ell(s)$ of the models which satisfy assumptions (1) through (4) is defined in Eq. (4.2) as

$$V_\ell(s) = A_\ell^{tu}(s) + \frac{1}{\pi} \int_{\text{L.H.C.}} ds' \frac{\text{Im } A_\ell^{st}(s') + \text{Im } A_\ell^{su}(s')}{s' - s}.$$

If assumption (5) is imposed, we see that

$$\lim_{s \rightarrow +\infty} V_\ell(s) = \lim_{s \rightarrow +\infty} A_\ell^{tu}(s).$$

Since the asymptotic behavior of $A_\ell^{tu}(s)$ can be expressed, by partial-wave projecting the Regge asymptotic behavior of A_{tu} , in terms of Regge parameters which can be obtained from phenomenological fitting of the high-energy scattering data, so is the high-energy behavior of $V_\ell(s)$. To obtain the behavior of $V_\ell(s)$ in the intermediate-energy region, we must assume that partial-wave duality holds down to that energy region, i.e., then we can write

$$V_\ell(s) \sim A_\ell^{tu}(s)$$

in the intermediate-energy region. This method is used in Ref. (16) to express the behavior of the $\pi\pi$ p-wave potential in terms of Pomanchuk parameters in the intermediate and high-energy region.

The Veneziano model satisfies²³ the assumption of partial-wave duality, since the discontinuity of $A_s^{\pi\pi}(s)$ ($x = t, u$) across the left-hand cut decreases fast enough to allow Eq. (6.2) to hold, and Eq. (6.3) holds since all the Regge trajectories of that model are linear. These arguments can be generalized to Wong's model with the Pomanchuk intercept less than one.

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FOOTNOTES AND REFERENCES

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PART II. A METHOD TO SMOOTH EXPERIMENTALLY-
DETERMINED PARTIAL-WAVE PHASE SHIFTS *

1. INTRODUCTION

Experimentally-determined phase shifts for elastic scattering processes usually contain ambiguities. For example, πN partial-wave phase shifts from various groups¹⁻⁴ are bumpy in the energy region where the inelasticity builds up. The bumpy behavior of the phase shifts suggests the existence of nearby singularities, e.g., resonance poles, but there is naturally fear that some of this bumpy behavior may be caused by experimental error and not correspond to genuine singularities of the analytic S matrix. One category of spurious singularities are those with complex positions on the first sheet (usually called "causality violating singularities"). Smoothing procedures typically fit the data to analytic functions that lack such inadmissible singularities, being singular only in regions permitted by general S-matrix principles. The purpose of this article is to propose a method of smoothing, based on circular application of the N/D method. This method leads to a simpler smoothing procedure than do previous N/D methods. We also demonstrate that CDD poles can be handled without ambiguity.

We need to review briefly previous attempts to use N/D integral equations in order to make clear the differences between this and the previous works. The standard way to perform an N/D calculation⁵ is first to estimate the nearby "left-hand" singularities of partial-wave amplitude from experimental data on cross channel reactions. The distant "left-hand" singularities, the input inelastic function and CDD pole parameters are considered to be adjustable parameters. Inserting these inputs into the N/D integral equations, one tries by adjustment of the input parameters to obtain an output phase shift which is a good fit to experiment. There have also been investigations

different from this standard method but parallel to it. We mention three: One method^{6,7} avoids usage of the N/D integral equations. Instead, the partial-wave dispersion relation is solved directly by some variational technique implementing unitarity, though the input potential is parametrized as in the standard N/D approach. The second approach uses Blankenbecler-Sugar equation⁸⁻¹¹ to include multi-particle channels. The third approach we want to mention is a multi-channel N/D framework,¹²⁻¹⁴ instead of the single-channel formalism of the above described standard approach. The parametrization of the left-hand singularities of all the coupled reactions replaces both the input inelastic function and the CDD pole parameters. (In practice, of course, it is impossible to include all coupled channels.)

The adjustment of input parameters in the above-mentioned methods is coupled to the procedure for solving the integral equations, that is, all the input parameters must be adjusted simultaneously and the integral equation must be solved once after each adjustment. This situation makes the practical calculation very complicated if we require the output phase shift to become so accurate that we can determine whether or not the bumpy behavior of the experimental phase shift is caused by causality-violating singularities. Our proposal is based on the observation that the N/D method is circular and that the standard approach corresponds only to half of the circle.

