MASTER

THREE MODELS OF VIBRATING LONG BONES

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MODEL IINTRODUCTION

Previous papers have reported the ulnar resonant frequencies of normal subjects and of various patient populations [5-7,9]. As a conceptual model for these studies, the ulna was treated as a cylindrical tube attached to rigid supports at each end by hinges which simulate the joints. This report evaluates the ability of the model to predict the ulnar resonant frequency of patients for whom the ulnar length was measured and for whom the ulnar mineral content and width was determined at a specific location by the photon absorptiometric method [11].

THEORY

The ulna is considered to be a uniform cylindrical tube of length L , outside diameter $2a_p$, and inside diameter $2a_e$ (Fig. 1).

The tube is attached at each end to rigid supports by hinges which allow deflection in the x-y plane only (Fig. 2). The material of which the tube is composed is treated as a homogeneous, isotropic substance of density ρ with a Young's modulus of elasticity of Y .

It can be shown [8] that the transverse deflection y from the equilibrium position at a location x along the axis of the vibrating tube is given by

$$(1) \quad y = \cos(\omega t + \phi) [A \cosh(\omega x/v) + B \sinh(\omega x/v) + C \cos(\omega x/v) + D \sin(\omega x/v)] .$$

In this equation, ω is the angular frequency of vibration, t is time, ϕ is an arbitrary epoch, and v is the phase velocity of the transverse wave component with angular frequency ω along the length of the tube. A , B , C , and D are arbitrary constants determined from boundary conditions.

The phase velocity is given by

$$(2) \quad v = \sqrt{\omega c K} ,$$

where c is the speed of longitudinal wave propagation through the

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material of the tube and K is the radius of gyration of the cross section of the tube about the neutral axis. The speed of longitudinal wave propagation is given by

$$(3) \quad c = \sqrt{Y/\rho} ,$$

and the radius of gyration is given by

$$(4) \quad K = 0.5 \sqrt{a_p^2 + a_e^2} .$$

The angular frequency of vibration (ω) is related to the cyclic frequency of vibration (f) by

$$(5) \quad \omega = 2\pi f .$$

At each end of the hinged bar, the displacement and internal moment must vanish [8]. Therefore,

$$(6) \quad y = 0 \quad \text{and} \quad (7) \quad \frac{\partial^2 y}{\partial x^2} = 0 \quad \text{at } x = 0 \text{ and } x = L .$$

Substitution of Equation 1 into Equations 6 and 7 at $x = 0$ and simplifying shows that the constants A and C vanish. Therefore,

$$(8) \quad y = \cos(\omega t + \phi) [B \sinh(\omega x/v) + D \sin(\omega x/v)] .$$

Substitution of Equation 8 into Equations 6 and 7 at $x = L$, eliminating $\sinh(\omega L/v)$ and simplifying yields

$$(9) \quad \sin(\omega L/v) = 0 .$$

The fundamental nontrivial solution of Equation 9 is

$$(10) \quad \omega_0 L = \pi v ,$$

where ω_0 is the allowed angular frequency of vibration. If f_0 is defined as the allowed cyclic frequency of vibration,

$$(11) \quad f_0 = \frac{\pi c}{4L^2} \sqrt{a_p^2 + a_e^2}$$

is obtained by combining Equations 2, 4, 5, and 10 and then simplifying. Equation 11 forms the basis for predicting ulnar resonant frequency given L , M_B , and W . M_B and W are the mineral content (expressed in g/cm) and width (expressed in cm) of the ulna at a scan site proximal to the ulnar head by $0.3L$. It is assumed that this location provides a representative cross section of the ulna (Fig. 3).

From experimental data relating ash content to density of compact bone [10], one finds that

$$(12) \quad b = 1.1806\rho - 1.1945 ,$$

where b is the ash or mineral content (expressed in g/cm^3) of fresh cortical bone. If Equation 12 is multiplied by the cross sectional area of the bone, one finds that, since

$$(13) \quad M_B = \pi b(a_p^2 - a_e^2) ,$$

$$(14) \quad M_B = \pi(a_p^2 - a_e^2)(1.1806\rho - 1.1945) .$$

Since the width of the bone at the scan site (W) is given by $2a_p$, one may solve Equation 14 for a_e^2 , combine the result with Equation 11, substitute the expression for W , and simplify:

$$(15) \quad f_o = \frac{\pi c}{4L^2} \sqrt{\frac{W^2}{2} - \frac{M_B}{\pi(1.1806\rho - 1.1945)}} .$$

