ASYMPTOTIC BEHAVIOR AND THE POSSIBILITY OF DUALITY WITHOUT
GHOSTS IN FEYNMAN DIAGRAM MODELS*

by

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Abstract

Elastic scattering of two spinless particles with equal or unequal masses is considered in Feynman diagram models of the Van Hove type with arbitrary spectra of masses, spins and coupling constants including infinitely large masses and spins. It is shown that if the amplitude defined by analytic continuation of the sum of all lowest order diagrams with s-channel poles has no unphysical or u-channel singularities, then for fixed t this amplitude is not bounded by any power of s as $|s| \rightarrow \infty$ in any infinitely multiply connected domain which excludes only neighborhoods of the poles. As a consequence, models of the Van Hove type cannot exhibit duality. The result is independent of the order of summation of diagrams. It depends crucially on the requirement that coupling constants be real.

* Work supported by the U.S. Atomic Energy Commission, Contract No. AT (11-1)-1764.

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I. INTRODUCTION

It is well known that it is possible to construct Feynman diagram models¹⁻⁴ with simple couplings which exhibit Regge asymptotic behavior in one channel. At first, one might hope that by introducing a very large number of particles and adjusting all the coupling constants, a model of the Van Hove type¹⁻⁴ possessing Regge behavior in two channels could be found. The existence of dual resonance models^{5,6} would tend to support this hope if implicit poles in t were allowed in the sum of s -channel pole diagrams. However, we will prove that the asymptotic behavior of any such generalized Van Hove model is unacceptable from the point of view of duality. The factors which are primarily responsible for this negative result are (1) the absence of ghosts and (2) the simple couplings of the arbitrarily high spin particles which are present in the model. We prove that if the sum of all lowest order diagrams with s -channel poles has no implicit unphysical or u -channel singularities then for fixed t , the amplitude defined by this sum is not bounded by any power of s as $|s| \rightarrow \infty$ in any infinitely multiply connected domain which excludes only neighborhoods of the poles. No assumption is made about

the domain of convergence of the sum of diagrams other than its existence.

From the point of view of applications of the Van Hove model, the result indicates that outside the usually accepted region of applicability the model is quite badly behaved.

From the point of view of duality the result is mildly unfortunate because of its generality. It implies that no dual resonance theory with trajectories and daughters of any shape or spacing can correspond to a ghost-free theory with the simple and conventional couplings of the model.

Although ghosts plague the generalized Veneziano model, one would like to imagine that some dual multiparticle theory without ghosts exists. In this regard, the present work indicates that it would be desirable to at least show that within the context of Feynman diagrams, ghost-free theories containing arbitrarily high spins can exist with reasonable couplings which do not immediately generate bad asymptotic behavior in the Born approximation.

Abarbanel⁷ has considered the possibility that a rather general Born term can exhibit duality for special sets of coupling constants and masses in a strictly localizable field theory. He shows by example that at least one dual

resonance model, the Veneziano model, is contained in his Born term as a special case. However, problems with ghosts are known to occur in the Veneziano model,⁵ and in this example Abarbanel does not explicitly require that all coupling constants be real. The Born term considered by Abarbanel differs from the one treated in this paper.

The general idea of the proof presented here is that power boundedness implies the existence of a convergent series (partial fraction expansion) and that the infinite sum of Feynman diagrams can be conveniently and rigorously studied by comparison with this convergent series. If all the coupling constants are real, then the comparison leads to an inconsistency which involves the couplings of the arbitrarily high spin particles. Roughly speaking, the difficulty, which is quite severe, comes from the numerator s -dependence of the Feynman diagrams together with the fact that positivity prevents cancellation of this s -dependence when diagrams are summed. This description of the origin of the difficulty is considerably oversimplified in that it makes no reference to the crucial questions of convergence which are analyzed in detail in Sec. III. To put the situation in perspective we include two further observations. The first concerns pole models which are obtained by replacing numerator factors

of s with the appropriate masses-squared. The methods used in this paper will reveal no problems in connection with such models. Our second observation is that the difficulty might also disappear if we inject more s -dependence through the introduction of form factors.