Our method starts from a tentative phase shift based on experiment instead of on the parametrization of the input potential. This experimental phase shift is used to calculate an "experimental potential," which is defined to be the amplitude minus its physical (right-hand) cut and which will contain causality violating singularities if the tentative phase shift contains these singularities. In

the next step the experimental potential is fitted by a function containing the well known nearby unphysical region singularities,^{15,16} plus some properly-parametrized distant singularities whose positions are restricted by general principles. The fitted potential is, of course, free from causality violating singularities. If used as the input potential to the N/D integral equation, the output solution is generated to be free from causality-violating singularities so long as some circularity constraints are satisfied by adjusting the input inelastic quantities. In this procedure all the parameters of the input potential are fixed before we solve the N/D integral equations. Only the inelastic function and CDD pole parameters need to be adjusted from their experimental value during the procedure to solve the integral equations, and thus we can obtain a high accuracy output solution with relatively less effort in the numerical analysis.

We note that there is no CDD pole ambiguity in this method. In the R-function method¹⁷ CDD poles correspond to zeros of the partial-wave amplitude in the physical region, and in the Frye-Warnock method¹⁸ they are the zeros of the partial-wave S matrix in the physical region. Thus all CDD parameters are fixed by giving the experimental phase shift (see the next section for a review of this subject). We also note that our method requires a tentative experimental phase shift for the entire range of physical energies up to infinity. The high energy part is obtained by partial-wave projection of some high energy extrapolation of the elastic scattering amplitude (e.g., we may use a Regge pole fit). Thus our smoothing procedure also removes any possible causality-violating singularities (if any) caused by non-smooth transition from high energy to low energy.

In this article we formulate the smoothing procedure in detail for the πN phase shift and restrict consideration to the R-function method. The kinematics and the definition of various amplitudes for πN elastic scattering are summarized in the Appendix. In Sec. II, the reformulation of the N/D method and the handling of CDD poles are reviewed, while the circular character of the N/D method is discussed in Sec. III. The procedure of smoothing is outlined in detail in Sec. IV, and a rough but illustrative example is given by performing the first part of the smoothing of a πN P_{33} -wave in Sec. V. Some overall assessment of the proposed procedure is given in the last section.

2. CDD POLE PARAMETERS AND THE GENERALIZED N/D METHOD

The N/D method has recently been reconstructed,¹⁹⁻²² eliminating the assumption of power-boundedness for partial-wave amplitudes. Here we follow the style of Ref. 21 and review the reconstruction briefly in order to demonstrate the way to determine CDD pole parameters from a given phase shift. We use 0^- , $\frac{1}{2}^+$ kinematics (e.g., πN), summarized in the Appendix, and restrict consideration to the case of one CDD pole, the generalization to any number of CDD poles being straightforward.

A phase function $\theta_{\ell\pm}(w)$ is introduced from the parametrization

$$f_{\ell\pm}(w) = |f_{\ell\pm}(w)| e^{i\theta_{\ell\pm}(w)}, \quad (0 \leq \theta_{\ell\pm}(w) \leq \pi)$$

while a $\tilde{D}$ function is defined as

$$\tilde{D}(w) = \exp \left\{ -\frac{w}{\pi} \int_{w_0}^{\infty} dw' \frac{\theta(w')}{w'(w' - w)} \right\}, \quad (2.1)$$

where

$$w_0 = m + \mu,$$

the subscript $\ell\pm$ being omitted as understood. The function $\tilde{D}(w)$ will have a pole at $w = w_c$, if

$$\lim_{\epsilon \rightarrow 0_+} \theta(w_c - \epsilon) = \pi$$

and

$$\lim_{\epsilon \rightarrow 0_+} \theta(w_c + \epsilon) = 0.$$

We assume that one such pole in fact appears in the partial-wave amplitude. A new function $D(W)$ is defined as

$$D(W) = (W_0 - W) \tilde{D}(W) \quad (2.2)$$

so as to be free from the pole at $W = W_0$.