Experimental measurements of c , ρ , and Y in excised strips of fresh cortical bone [1] allow one to derive (in cgs units)

$$(16) \quad c = 2.1427 \cdot 10^5 + 6.5333 \cdot 10^{-7} Y , \quad \text{and} \quad (17) \quad \rho = 1.2586 + 3.6429 \cdot 10^{-12} Y .$$

If these equations are substituted into Equation 15 and the result simplified, one obtains

$$(18) \quad f_o = \frac{\pi}{4L^2} (a_1 + b_1 Y) \sqrt{\frac{W^2}{2} - \frac{M_B}{\pi(a_2 + b_2 Y)}} ,$$

where $a_1 = 2.1427 \cdot 10^5$, $a_2 = 0.2914$, $b_1 = 6.5333 \cdot 10^{-7}$, and $b_2 = 4.3008 \cdot 10^{-12}$. Equation 18 can be used in either of 2 ways. First, it can be used to determine the value of f_o predicted by the model given L , M_B , and W . In this case, some accepted mean value of Y must be used. Then, the predicted value of f_o can be compared to the measured value of this parameter. Second, Equation 18 can be used to determine the value of Y predicted by the model given the values of f_o , L , M_B , and W .

METHOD AND RESULTS

A group of 118 schoolchildren, ranging in age from 6 to 11 years, was selected for this study. First, assuming a constant Young's

modulus of $1.6865 \cdot 10^{11}$ [1], and given L, W, and M_B for each child, Equation 18 was used to predict f_0 . The predicted f_0 values are plotted against the measured values in Fig. 4. Second, Equation 18 was rearranged to obtain a third order polynomial expression for Y:

$$(19) \quad A_1 Y^3 + A_2 Y^2 + A_3 Y + B = 0,$$

$$\text{where } A_1 = \pi^2 W^2 b_1^2 b_2, \quad A_2 = \pi^2 W^2 b_1^2 a_2 + 2\pi^2 W^2 b_1 b_2 a_1 - 2\pi M_B b_1^2,$$

$$A_3 = 2\pi^2 W^2 b_1 a_1 a_2 + \pi^2 W^2 b_2 a_1^2 - 4\pi M_B b_1 a_1 - 32f_0^2 b_2 L^4, \text{ and}$$

$$B = \pi^2 W^2 a_1^2 a_2 - 2\pi M_B a_1^2 - 32f_0^2 a_2 L^4.$$

This cubic was solved by a standard formula which yields only the real root(s). The values of Y which are obtained by solution of Equation 19 are plotted as a function of age for the schoolchildren in Fig. 5. The RMS values (root mean square of the percentage discrepancies of the predicted values in terms of the actual values) of resonant frequency and Young's modulus are rather large. This suggests that either the uncertainty in the experimentally measured quantities caused poor behavior of the model, or the model itself is not applicable.

The calculated value of Young's modulus based on this model was determined for each member of the 6 patient populations described in Table I. Figs. 6-11 show the predicted Y values as a function of age for these groups.

ERROR ANALYSIS

For an analytic function y, where $y = y(x_1, x_2, x_3, \dots)$ with uncertainties of $\Delta x_1, \Delta x_2, \Delta x_3$, etc. associated with x_1, x_2, x_3 , etc., respectively, the uncertainty Δy in y is given by

$$\Delta y = y \sqrt{\left(\frac{\partial y}{\partial x_1} \cdot \frac{\Delta x_1}{y}\right)^2 + \left(\frac{\partial y}{\partial x_2} \cdot \frac{\Delta x_2}{y}\right)^2 + \left(\frac{\partial y}{\partial x_3} \cdot \frac{\Delta x_3}{y}\right)^2 + \dots}$$

In order to estimate the uncertainty of f_0 and Y associated with the model, we calculated Δf_0 and ΔY using data for a typical subject:

$$M_B = 0.80 \pm 0.05 \text{ g/cm (uncertainty 6.3\%)},$$

$$W = 1.00 \pm 0.10 \text{ cm (uncertainty 10.0\%)},$$

$L = 25 \pm 1$ cm (uncertainty 4.0%) , and $Y = 1.6865 \cdot 10^{11} \pm 1.6865 \cdot 10^{10}$ dynes/cm² (uncertainty 10.0%) . The predicted value of the resonant frequency is then $f_o = 204 \pm 47$ Hz (uncertainty 23%) . Similarly, the uncertainty in Y was calculated using the same assumed values and uncertainties for M_p , W , and L . If the measured f_o is assumed to be 204 ± 10 Hz (5%), then the calculated value of Young's modulus is found to be $1.6865 \cdot 10^{11} \pm 1.4323 \cdot 10^{10}$ dynes/cm² (8.5%) . These calculations clearly demonstrate that a small uncertainty in Young's modulus introduces a larger uncertainty in the resonant frequency predicted by the model.