II. DUALITY CONDITIONS AND THE MODEL

We consider the elastic scattering of two spinless particles with masses m_1 and m_2 which couple⁸ to an infinite set of particles. As in the Van Hove model, we examine the sum of lowest order diagrams with poles in one channel. The explicit expression for this sum is²⁻⁴

$$M(s, t) = \sum_{i=1}^{\infty} \frac{(2J_i + 1) g_i^2}{s_i - s} (q_i^2)^{J_i} P_{J_i}(z_i) \quad (1)$$

where

$$s = (p_1 + p_2)^2, \quad s_i = M_i^2, \quad t = (p_1 - p_3)^2,$$

$$4 q_i^2 = s - a_i, \quad (2)$$

$$a_i = 2m_1^2 + 2m_2^2 - \frac{(m_1^2 - m_2^2)^2}{s_i}, \quad (3)$$

$$z_i = 1 + \frac{2t}{s - a_i} \quad (4)$$

and the g_i are coupling constants. In Eq. (1), the P_ℓ are Legendre polynomials which can be expressed in terms of Gauss' hypergeometric series⁹ as follows:¹⁰

$$P_{\ell}(z) = F(-\ell, \ell + 1; 1; (1-z)/2) . \quad (5)$$

Substituting Eqs. (2-5) in Eq. (1) we find that

$$M(s, t) = \sum_{i=1}^{\infty} (s_i - s)^{-1} \sum_{n=0}^{J_i} C(J_i, n) t^n (s - a_i)^{J_i - n} \quad (6)$$

where

$$C(J_i, n) = \frac{(2J_i + 1) g_i^2}{4^i (n!)^2} \frac{(J_i + n)!}{(J_i - n)!} . \quad (7)$$

We observe that reality of the coupling constants g_i and the positivity of the factors which come from the expansion of P_{ℓ} as a power series (5) in t are responsible for the fact that

$$C(J_i, n) > 0 . \quad (8)$$

We wish to examine the consequences of imposing five conditions which are a subset of the necessary conditions one would associate with a dual multiparticle theory. The first four are conditions which one might associate with the Van Hove model alone.

(i) It is required that the infinite sum in Eq. (6) be uniformly convergent in some unspecified domain of s and t , and that the Born term $M(s,t)$ can be defined elsewhere by analytic continuation.

(Note that the order in which diagrams are summed has been left arbitrary by not having specified the dependence of s_i and J_i on i . Here we continue to allow for the possibility that the order of summation is important by not demanding absolute convergence.)

(ii) It is required that J_i be unbounded.

(If J_i were bounded then $M(s,t)$ would be a polynomial in t . This would correspond to behavior as $|t| \rightarrow \infty$ for fixed s which is uninteresting from the point of view of either the Van Hove model or duality.)

(iii) Particles with finite mass are required to have finite spins and finite degeneracy.

(This condition is satisfied^{11,12} by dual multiparticle theories. Except for this requirement the spin content and degeneracy are completely arbitrary.)

(iv) For any fixed t with the possible exception of a set of isolated points, $M(s,t)$ is required to be a meromorphic function of s whose poles and residues are just poles and residues of terms in the infinite sum in Eq. (6).

(There is no restriction on possible implicit singularities in t at isolated points because it is an essential requirement of duality that we allow for the possibility of implicit poles in t or u . Having allowed for the possibility of singularities in t , it is convenient to exclude u -channel singularities by considering only processes in which there are no u -channel contributions to the Born term. This eliminates one source of implicit singularities in the variable s when t is held fixed. We also exclude as "unphysical" any non- u singularities whose position in s is t dependent. Such singularities have no interpretation as particle contributions to our Born term and should not be present in any dual model. Finally, we note that no s -channel singularities are supposed to be "dual" to other s -channel singularities. Therefore, in condition (iv) we forbid implicit singularities in s while imposing no restrictions on the explicit "physical" s -channel poles of Eq. (6).)

(v) It is tentatively required that for any fixed t with the possible exception of isolated points, $M(s, t)$ is bounded by some power of s as $|s| \rightarrow \infty$ in a multiply connected domain excluding only neighborhoods of the poles but including infinitely many segments of the positive real s -axis.

(Note that neighborhoods of all poles must be

excluded. On the other hand, segments of the positive real s -axis out to infinity are included because dual theories are supposed to have some relevance in the asymptotic part of the physical region. The resulting infinitely multiply connected domain need not include segments of the s -axis between every pair of adjacent poles.)

III. INCOMPATIBILITY OF THE DUALITY CONDITIONS AND THE MODEL

For subsequent purposes we define a function $\bar{M}(s, t)$ which differs from the amplitude $M(s, t)$ only through the omission of those diagrams whose poles are below threshold.