For the convenience of later application in Sec. 5, and to simplify the present discussion we assume that the N elastic scattering amplitude satisfies the specific Regge parametrisation of Ref. 23; then the asymptotic behavior of a partial-wave amplitude, neglecting the nonessential logarithmic factors, becomes

$$f_{\ell\pm}(W) \xrightarrow{W \rightarrow \infty} \frac{1}{W}.$$

$$\tan \theta(W) = \frac{\text{Im } f_{\ell\pm}(W)}{\text{Re } f_{\ell\pm}(W)} \xrightarrow{W \rightarrow \infty} C \quad (C < 0).$$

The assumption of this specific asymptotic behavior for the partial-wave amplitude is not essential to our discussion,²¹ but makes it simple. With such asymptotic behavior for the phase, the D function behaves asymptotically like

$$D(W) \xrightarrow{W \rightarrow \infty} W^{1+\alpha} \quad \left(\frac{1}{2} < \alpha < 1\right). \quad (2.3)$$

An N function is defined to be

$$N(W) = D(W) f_{\ell\pm}(W) \quad (2.4)$$

while the potential is

$$V(W) = f_{L\pm}(W) - \frac{1}{\pi} \int_{W_0}^{\infty} dW' \frac{\text{Im } f_{L\pm}(W')}{W' W} \quad (2.5)$$

Also needed is a C function:

$$C(W) = N(W) - D(W) V(W) = \frac{D(W)}{\pi} \int_{W_0}^{\infty} dW' \frac{\text{Im } f_{L\pm}(W')}{W' W} \quad (2.6)$$

The partial-wave unitarity relation can be expressed as

$$\text{Im} \left\{ \frac{1}{f_{L\pm}(W)} \right\} = -q_R R(W)$$

or

$$R(W) = \frac{\text{Im } f_{L\pm}(W)}{q_R |f_{L\pm}(W)|^2} \quad (2.7)$$

where $R(W) = 1$ in the elastic region and $R(W) \geq 1$ in the inelastic region. From partial-wave unitarity and the definitions of $N(W)$ and $C(W)$, i.e., Eqs. (2.4) and (2.6), we have

$$\text{Im } D(W) = \begin{cases} -q_R R(W) N(W) & W \geq W_0 \\ 0 & W < W_0 \end{cases} \quad (2.8)$$

and

$$\text{Im } C(W) = \begin{cases} q_R R(W) \cdot V(W) \cdot N(W) & W \geq W_0 \\ 0 & W < W_0 \end{cases}$$

The $\tilde{D}$ function satisfies a dispersion relation with two subtractions, according to its definition Eqs. (2.1) and (2.2) and its asymptotic behavior Eq. (2.3), while the C function satisfies a dispersion relation with one subtraction. Therefore we need three subtraction constants. We choose these to be as $D(0)$, $D'(0)$, and $C(0)$. The constant $D(0)$ is a normalization constant for both $D(W)$ and $H(W)$, and will not appear in their ratio. The constants $D'(0)$ and $C(0)$ come into the problem because of the physical region zero of the partial-wave amplitude $f_{\pm}(W)$ at $W = W_c$. If $f_{\pm}(W)$ does not have any zero in the physical region, only the constant $D(0)$ appears. Since $D'(0)$ and $C(0)$ constitute two parameters which must be supplied in addition to the potential $V(W)$ and the R function in the H/D integral equations (see below), we refer to them as a pair of CDD pole parameters. This definition of CDD pole is reasonable since in the $\tilde{D}$ function, there is a pole at $W = W_c$ due to the zero of $f_{\pm}(W)$ at $W = W_c$. The values of $D'(0)$ and $C(0)$ can be written down from Eqs. (2.1), (2.2), and (2.6) to be

$$\alpha \equiv D'(0) = - \left\{ \frac{1}{W_c} + \frac{1}{\pi} \int_{W_0}^{\infty} dW' \frac{\varrho(W')}{W'^2} \right\} \quad (2.9)$$

$$\beta \equiv C(0) = \frac{1}{\pi} \int_{W_0}^{\infty} dW' \frac{\text{Im } f_{\pm}(W')}{W'}$$

where the D function is normalized to be unity at $W = 0$. From Eq. (2.9) we see explicitly that the CDD pole parameters are determined uniquely if the partial-wave amplitude $f_{\pm}(W)$ (and thus the

phase shift) is given. The dispersion relations for $D(W)$ and $C(W)$ can be written as

$$D(W) = 1 + \gamma W - \frac{W^2}{\pi} \int_{W_0}^{\infty} dW' \frac{q_s(W') R(W') N(W')}{W'^2 (W' - W)}$$

and

$$C(W) = L(W) - V(W) D(W) = \beta + \frac{W}{\pi} \int_{W_0}^{\infty} dW' \frac{q_s(W') R(W') V(W') N(W')}{W' (W' - W)}$$

