DUAL RESONANCE MODELS - P. H. Frampton

PART THREE: OPERATOR FORMALISM*

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We probe more deeply into the properties of the multiparticle dual amplitude, by fully exploiting the operator formalism. In particular, by realizing the projective group on the Fock space spanned by the harmonic-oscillator-like operators we find the gauge invariances of the theory; the latter are especially powerful if the intercept is exactly one which is the value for which we have already seen simplifications in the four-particle Euler B function. Combining the projective group and the operator formalism enables us to rewrite the N-point function in a manner such that both cyclic symmetry and factorizability are manifest.

For the case of unit intercept, it is shown that the large gauge invariance group enables elimination of all ghost states arising from the relativistic metric. This is essential if we are to preserve the usual quantum mechanical interpretation of the resonances.

Finally we discuss other operator constructs. Specifically we introduce the twisting operator and the twisted propagator. These have no direct equivalent in local field theories. Then we discuss the multiregion vertex, in which the projective group properties enable us to keep control over factorization and cyclic symmetric properties.

3a. Projective Group

The cyclic symmetry and duality properties, including the absence of incompatible poles, of the N -point function are most evident in the Koba-Nielsen representation of the amplitude, where the integrand is invariant under a group of projective transformations. On the other hand, the factorization properties and the density of levels are easily derived by using the harmonic oscillator operator formalism. Our objective now will be to study more deeply the properties of B_N by combining the two approaches such that eventually we can write B_N in a form where both cyclic symmetry and factorization are evident; to do this we first need to construct unitary irreducible representations (UIR) of the projective group $SU(1,1)$, or what is essentially the same thing up to a similarity transformation $SL(2,R)$, realized on the Fock space spanned by the operators $a^{(n)}$, $a^{(n)\dagger}$ satisfying

$$[a_\mu^{(n)}, a_\nu^{(m)\dagger}] = -g_{\mu\nu} \delta_{mn} \quad (3.1)$$

Fortunately the UIR of $SU(1,1)$ have been completely classified.^(1,2) The first thing we learn from the literature is that because $SU(1,1)$ is noncompact, all the UIR's are infinite dimensional; this is a happy event, since we have an infinite number of modes $n = 1, 2, 3, \dots$ on which to map these infinite-dimensional representations.

Pauli matrices are defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (3.2)$$

and putting angular momentum $\underline{J} = \frac{1}{2} \underline{\sigma}$, $J_{\pm} = (J_1 \pm iJ_2)$ we satisfy the SU(2) algebra

$$[J_+, J_-] = 2J_3 \quad (3.3)$$

$$[J_{\pm}, J_3] = \mp J_{\pm} \quad (3.4)$$

We can equally use the Pauli matrices to represent the SU(1,1) Lie algebra by identifying

$$L_1 = -J_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \quad (3.5)$$

$$L_{-1} = J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (3.6)$$

$$L_0 = J_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (3.7)$$

which satisfy

$$[L_1, L_{-1}] = 2L_0 \quad (3.8)$$

$$[L_{\pm 1}, L_0] = \pm L_{\pm} \quad (3.9)$$

as required. The reason for the identification of L_{-1} with J_+ rather than J_- is (i) that it conforms with the most common usage in the literature and (ii) that later we will want to use $L_{-1} = (L_1)^\dagger$ as a raising operator (like $a^{(n)\dagger}$) and L_1 as a lowering operator (like $a^{(n)}$) inside the Fock space.

We wish to deal with projective transformations on the Kobayashi-Nielsen variables z according to

$$z' = \frac{az+b}{cz+d} \quad ad-bc = 1 \quad (3.10)$$

We define $z = \zeta_1 / \zeta_2$ whereupon

$$\zeta_1' = a\zeta_1 + b\zeta_2 \quad (3.11)$$

$$\zeta_2' = c\zeta_1 + d\zeta_2 \quad (3.12)$$

transform as the components of a two-component spinor. A suitable orthonormal basis for a give UIR can be written

$$|J k m\rangle = N(J k m) (\xi_1 \xi_2)^J (\xi_1 / \xi_2)^{k+m} \quad (3.13)$$

where $J(J+1)$ is the eigenvalue of the Casimir operator $L_0(L_0+1) - L_1 L_{-1}$ and k is a second label of the UIR. Finally m (related to the eigenvalue of L_0) labels states within a given UIR, and $N(J k m)$ is a normalization factor.