Thus, we write

$$\bar{M}(s, t) = \sum_{i=1}^{\infty} \theta(s_i - s_0) (s_i - s)^{-1} \sum_{n=0}^{J_i} C(J_i, n) t^n (s - a_i)^{J_i - n} \quad (9)$$

where

$$s_0 = (m_1 + m_2)^2,$$

$$\theta(x) = 1 \quad \text{for } x \geq 0,$$

$$\theta(x) = 0 \quad \text{for } x < 0.$$

The requirements of meromorphy and finite degeneracy imply that only a finite number of diagrams have poles below threshold. Thus, the domain of convergence of the infinite series in Eq. (9) completely overlaps the domain of convergence of the infinite series in Eq. (6). If we denote the maximum value of spin for the particles below threshold by $\bar{J}$, then we see that the function $(M - \bar{M}) \sim s^{\bar{J}-1}$ as $|s| \rightarrow \infty$ with t held fixed. Thus, $\bar{M}(s, t)$ is bounded along with $M(s, t)$ by some power of s in the domain described in condition (v).

The ordered sequence of points at which the s-channel poles of $\bar{M}(s,t)$ are located will be denoted by σ_p , $p=1,2,3,\dots$ where $\sigma_p < \sigma_{p+1}$. Since essential singularities are not allowed, we must have $\sigma_p \rightarrow \infty$ as $p \rightarrow \infty$. In general, any given pole will appear in more than one diagram. The set of values of i for which $s_i = \sigma_p$ will be denoted by $I(p)$ and for these values of i we write $a_i = b_p$.

We next consider the contour integral

$$I_m(s,t) = \frac{1}{2\pi i} \int_{C_m} ds' \frac{\bar{M}(s',t)}{(s'-s)^{k+1}} \quad (10)$$

taken along a circle C_m which has its center at the origin and which encloses the first m pole positions $\sigma_1 \dots \sigma_m$. Here k is an integer, $k \geq K = K(t)$ and $K(t)$ is the least integer for which $\bar{M}(s,t)/s^K \rightarrow 0$ as $|s| \rightarrow \infty$ in the domain described in condition (v). Condition (v) guarantees that we can find an infinite sequence of circles¹³ C_m on which $|I_m(s,t)| \rightarrow 0$ as $m \rightarrow \infty$ for any finite s away from the poles. This in turn implies that we can obtain a uniformly convergent series¹³ by evaluating the residues of poles of the integrand in Eq. (10) and considering the limit $m \rightarrow \infty$. In this way we obtain

$$\bar{M}^{(k)}(s, t) = k! \sum_{p=1}^{\infty} (\sigma_p - s)^{-k-1} \sum_{i \in I(p)} \sum_{n=0}^{J_i} C(J_i, n) t^n (\sigma_p - b_p)^{J_i - n} \quad (11)$$

where the k -th derivative of $\bar{M}$ with respect to s is denoted by $\bar{M}^{(k)}$. In arriving at Eq. (11), we have invoked condition (iv).

We next make use of the convergence of the p -sum in Eq. (11) at $s = \sigma$ and $t = \tau$ where σ and τ are positive real numbers, $k \geq K(\tau)$ and $\sigma \neq \sigma_p$, $p = 1, 2, \dots$. In this case all terms in the infinite series with $p > p_0$ are positive, p_0 being the least integer for which $\sigma_p \geq \sigma$. Note that because poles below threshold do not appear in $\bar{M}(s, t)$, we have $\sigma_p \geq (m_1 + m_2)^2$, and consequently

$$\sigma_p \geq 2m_1^2 + 2m_2^2 - \frac{(m_1^2 - m_2^2)^2}{\sigma_p} \quad (12)$$

or in terms of the quantities which appear in Eq. (11)

$$(\sigma_p - b_p) \geq 0. \quad (13)$$

Thus, the infinite series in Eq. (11) with $s = \sigma$ and $t = \tau$ converges absolutely. Absolute convergence for other values of s and t can be inferred from

$$\begin{aligned}
& \left| (\sigma_p - s)^{-k-1} \sum_{i \in I(p)} \sum_{n=0}^{J_i} c(J_i, n) t^n (\sigma_p - b_p)^{J_i - n} \right| \\
& \leq |\sigma_p - s|^{-k-1} \sum_{i \in I(p)} \sum_{n=0}^{J_i} c(J_i, n) |t|^n (\sigma_p - b_p)^{J_i - n} \\
& \leq (\sigma_p - \sigma)^{-k-1} \sum_{i \in I(p)} \sum_{n=0}^{J_i} c(J_i, n) \tau^n (\sigma_p - b_p)^{J_i - n} \quad (14)
\end{aligned}$$