Inserting the dispersion relation of $D(W)$ into that of $C(W)$, we obtain the N/D integral equation as

$$g(W) = v(W) + \int_{W_0}^{\infty} dW' K(W, W') g(W') \quad (2.10)$$

and

$$D(W) = 1 + \gamma W - \frac{W^2}{\pi} \int_{W_0}^{\infty} dW' \frac{q_s(W') R(W')}{W'} \cdot \frac{g(W')}{W' - W} \quad (2.11)$$

where

$$g(W) = \frac{N(W)}{W},$$

$$v(W) = \frac{1}{W} [\beta + (1 + \gamma W) \cdot V(W)]$$

and

$$K(W, W') = \frac{1}{\pi} \cdot \frac{q_s(W') R(W')}{W'} \cdot \frac{W' V(W') - W V(W)}{W' - W}.$$

3. CIRCULARITY OF THE N/D METHOD

The N/D method outlined in the previous section has a circular nature. Starting from a partial-wave amplitude $f_{\pm}(W)$, we calculate the potential $V(W)$ from Eq. (2.5), the R function from Eq. (2.7), and the necessary CDD pole parameters from Eq. (2.9). Then inserting them into the N/D integral equations, an output partial-wave amplitude $f_{\pm}(W)$ is obtained. Of course the output partial-wave amplitude should be identical to the input one. By a careful study of the construction of the N/D integral equations, we see that this circular nature is rather trivial²⁴ in the sense that any partial-wave amplitude will satisfy this circle, if it satisfies the unitarity relation in the elastic region, the unitarity bound in the inelastic region, the Hölder continuity,¹⁹ and if its potential makes the N/D integral equation Fredholm.

The above mentioned circularity of the N/D method guarantees the absence of ghost zeros from the output D function. But the absence of ghost zeros from the D function becomes uncertain if we start from giving an input potential, a R function and CDD pole parameters instead from an input partial-wave amplitude. Suppose such a set of input informations is given, i.e., an input potential, a R function and CDD pole parameters from Eqs. (2.7), (2.5), and (2.9) respectively. We can reconstruct an output R function, an output potential and CDD pole parameters. The circularity between the input and the output R functions is trivial [it can be shown from Eqs. (2.4), (2.7), and (2.8)]. The circularity for the potential and CDD pole parameters is not obvious. Nevertheless if the latter circularity is also imposed, then the complete circularity of the N/D method guarantees the absence of ghost zeros from the D function.

We demonstrate the above argument by considering a simple example and investigate the nontrivial appearance of the circularity of the potential. We assume that there is no CDD pole, and the potential contains N poles only, so it can be written as

$$V(W) = \sum_{j=1}^N \frac{c_j}{W - W_j}$$

where

$$V^*(W) = V(W^*) .$$

The N/D integral equation becomes

$$N(W) = V(W) - \sum_{j=1}^N \frac{c_j W_j}{\pi} \cdot \frac{x_j}{W - W_j} , \quad (3.1)$$

where

$$x_j = \int_{W_0}^{\infty} dW' \frac{F(W') N(W')}{W' - W_j}$$

and

$$F(W') = \frac{q_s(W') R(W')}{W'}$$

Multiplying both sides by $F(W)/(W - W_1)$ and integrating over W from W_0 to ∞ , we obtain

$$x_i = t_i - \frac{1}{\pi} \sum_{j=1}^N a_{ij} c_j x_j W_j \quad (i = 1, \dots, N) , \quad (3.2)$$