A detailed analysis of $SU(1,1)$ reveals that there are three types of UIR: (i) $J \neq \pm k$, with eigenvalues of $L_0 (=k+m)$ unbounded and integer spaced (ii) $J = +k$, with eigenvalues of L_0 bounded from above and integer spaced and (iii) $J = -k$, with eigenvalues of L_0 bounded from below and integer spaced. We will eventually identify L_0 with the squared mass operator of the dual model, so only the representations of the third type will be used, and the corresponding representation matrices (for $J=-k$) will be written $D_{mn}^{(J,+)}(X)$.

In general a representation matrix for the transformation

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (3.14)$$

is defined by

$$|J k m\rangle' = N(J k m) (\xi_1' \xi_2')^J (\xi_1' / \xi_2')^{k+m} \quad (3.15)$$

$$= \sum_{n=0}^{\infty} D_{mn}^{(J,k)}(\Lambda) |J k n\rangle \quad (3.16)$$

which becomes, for $J = -k$ and $\xi_1 / \xi_2 = z$

$$N(J - J m) (az+b)^m (cz+d)^{2J-m} = \sum_{n=0}^{\infty} D_{mn}^{(J,+)}(\Lambda) N(J - J n) z^n \quad (3.17)$$

Now consider an infinitesimal transformation generated by L_0 according to

$$z \rightarrow z' = (1 + i\varepsilon L_0) z \quad (3.18)$$

the, using our Pauli matrix representation for L_0 ,

$$\begin{pmatrix} z'_1 \\ z'_2 \end{pmatrix} = z \begin{pmatrix} 1 + O(\varepsilon^2) \\ 0 \end{pmatrix} \quad (3.19)$$

$$\begin{pmatrix} z'_1 / z'_2 \end{pmatrix} = (1 + i\varepsilon) z + O(\varepsilon^2) \quad (3.20)$$

Therefore

$$N(Jkm) z^{J+k+m} (1+i\varepsilon)^{k+m} = N(Jkm) z^{J+k+m} (1+i\varepsilon(k+m) + O(\varepsilon^2)) \quad (3.21)$$

$$= \sum_{n=0}^{\infty} (\delta_{mn} + i\varepsilon D_{mn}^{(Jk)}(L_0)) N(Jkn) z^{J+k+n} + O(\varepsilon^2) \quad (3.22)$$

and it follows that (the normalization factor $N(Jkm)$ cancels for this case)

$$D_{mn}^{(Jk)}(L_0) = \delta_{mn} (m+k) \quad (3.23)$$

and, in particular,

$$D_{mn}^{(J+)}(L_0) = \delta_{mn} (m-J) \quad (3.24)$$

Now consider the generator L_{+1} , to find

$$\begin{aligned} & N(Jkm) (z - i\varepsilon z^2)^J (z + i\varepsilon z^2)^{k+m} \\ &= N(Jkm) z^{J+k+m} (1 + (k+m-J)i\varepsilon z) + O(\varepsilon^2) \end{aligned} \quad (3.25)$$

$$= \sum_{n=0}^{\infty} (\delta_{mn} + i\varepsilon D_{mn}^{(Jk)}(L_1)) z^{J+k+n} N(Jkn) \quad (3.26)$$

whence

$$D_{mn}^{(Jk)}(L_1) = \delta_{n,m+1} (k+m-J) \frac{N(Jkm)}{N(Jkn)} \quad (3.27)$$

The corresponding calculation for L_{-1} gives

$$D_{mn}^{(Jk)}(L_{-1}) = \frac{N(Jk m)}{N(Jk m-1)} \delta_{n, m-1} (k+m-J) \quad (3.28)$$