for $p \geq p_0$, $\text{Re}(s) \leq \sigma$ and $|t| \leq \tau$. Through the Weierstrass M-test,¹⁴ this inequality shows that the series in Eq. (11) converges uniformly and absolutely for $|t| \leq \tau$, $k \geq K(\tau)$, $\text{Re}(s) \leq \sigma$ and $s \neq \sigma_p$, $p = 1, 2, \dots$. Note that for $|t| < \tau$ we might have had $K(t) \leq K(\tau)$. By taking σ to be arbitrarily large, we see that we have established uniform and absolute convergence in the finite part of the s-plane excluding neighborhoods of the poles. In contrast, the domain of convergence in t is correlated with the index k and thus may be different for the series which represent the various functions $\bar{M}^{(k)}(s, t)$. However, we can take τ to be as large as we wish and still find some series (those with $k \geq K(\tau)$) which converge uniformly and absolutely inside a circle of radius τ .

We can now do some rearrangement of the series in Eq. (11). Absolute convergence of the repeated series

$\sum_p \sum_{i \in I(p)}$ taken in that order implies the absolute convergence of the corresponding double series¹⁵ whose terms can therefore be summed in any order¹⁶ as a single series without changing the value of the sum. We choose the new order of summation to be the same as the order of summation of diagrams in Eq. (9). Thus, the series representation

$$\bar{M}^{(k)}(s, t) = k! \sum_{i=1}^{\infty} \theta(s_i - s_0) (s_i - s)^{-k-1} \sum_{n=0}^{J_i} C(J_i, n) t^n (s_i - a_i)^{J_i - n} \quad (15)$$

converges absolutely and uniformly for $|t| \leq \tau$, $k \geq K(\tau)$, $|s| < \infty$ and $s \neq s_i$, $i = 1, 2, \dots$.

By differentiating Eq. (9) k times with respect to s , we obtain the series representation

$$\bar{M}^{(k)}(s, t) = \sum_{i=1}^{\infty} \theta(s_i - s_0) \sum_{n=0}^{J_i} C(J_i, n) t^n \left(\frac{\partial}{\partial s} \right)^k \frac{(s - a_i)^{J_i - n}}{s_i - s} \quad (16)$$

which for arbitrary k converges¹⁷ in the same unspecified domain of convergence as Eqs. (6) and (9).

Now let us focus on an arbitrary point s_0, t_0 in the unspecified domain of convergence of Eq. (16). If we simply choose τ sufficiently large so that $\tau > |t_0|$ then s_0, t_0

will also be a point in the domain of convergence of a series of the form (15) with $k \geq K(\tau)$. Thus, we can find a common domain of convergence of two series (15) and (16) which represent the same function $\bar{M}^{(k)}(s, t)$ provided that we have $k \geq K(\tau)$. If in this domain we subtract Eq. (16) from Eq. (15), we obtain a nontrivial representation of zero,¹⁸

$$0 = \sum_{i=1}^{\infty} \theta(s_i - s_0) \sum_{n=0}^{J_i} C(J_i, n) t^n H(J_i - n, i, k, s) \quad (17)$$

where

$$H(\ell, i, k, s) = \left[\frac{\partial}{\partial s} \right]^k \frac{(s_i - a_i)^\ell - (s - a_i)^\ell}{s_i - s} \quad (18)$$

$$= (s_i - a_i)^{\ell-k-1} \sum_{j=0}^{\ell-k-1} \frac{(j+k)!}{j!} \left[\frac{s - a_i}{s_i - a_i} \right]^j \quad (19)$$

for $\ell \geq k + 1$. For $\ell \leq k$, $H = 0$.

We next show that the domain of convergence of Eq. (17) actually extends over the full domain of convergence of the series in Eq. (15). This is accomplished by comparing moduli of terms in Eqs. (15) and (17).