where

$$\alpha_{ij} = \int_{W_0}^{\infty} dW' \frac{F(W')}{(W' - W_i)(W' - W_j)},$$

and

$$t_i = \sum_{j=1}^N \alpha_{ij} c_j.$$

Substituting Eq. (3.1) into the dispersion relation for $D(W)$,

$$D(W) = 1 - \frac{W}{\pi} \int_{W_0}^{\infty} dW' \frac{F(W') N(W')}{W' - W},$$

we have

$$D(W) = 1 - \frac{W}{\pi} \sum_{j=1}^N c_j \alpha_j(W) \cdot \left(1 - \frac{W_j x_j}{\pi}\right), \quad (3.3)$$

where

$$\alpha_j(W) = \int_{W_0}^{\infty} dW' \frac{F(W')}{(W' - W)(W' - W_j)}.$$

Taking the ratio N/D of Eqs. (3.1) and (3.3), and continue to

$W = W_1$, we have

$$\lim_{W \rightarrow W_1} (W - W_1) \frac{N(W)}{D(W)} = \frac{c_1 \left(1 - \frac{W_1 x_1}{\pi}\right)}{1 - \frac{W_1}{\pi} \sum_{j=1}^N c_j \alpha_{1j} \left(1 - \frac{W_j x_j}{\pi}\right)}.$$

substituting Eq. (3.2) for x_1 in the numerator, we have

$$\lim_{W \rightarrow W_1} (W - W_1) \frac{N(W)}{D(W)} = C_1 \quad (i = 1, \dots, N).$$

This result indicates that the input potential is really a part of the unphysical region singularities of the output partial-wave amplitude. In order to show the absence of the other unphysical region singularities (poles) from the output partial-wave amplitude, i.e., the absence of ghost zeros from the D function, the circularity of the potential must be imposed.

The circularity of the potential can be written, from Eq. (2.5),

as

$$\sum_{j=1}^N \frac{c_j}{W - W_j} = \frac{N(W)}{D(W)} - \frac{1}{\pi} \int_{W_0}^{\infty} dW' \frac{N(W')}{W' - W} \cdot \text{Im} \left\{ \frac{1}{D(W')} \right\} \quad (3.4)$$

where $N(W)$ and $D(W)$ are given by Eqs. (3.1) and (3.3) respectively. The parameters c_j 's and W_j 's may be considered as given, then the circularity equation of Eq. (3.4) becomes a nontrivial constraint for the R function, in other words, the R function must satisfy this constraint in order to guarantee the absence of ghost zeros from the D function.

4. THE PROCEDURE FOR SMOOTHING A PHASE SHIFT

We now outline our procedure for smoothing a phase shift based on the circularity of the N/D method. We discuss explicitly the πN phase shift as an example, the application to other kinds of phase shifts being straightforward.

Starting from an experimental low- to medium-energy phase shift for a certain partial wave, we combine this with the high-energy phase shift from the partial-wave projection of a phenomenological parametrization of the high-energy πN elastic amplitude, e.g., a Regge pole fit etc. We call this combined phase shift the experimental phase shift. The experimental input potential, R function, and CDD pole parameters (if their existence is indicated by the experimental phase shift) are calculated from their definitions by inserting the experimental phase shift into Eqs. (2.5), (2.7), and (2.9) respectively. In order to eliminate causality-violating singularities we next seek a "fitted" potential that approximates as well as possible the "input" potential but that contains only allowed unphysical region singularities.²⁵ The structure of the near-by unphysical singularities are known,^{15,16} but the distant unphysical-region singularities, whose positions are known, but whose magnitude is uncertain, must be parametrized and these parameters determined by a best fit to the experimental potential. This fitted potential as well as the experimental CDD pole parameters and the experimental R function are then used as the input to the N/D integral equation. Since the fitted potential is free from causality-violating singularities, the structure of the N/D integral equation guarantees a corresponding solution for the R function as long as the input R function is

The R function and the CDD pole parameters then must be adjusted from their experimental input to make the circularity for the potential and CDD pole parameters be satisfied. The output partial-wave amplitude, after the complete circularity is imposed, is guaranteed to be free from causality violating singularities.

We note that in this smoothing procedure the determination of the parameters of the fitted potential is carried out before the integral equations are solved. This represents a considerable simplification of the effort in numerical analysis compared to the standard N/D approaches discussed in Sec. I, where the adjustment of the potential parameters is coupled to the procedure for solving the integral equations. Further the absence of causality violating singularities is not guaranteed in those standard N/D approaches due to the ignorance of the circularity of the potential and -CDD pole parameters. Another characteristic of our smoothing procedure, which in principle also exists in the standard approach but usually is neglected, is the necessity to use an asymptotic high-energy phase shift. Thus our smoothing procedure automatically removes possible causality-violating singularities associated with nonsmooth transition from low energies to high energies.