MODEL II

INTRODUCTION

This model simulates the flexural vibration of a beam which is hinged at each end and is supported along its length by an elastic foundation. This model is similar to Model I except for the addition of the elastic foundation which simulates the elastic properties of the soft tissue (mostly muscle) which surrounds the ulna. Conceptually, this model is a better approximation of the vibrating ulna than is Model I, but it is not satisfactory for several inherent reasons.

THEORY

Consider the beam shown in Fig. 12. For the first and second mode of vibration, the deflection curve may be approximated by

$$(20) \quad y = \alpha_n \sin(n\pi x/L) , \text{ where } (21) \quad \alpha_n = \alpha_o \sin(\omega_n t) .$$

The kinetic (E_k) and potential (E_p) energy of the vibrating bar may be written as

$$(22) \quad E_k = \frac{M}{2L} \int_0^L \left(\frac{\partial y}{\partial t} \right)^2 dx , \text{ and } (23) \quad E_p = \frac{YI}{2} \int_0^L \left(\frac{\partial^2 y}{\partial x^2} \right)^2 dx + \frac{k}{2L} \int_0^L y^2 dx ,$$

respectively. In these equations, M is the mass of the bar, I is the moment of inertia of the cross section of the bar, and k is the spring constant of the foundation (soft tissue). By substitution of the proper derivatives into Equations 22 and 23, one can maximize E_k and E_p to obtain

$$(24) \quad E_{k_{\max}} = 0.25 M \omega_n^2 \alpha_o^2 , \text{ and } (25) \quad E_{p_{\max}} = \frac{\alpha_o^2}{4} (k + Y I n^4 \pi^4 / L^3) .$$

Equating $E_{k_{\max}}$ and $E_{p_{\max}}$ yields the nondimensional equation

$$(26) \quad \omega_n^2 = YI \frac{(n\pi)^4}{ML^3} + \frac{k}{M}.$$

In Equation 26, n has allowed values of 0, 1, or 2. It can be shown that

$$(27) \quad M = M_B L \rho / b, \text{ and} \quad (28) \quad I = \frac{M_B}{2b} \left(\frac{W^2}{2} - \frac{M_B}{\pi b} \right).$$

In order to obtain an estimate of k , the foundation spring constant, we assume that the bone is surrounded by a muscle-like material of thickness 2.5 cm and Young's modulus of 52,000 dynes/cm² [3]. With this assumption, one finds that

$$(29) \quad k = 4.15 \cdot 10^4 LW.$$

By substituting Equations 27-29 into Equation 26 for $n = 1$, one can calculate the allowed frequency of vibration.

RESULTS

This model was tested with the same data as was used for Model I. In general, the predicted resonant frequencies were larger than for the first model. This is predominantly a result of the term k/M in Equation 26 (representing the foundation or soft tissue elasticity).

There are 2 additional difficulties with this model. First, the approximation that is made for the flexural deflection is not easily justified. Second, using the exact expression of flexural deflection for solving the equations of motion for the model is so difficult as to be impractical.

MODEL III

INTRODUCTION

The previous models were inadequate in some respects. Thus, we evaluated a third model which assumes that the vibrating bar is attached to rigid supports by springs rather than hinges at each end. These springs simulate the elastic properties of the articular cartilage found at each end of the ulna.

THEORY

The only difference between this model and Model I is the change in boundary conditions. The force constants of the 2 springs are K_e and K_w for the "elbow" and "wrist" respectively. Fig. 13 illustrates this model.

At each end of the bar, the internal moment vanishes. Thus,

$$(30) \quad \frac{\partial^2 y}{\partial x^2} = 0 \quad \text{at } x = 0 \text{ and } x = L.$$

Substitution of Equation 1 into Equation 30 at $x = 0$ and simplifying shows that

$$(31) \quad A = C.$$

Substitution of Equation 31 into Equation 1 and substitution of the result into Equation 30 at $x = L$ and simplifying yields

$$(32) \quad D = A \frac{\cosh(\omega L/v) - \cos(\omega L/v)}{\sin(\omega L/v)} + B \frac{\sinh(\omega L/v)}{\sin(\omega L/v)}.$$