For sufficiently large s_i , i.e., $s_i > R$, we have

$$\left| \frac{s-a_i}{s_i-a_i} \right| < \epsilon < \frac{1}{k+1} . \quad (20)$$

This condition also holds for sufficiently large i since we can choose $i > \bar{i}$ where $\bar{i}$ is the largest index of any s_i with $s_i \leq R$. From Eq. (19) we see that

$$|H(\ell, i, k, s)| < (s_i - a_i)^{\ell-k-1} \sum_{j=0}^{\ell-k-1} \frac{(j+k)!}{j!} \epsilon^j \quad (21)$$

for $i > \bar{i}$.

If we call the j -th term in this last sum C_j , then we observe that $C_{j+1} \leq (k+1)\epsilon C_j$ and consequently

$$|H(\ell, i, k, s)| < (s_i - a_i)^{\ell-k-1} \frac{k!}{1-(k+1)\epsilon} . \quad (22)$$

Furthermore, it is easy to show that

$$|H(\ell, i, k, s)| < \frac{(s_i - a_i)^\ell}{|s_i - s|^{k+1}} (1+\epsilon)^{k+1} \frac{k!}{1-(k+1)\epsilon} \quad (23)$$

or

$$|H(\ell, i, k, s)| < \text{const. } k! \frac{(s_i - a_i)^\ell}{|s_i - s|^{k+1}} \quad (24)$$

for sufficiently large i . The important feature of this

last inequality is that the constant does not depend on i , ℓ , or s . The inequality (24) involves the s dependent factors which appear in Eqs. (15) and (17). We now reidentify ℓ with $J_i - n$ and use our last result (24) to establish that the modulus of the i -th term in Eq. (17) with $|t| \leq \tau$ is less than some constant times the modulus of the i -th term in Eq. (15) with $t = \tau$ for all $i > \bar{i}$. Thus, the series in Eq. (17) is found to converge absolutely and uniformly in the same domain as Eq. (15).

We note that within our "enlarged" domain of convergence of Eq. (17), s and t can simultaneously take on positive real values and the only restriction on s is that $s \neq s_i$, $i = 1, 2, \dots$. For real $s > 2m_1^2 + 2m_2^2$, we have $s - a_i > 0$ and we observe from Eq. (19) that $H(\ell, i, k, s) \geq 0$ for all ℓ and i . It should be recalled that the constants $C(J_i, n)$ are also positive. The net result is that for real $t > 0$ and real $s > 2m_1^2 + 2m_2^2$, each nonvanishing contribution to Eq. (17) is positive. At the same time the series in Eq. (17) has been shown to be convergent and its sum must be zero. Thus, we have arrived at a contradiction.

last inequality is that the constant does not depend on i , ι , or s . The inequality (24) involves the s dependent factors which appear in Eqs. (15) and (17). We now reidentify ι with $J_i - n$ and use our last result (24) to establish that the modulus of the i -th term in Eq. (17) with $|t| \leq \tau$ is less than some constant times the modulus of the i -th term in Eq. (15) with $t = \tau$ for all $i > \bar{i}$. Thus, the series in Eq. (17) is found to converge absolutely and uniformly in the same domain as Eq. (15).

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IV. DISCUSSION

Since most of the proof involved comparison with a series whose convergence relies on the polynomial boundedness condition (v), we have shown that any model which satisfies the other conditions cannot also be polynomially bounded in the infinitely multiply connected domain described in condition (v).

A physically acceptable way of avoiding the contradiction found in Sec. III would be to replace the coupling constants with form factors. It would then be interesting to see what properties the form factors must have in order to generate acceptable behavior as $|s| \rightarrow \infty$ for fixed t .

It should be noted that the contradiction might disappear in a theory with ghosts in which case the $C(J_i, n)$ would not all be positive. Since there can be no problem with polynomial boundedness in s if we restrict ourselves to finite spins, it is worthwhile to point out what happens to Eq. (17) when only finite spins are present. In this case, we can easily find a finite value of k , e.g. any value greater than the maximum spin, such that Eq. (17) is satisfied simply because all the $H(J_i - n, i, k, s) = 0$ for $k \geq J_i$ (See Eq. (18)). This last observation also shows that the lack of polynomial boundedness is not due to the finite spin particles and could not be remedied by introducing ghosts with only finite spin.

Since the conditions which were imposed in Sec. II are well accepted features of dual multiparticle theories, we see that Feynman diagram models of the Van Hove type cannot exhibit duality.

ACKNOWLEDGMENT

I would like to thank Professor S. Gasiorowicz,
Professor D. Geffen, Professor H. Suura and Dr. R. Goble
for some very helpful discussions.

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