Numerical study of the discrete minimal
surface equation in a nonconvex domain

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ABSTRACT

A discrete approximation to the minimal surface equation is solved numerically on an L-shaped domain with Dirichlet boundary conditions. The nature of the solution discontinuity at the vertex of the reentrant corner is depicted graphically.

1. In this study, a discrete approximation to the minimal surface equation

$$(1) \quad \frac{\partial}{\partial x} \left(\frac{u_x}{W} \right) + \frac{\partial}{\partial y} \left(\frac{u_y}{W} \right) = 0, \quad W = (1 + u_x^2 + u_y^2)^{\frac{1}{2}},$$

is solved numerically on an L-shaped domain D with Dirichlet boundary conditions (see Fig. 1a). The domain D is the unit square $(0,1) \times (0,1)$ with the smaller square $[\frac{1}{2}, 1) \times [\frac{1}{2}, 1)$ deleted from it, and the boundary conditions are that $u = 0$ on the outside border of D and that u increase linearly to the value c at $(\frac{1}{2}, \frac{1}{2})$ along the reentrant legs. That is,

$$(2) \quad u = 0 \quad \text{on} \quad \begin{cases} x=0 \\ y=0 \\ x=1, 0 < y < \frac{1}{2} \\ y=1, 0 < x < \frac{1}{2} \end{cases}$$

$$u = 2c(1-x) \quad \text{on} \quad y=\frac{1}{2}, \quad \frac{1}{2} \leq x \leq 1,$$

$$\text{and} \quad u = 2c(1-y) \quad \text{on} \quad x=\frac{1}{2}, \quad \frac{1}{2} \leq y \leq 1.$$

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The analogous Dirichlet problem on the nonconvex quadrilateral considered by Radó [5, 6] is discussed by Nitsche [3, 4], who proved that a solution $u \in C^2(D) \cap C^0(\bar{D})$ does not exist for any value of $c > c_0$. A solution does exist, however, if it is permitted to have a discontinuity at the vertex of

the reentrant corner. It is the purpose of the present study to indicate the behavior of such a solution surface in the neighborhood of a reentrant corner. An L-shaped domain, rather than the quadrilateral one, is chosen here because of its practical convenience: it can be subdivided by means of a uniform square mesh with no special treatment required at the boundary.

2. The discrete approximation used is that given in [1, 2]. Let U_{ij} denote the approximating value to $u(x,y)$ at the node point $x_i = ih, y_j = jh$ on the uniform square mesh of width $h = \frac{1}{N}$ (N an integer); then in place of (1) one has the system of nonlinear algebraic equations

$$\begin{aligned} F_{ij} = & \frac{1}{W_{ij}} (2U_{ij} - U_{i-1,j} - U_{i,j-1}) + \frac{1}{W_{i+1,j}} (2U_{ij} - U_{i+1,j} - U_{i,j-1}) \\ (3) \quad & + \frac{1}{W_{i,j+1}} (2U_{ij} - U_{i-1,j} - U_{i,j+1}) + \frac{1}{W_{i+1,j+1}} (2U_{ij} - U_{i+1,j} - U_{i,j+1}) = 0. \end{aligned}$$

In (3)

$$(4) \quad W_{ij} = (1 + [u_x^2 + u_y^2]_{ij})^{\frac{1}{2}}$$

denotes W for the mesh cell with center $(i-\frac{1}{2}, j-\frac{1}{2})$, evaluated by use of

$$\begin{aligned} [u_x^2 + u_y^2]_{ij} = & \frac{1}{2h^2} \left[(U_{ij} - U_{i-1,j})^2 + (U_{ij} - U_{i,j-1})^2 + (U_{i,j-1} - U_{i-1,j-1})^2 \right. \\ & \left. + (U_{i-1,j} - U_{i-1,j-1})^2 \right]. \end{aligned}$$

The system of equations (3) can be derived directly from the variational integral $A = \iint_D W dx dy$ by using (4) to obtain the corresponding discrete sum and then by setting equal to zero the partial derivatives with respect to the unknown nodal values of U . Equation (3) is to be solved for the interior nodal values of U subject to (2) at the boundary nodes.