Therefore,

$$(33) \quad y = \cos(\omega t + \phi) \left[A \left\{ \cosh(\omega x/v) + \cos(\omega x/v) + \frac{\sin(\omega x/v)}{\sin(\omega L/v)} [\cosh(\omega L/v) - \cos(\omega L/v)] \right\} \right. \\ \left. + B \left\{ \sinh(\omega x/v) + \frac{\sin(\omega x/v)}{\sin(\omega L/v)} \sinh(\omega L/v) \right\} \right].$$

Also, it can be shown that the elastic force [8] is

$$(34) \quad K_e y = (a_p^4 - a_e^4) \frac{\pi Y}{4} \frac{\partial^3 y}{\partial x^3} \quad \text{and} \quad (35) \quad K_w y = -(a_p^4 - a_e^4) \frac{\pi Y}{4} \frac{\partial^3 y}{\partial x^3}$$

at $x = 0$ and $x = L$, respectively. Substitution of Equation 33 into Equations 34 and 35, combining and simplifying the result yields

$$(36) \quad \frac{8K_w}{\pi Y \alpha_1 X^3} \left(\sinh(XL) - \frac{\alpha_2 \cosh(XL)}{\alpha_3 - \alpha_5} \right) = \cot(XL) \sinh(XL) - \cosh(XL) + \frac{\alpha_2 \alpha_4}{\alpha_3 - \alpha_5},$$

where

$$(37) \quad X = \omega/v,$$

$$(38) \quad \alpha_1 = a_p^4 - a_e^4,$$

$$(39) \quad \alpha_2 = \sin(XL) - \sinh(XL),$$

$$(40) \quad \alpha_3 = \cos(XL) - \cosh(XL),$$

$$(41) \quad \alpha_4 = \alpha_3 \cot(XL) + \sin(XL) + \sinh(XL), \text{ and}$$

$$(42) \quad \alpha_5 = 8K_e \sin(XL) / (\pi Y \alpha_1 X^3).$$

As for the case of Model I, the variables in Equation 36 may be expressed in terms of W , L , M_B , Y , f_o , etc. Young's modulus of

cartilage was derived from the literature [2] ($3.385 \cdot 10^8$ dynes/cm²). Based on the assumption that the articular cartilage is 2 mm thick at both wrist and elbow, and that the bearing surface is 2 cm² and 1 cm² at the elbow and wrist, respectively, K_e and K_w were estimated to have values of

$$(43) \quad K_e = 6.78 \cdot 10^9 \text{ dynes/cm}, \text{ and} \quad (44) \quad K_w = 1.69 \cdot 10^9 \text{ dynes/cm}.$$

In order to calculate the resonant frequency of the ulna based on this model, one must solve Equation 36 for X or ω/v . Unfortunately, there is no explicit solution of the function

$$(45) \quad X = X(Y, L, W, M_B, K_e, K_w).$$

Therefore, one must solve this function numerically with a digital computer. This function was initially solved by Newton's method [4], but the convergence was too slow. We then used Δ^2 -Aitken's accelerating method. This improved convergence markedly.

A typical plot of Equation 36 is shown in Fig. 14. The resonant frequency predicted by the model can be found from the value of X where the left hand side (LHS) and right hand side (RHS) of Equation 36 intersect. The fundamental resonance is obtained from the lowest value of X at intersection if several intersections occur. In Fig. 14, the first intersection occurs at $X \approx 0.13$.

RESULTS

Generally, this model is similar to the hinged-hinged model (see Fig. 15). One expects Models I and III to yield increasingly similar results as the values of K_e and K_w increase. Experimentally, this was shown to be the case. The behavior of the model is quite satisfactory for realistic values of K_e and K_w . At first, this model was compared to Model I on the basis of the data collected from the schoolchildren. Initially, the RMS of frequency was 10% larger for Model III than for Model I. However, by systematically changing W , M_B , K_e , and K_w , we were able to force a marked improvement in RMS of predicted frequency. Table II summarizes the behavior of Model III as the parameters mentioned above were changed.

From the above, it is apparent that the predicted resonant frequency based on this model is largely dependent on bone width and is not critically dependent on the other parameters. As mentioned earlier, we assumed that W at the standard 0.3L scan site represented an average width for the ulna. However, this assumption may not be valid.

We found that using $1.1W$ instead of W in the model improved the RMS of frequency (Fig. 16).

DISCUSSION

Model I and Model III were also compared for several patient populations. Our findings are summarized in Table III. From this table, one can see that the spring-spring model (Model III) more accurately predicts resonant frequency than does the hinge-hinge model (Model I), but this fit is still far from ideal. This by no means implies that the model is invalid, but may be a result of the rather large uncertainty in some of the measured variables. For example, as we have seen previously, the uncertainty in the average width measurement of the ulna introduces a large uncertainty in the predicted frequency of ulnar resonance (f_0 is critically dependent on W).