There is a more effective method to check the absence of the ghost zeros (causality violating zeros or zeros below the elastic threshold) from the output D function than the explicit check of the circularity of the potential. Suppose the dispersion relation of D requires $N + 1$ subtractions, that is, we are considering the case with N CDD poles. A δ function can be constructed by Eq.(2.1) from the phase of the output D function. This phase should have N discontinuous jumps from π to 0 in the physical region in order to

generate N poles in $\tilde{D}$. A new D function is then obtained by multiplying the $\tilde{D}$ function with N zeros to cancel N poles of $\tilde{D}$. This new D function is apparently free from ghost zeros. The ratio between the output D and this new D function is regular on the whole complex energy plane. If this entire function approaches a constant asymptotically at any direction, the output D function is proportional to the new D function, and it is also free from ghost zeros.

5. AN ILLUSTRATIVE EXAMPLE (SMOOTHING OF A πN P_{33} -WAVE)

We smooth a πN P_{33} -wave phase shift in this section as an example illustrative of the procedure outlined in the previous section. The fitting technique used in this example is primitive and can be greatly improved by advanced numerical analysis. The last step of the smoothing procedure, i.e., the check of the circularity of the potential and CDD pole parameters has not been performed in this simple calculation. Instead the behavior of the output D function is investigated explicitly. It turns out that there is no obvious causality violating zeros (ghost zeros) in our solution.

The low-energy πN P_{33} -wave phase shift is taken from the CERN analysis,¹ while the high-energy phase shift is obtained from a partial-wave projection of Barger and Phillips' five Regge pole fit²³ to the πN elastic scattering amplitude. A portion of the combined experimental phase shift (from the πN threshold up to 3.3 GeV in center of mass energy) is shown in Fig. 1 by the dotted points. The transition from the CERN phase shift to the Regge phase shift occurs around 2.2 GeV. Substituting this combined experimental phase shift into Eq. (2.5), an experimental potential is obtained. To simplify the calculation we fit the experimental potential by a finite number of poles. In any serious attempt to implement this smoothing procedure, the known structure of nearby unphysical region singularities (they are usually logarithmic) should be used explicitly, instead of just poles. For our fitted potential in this example we use twenty-one poles; three in the interval between the origin of the W plane and the πN elastic threshold, one pair of complex conjugate poles at the position of the circle singularities²⁵ associated with the

t channel, and nine pairs of complex conjugate poles along the imaginary axis of the W plane. The positions of the poles are fixed, but their residues are considered as the adjustable parameters. Thus we have twelve adjustable parameters (the residues of a pair of complex conjugate poles are complex conjugate). These twelve adjustable parameters are then determined by a least square fit to the experimental potential. A portion of the fitted potential and the experimental one are shown in Fig. 2 up to about 3.5 GeV. A better fit can be obtained by an advanced numerical analysis, e.g., also take the positions of poles as adjustable parameters, and by considering some logarithmic singularities explicitly.

The experimental phase shift from which we started requires one CDD pole at infinity, whose presence reflects our definition of the partial-wave amplitude combined with the Regge behavior of Ref. 25, our partial-wave amplitude vanishing at infinity. This CDD pole could be removed by changing the definition of the partial-wave amplitude to let it approach a nonvanishing asymptote, but then a subtraction constant must be included in the definition of the potential. The physics would be unchanged. Using the notion of a CDD pole at infinity in this example, we obtain the CDD pole parameters (one subtraction constant for the C function and one for the D function). An experimental R function is also calculated from the combined experimental phase shift. The fitted potential, the experimental R function, and the experimental CDD pole parameters are used as the input to the N/D integral equations. The output solution of the integral equations has a Δ_{33} resonance with a mass shifted about 80 MeV below the experimental mass (1236 MeV).