3. The numerical solution of (3) was carried out using the technique of block nonlinear successive overrelaxation described in [2]. Approximate solutions, accurate to within $10^{-5}c$, were obtained for $c = 0.1(0.1)1.0$ and for mesh spacings $h = \frac{1}{10}, \frac{1}{20}, \frac{1}{40},$ and $\frac{1}{80}$. The results are displayed graphically in Figs. 2-9. (The automatic plotter that prepared the figures interpolated linearly between the data points.)

In Fig. 2 are plotted the values of $U_{1,j}$ for $c = 1.0$ along the line segment $l_1(x-y = 0 - \text{see Fig. 1b})$ for the four mesh spacings used. These curves indicate the behavior of the numerical solution as h is reduced. Figs. 3 and 4 depict the corresponding values of $U_{1,j}$ for $c = 1.0$ along the line segments l_2 and l_3 ($y = \frac{1}{2}$ and $x + y = 1$, resp. - see Fig. 1b). Note that, as one would expect for the original problem, the solutions of the discrete problems have the greatest jump in approaching $(\frac{1}{2}, \frac{1}{2})$ along the direction of l_1 .

Figs. 5-7 depict the analogous graphs for $c = 0.1$.

In Fig. 8 are depicted the values of $U_{1,j}$ for $h = \frac{1}{40}$ along the line l_1 for $c = 0.1(0.1)1.0$. Note that, as in the previous graphs, the vertical scale is normalized to c . These curves illustrate the behavior of the jump discontinuity at $(\frac{1}{2}, \frac{1}{2})$ as a function of c .

Finally, in Fig. 9 is shown the extrapolated estimate as a function of c for the limiting value $u(\frac{1}{2}^-, \frac{1}{2}^-)$ along the line l_1 . The limiting values $u(\frac{1}{2}^-, \frac{1}{2}^-)$ in this figure were obtained for each c by passing a parabola through the computed approximation to $u(\frac{1}{2}-h, \frac{1}{2}-h)$ for the cases with $h = \frac{1}{20}, \frac{1}{40},$ and $\frac{1}{80}$ and extrapolating to $h = 0$. In addition to the values for $c = 0.1(0.1)1.0$, those for $c = 0.05$ and $c = 1.5$ were calculated as well.

4. A rigorous estimate for the accuracy with which Fig. 9 represents the solution of the original non-discrete problem (1, 2) would be very difficult to obtain, because of the complications introduced by the discontinuity at $(\frac{1}{2}, \frac{1}{2})$. However simple heuristic checks give evidence that the extrapolation of the numerical results to $h = 0$ can legitimately be carried out. One such check that was used was to include the value of $u(\frac{1}{2}-h, \frac{1}{2}-h)$ for $h = \frac{1}{10}$ in the extrapolation to $h = 0$, using a cubic polynomial to estimate the limit. These limits differed from those plotted in Fig. 9 by less than one percent.

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Figure 1a

The L-shaped domain

Figure 1b

Line segments l_1 , l_2 , l_3 along
which tabular data are plotted

Figure 2. U as a function of distance along k_1 , for $c = 1.0$. Curves from top to bottom are for $h = \frac{1}{10}, \frac{1}{20}, \frac{1}{40}$, and $\frac{1}{80}$.

Figure 3. U as a function of distance along l_2 for $c = 1.0$. Curves from top to bottom are for $h = \frac{1}{10}, \frac{1}{20}, \frac{1}{40},$ and $\frac{1}{80}$.

Figure 4. U as a function of distance along z_3 for $c = 1.0$. Curves from top to bottom are for $h = \frac{1}{10}, \frac{1}{20}, \frac{1}{40},$ and $\frac{1}{80}$.

Figure 5. Same as Fig. 2 except that $c = 0.1$

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Figure 6. Same as Fig. 3 except that $c = 0.1$

Figure 7. Same as Fig. 4 except that $c = 0.1$

Figure 8. U as a function of distance along l_1 for $h = \frac{1}{40}$. Curves from top to bottom are for $c = 0.1, 0.2, \dots, 1.0$.

Figure 9. Estimate of $\lim_{h \rightarrow 0} u(1/2-h, 1/2-h)$ as a function of c .