Based on the above analysis, a better model can be proposed that is independent of W . This model would be essentially a spring-spring model like Model III, but a truncated conical bar or tube would be substituted for the cylindrical bar. This proposed model would more closely approximate the anatomical shape of the ulna, but has the disadvantage that one must know the width of the bone at 2 different scan sites in order to completely define the dimensions of the bar. Theoretically, we can then expect a better resonant frequency approximation to the measured frequency of ulnar resonance.

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TABLE I
Clinical Groups Evaluated

<u>SEX</u>	<u>CLINICAL DESCRIPTION OF GROUP</u>	<u># IN GROUP</u>
M	Normal	91
F	Normal	97
M	Rheumatoid Arthritis Without Steroids	17
F	Rheumatoid Arthritis Without Steroids	39
M	Rheumatoid Arthritis With Steroids	11
F	Rheumatoid Arthritis With Steroids	34

TABLE II
Effect of Parameter Variation on Frequency RMS for Schoolchildren

<u>PARAMETER(S)</u>	<u>CHANGE</u>	<u>CHANGE IN RMS</u>
K_e, K_w	+13%	-1%
K_e, K_w	-20%	+3%
M_B	+15%	+9%
W	+10%	-16%
M_B, W	+15, 10%	+4%
M_B, W	-15, +10%	+1%
W, Y*	+10%	-13%

*Y corrected for age based on Fig. 5.

TABLE III
Reliability of Resonant Frequency Prediction

SEX	GROUP	MODEL I				MODEL III*			
		W		1.1W		W		1.1W	
		#	RMS	#	RMS	#	RMS	#	RMS
M+F	Schoolchildren	118	19.249	118	23.300	111	21.368	118	17.700
F	No Rheumatoid Arthritis No Steroids No Osteoporosis	41	28.410	41	35.700	20	19.820	38	29.730
F	Rheumatoid Arthritis No Steroids	39	33.883	39	36.700	18	26.092	31	32.360
F	Rheumatoid Arthritis With Steroids	34	47.299	34	58.300	21	42.090	28	46.900
M	No Rheumatoid Arthritis No Steroids No Osteoporosis	29	25.949	28	26.000	20	29.038	28	20.700
M	Rheumatoid Arthritis No Steroids	17	28.840	17	35.800	13	222.640	17	45.200
M	Rheumatoid Arthritis With Steroids	11	30.782	11	38.800	7	32.628	11	31.100
F	No Rheumatoid Arthritis With Steroids	9	28.440	9	38.700	6	24.737	9	28.700
M	No Rheumatoid Arthritis With Steroids	4	25.153	5	31.600	3	29.683	4	24.500
F	No Rheumatoid Arthritis No Steroids With Osteoporosis	6	33.459	6	49.500	6	25.200	6	34.300

* When Model III was used for the Schoolchildren with 1.1W and with Y corrected for age, the RMS of the predicted resonant frequency was 15.500%.

Fig 1: Conceptual model of the ulna.

Fig 2: Model I boundary conditions.

Fig 3: Absorptiometric scan site location.

Fig 4: Predicted f_0 vs measured f_0 based on Model I for 118 schoolchildren. The RMS value (root mean square of the percentage discrepancy of predicted f_0 in terms of the measured f_0) is 19.25%.

Fig 5: Calculated Y vs age based on Model I for 118 schoolchildren.

Fig 6: Y vs age for 91 normal male subjects. The range bars enclose ± 1 standard error of the mean.

Fig 7: Y vs age for 97 normal female subjects.

Fig 8: Y vs age for 17 males with rheumatoid arthritis and no history of corticosteroid therapy.

Fig 9: Y vs age for 39 rheumatoid females without steroid therapy.

Fig 10: Y vs age for 11 rheumatoid males with a history of steroid therapy.

Fig 11: Y vs age for 34 rheumatoid females with a history of steroid therapy.

Fig 12: Model II boundary conditions.

Fig 13: Model III boundary conditions.

Fig 14: Typical plot of Equation 36. The fundamental resonance occurs at $X \approx 0.13$.

Fig 15: Predicted f_0 vs measured f_0 based on Model III for 118 schoolchildren. 1.1W was substituted for the measured value of W in Equation 36.

Fig 16: RMS vs substituted value for W in Equation 36 for 118 schoolchildren.