The CDD pole parameters are varied about 10% from their experimental value to bring the output mass of the Δ_{33} resonance to its experimental value. The R function is adjusted by hand, starting from the experimental value to produce a reasonable fit to the input phase shift. This adjusted R function has not been listed, instead the output solution translated to the output phase shifts are shown in Fig. 2 as the solid curve [we note that due to the trivial circularity of the R function, our input R function can be easily obtained from the output phase shift through Eq. (2.7)]. The circularity of the potential and CDD pole parameters is not checked in this simple example, but the values of the output D function at various energies are plotted in Fig. 3. The lack of clockwise circle in this figure indicates the absence of nearby ghost zeros from the D function. This can also be seen from the negativity of the imaginary part of the D function in the physical region, combining with the dispersion relation of D (the imaginary part of D at complex W will never vanish). We note that no systematic effort has been made in this example to obtain a best fit to the input phase shift.

6. DISCUSSION

The smoothing procedure outlined in the previous sections has been constructed on the ordinary partial-wave amplitude. We note that a similar smoothing procedure can be constructed on other partial-wave amplitudes,^{26,27} with which the partial-wave expansion converges faster than the ordinary partial-wave expansion.

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APPENDIX

We review the kinematics of πN scattering in this appendix. The notations follow that of Ref. 25 and Ref. 28. Let the four momenta of the incident and outgoing pions be q_1 and q_2 respectively, while those of the initial and final nucleons are p_1 and p_2 . The masses of pion and nucleon are denoted as μ and m . The Mandelstam variables s , t , and u are defined to be

$$s = (p_1 + q_1)^2$$

$$t = (q_2 - q_1)^2$$

$$u = (p_1 - q_2)^2$$

The magnitude of the three momentum vector in the center of mass frame, q_s , and the cosine of the scattering angle, $\cos \theta$, are

$$q_s^2 = \frac{[s - (m + \mu)^2][s - (m - \mu)^2]}{4s}$$

and

$$\cos \theta = 1 + \frac{t}{2q_s^2}.$$

Two amplitudes f_1 and f_2 are defined²⁸ in terms of the invariant amplitudes $A(s, t, u)$ and $B(s, t, u)$ as

$$f_1 = \frac{E + m}{2W} [A + (W - m)B]$$

$$f_2 = \frac{E - m}{8\pi W} (-A + (W + m)B),$$

where

$$W = (s)^{\frac{1}{2}}$$

and

$$E = (m + q_s)^{\frac{1}{2}}.$$

The partial-wave expansion of f_1 and f_2 ²⁸ are

$$f_1 = \sum_{\ell=0}^{\infty} (f_{1+}(W) P'_{\ell+1}(\cos \theta) - f_{1-}(W) P'_{\ell-1}(\cos \theta))$$

and

$$f_2 = \sum_{\ell=1}^{\infty} (f_{2-}(W) - f_{2+}(W)) P'_{\ell}(\cos \theta).$$

By the help of the orthogonal relation

$$\int_{-1}^1 d(\cos \theta) P'_k(\cos \theta) (P'_{\ell+1}(\cos \theta) - P'_{\ell-1}(\cos \theta)) = \delta_{k\ell}$$

we obtain

$$\begin{aligned} f_{1\pm}(W) = & \frac{1}{16\pi W^2} [(W + m + \mu)(W + m - \mu) \cdot [A_{\ell}(s) + (W - m)B_{\ell}(s)] \\ & + (W - m + \mu)(W - m - \mu) \cdot \{-A_{\ell}(s) + (W + m)B_{\ell+1}(s)\}] \end{aligned}$$

where

$$A_2(s) = \int_{-1}^1 d(\cos \theta) A(s, t, u) P_2(\cos \theta)$$

and

$$B_2(s) = \int_{-1}^1 d(\cos \theta) B(s, t, u) P_2(\cos \theta) .$$

FOOTNOTES AND REFERENCES

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FIGURE CAPTIONS

- Fig. 1. The dotted points below 2.2 GeV are the CERN experimental phase shift for πN P_{33} -wave and those above 2.2 GeV are the Regge phase shift from the partial-wave projection of the Regge pole fit of Ref. 23. The solid curves are our output phase shift.
- Fig. 2. The dotted points represent the value of the experimental potential, and the solid curve is the fitted one.
- Fig. 3. The values of the output D function are plotted for various energies.

[Fig. 1]

XBL728-3750

[Fig. 2]

XBL726 - 2749

XBL 728-3748

[Fig. 3]