The normalization constants follow from the requirement that

$$(L_1)^\dagger = L_{-1} \quad (3.29)$$

or (for $J=-k$)

$$D_{m, m+1}^{(J+)}(L_1) = D_{m+1, m}^{(J+)}(L_{-1})^* \quad (3.30)$$

whereupon

$$(m-2J) \frac{N(J-J m)}{N(J-J m+1)} = \left[\frac{N(J-J m+1)}{N(J-J m)} \right]^* (m+1) \quad (3.31)$$

and therefore

$$\left| \frac{N(J-J m)}{N(J-J 0)} \right|^2 = \frac{\Gamma(m-2J)}{\Gamma(-2J) \Gamma(m+1)} \quad (3.32)$$

Taking an arbitrary phase to be one, and defining

$$N(J-J 0) = \sqrt{\Gamma(-2J)} \quad (3.33)$$

one arrives at

$$N(J-J m) = \sqrt{\frac{\Gamma(m-2J)}{m!}} \quad (3.34)$$

Collecting together our results so far we have, for the UIR's bounded from below

$$D_{mn}^{(J+)}(L_0) = S_{mn}(-J+n) \quad (3.35)$$

$$D_{mn}^{(J+)}(L_1) = \sqrt{n(n-1-2J)} \delta_{m, n-1} \quad (3.36)$$

$$D_{mn}^{(J+)}(L_{-1}) = \sqrt{(n+1)(n-2J)} \delta_{m, n+1} \quad (3.37)$$

Now we come to the identification of the realization on the Fock space according to

$$L_i = - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{\mu}^{(m-j)} D_{mn}^{(j+)} (L_i) a_{\mu}^{(n-j)} \quad (3.38)$$

with the operators of the dual resonance model; in particular the identification of L_0 with the combination occurring in the propagator denominator, namely

$$R - p^2 = - \sum_{n=1}^{\infty} n a_{\mu}^{(n)\dagger} a_{\mu}^{(n)} - p^2 \quad (3.39)$$

To make the identification $L_0 = R - p^2$, we take $J \rightarrow 0$ from below, that is $J = -\epsilon$ ($\epsilon > 0$) and $\epsilon \rightarrow 0$. Note, however, that when $\epsilon \rightarrow 0$ our normalization constant blows up

$$N(-\epsilon, +\epsilon, 0) = \sqrt{\Gamma(2\epsilon)} \xrightarrow{\epsilon \rightarrow 0} \infty \quad (3.40)$$

This corresponds to the fact that no scalar $J=k=0$ representation of $SU(1,1)$ exists, strictly speaking. This technical difficulty was first discovered and overcome by Fubini and Veneziano in a well-known paper.⁽³⁾ [Concerning the projective group properties see also References 4,5]. A further difficulty is to identify the zeroth mode operators $a_{\mu}^{(\epsilon)}$ and $a_{\mu}^{(\epsilon)\dagger}$ in the limit $\epsilon \rightarrow 0$ with the momentum p_{μ} and its conjugate the position operator Q_{μ} .

For $J = -\epsilon$ we have

$$L_0 = - \sum_{n=1}^{\infty} (n+\epsilon) a^{(n)\dagger} a^{(n)} - \epsilon a^{(\epsilon)\dagger} a^{(\epsilon)} \quad (3.41)$$

$$L_1 = - \sum_{n=1}^{\infty} \sqrt{(n+1)(n+2\epsilon)} a^{(n)\dagger} a^{(n+1)} - \sqrt{2\epsilon} a^{(\epsilon)\dagger} a^{(1)} \quad (3.42)$$

$$L_{-1} = (L_1)^{\dagger} \quad (3.43)$$

The limits $\epsilon \rightarrow 0+$ in the summations is harmless. We write

$$[a_{\mu}^{(\varepsilon)}, a_{\nu}^{(\varepsilon)\dagger}] = -g_{\mu\nu} \quad (3.44)$$

and identify

$$a_{\mu}^{(\varepsilon)} = \frac{i}{\sqrt{2\varepsilon}} (P_{\mu} - i\varepsilon Q_{\mu}) \quad (3.45)$$

$$a_{\mu}^{(\varepsilon)\dagger} = \frac{-i}{\sqrt{2\varepsilon}} (P_{\mu} + i\varepsilon Q_{\mu}) \quad (3.46)$$

with

$$[Q_{\mu}, P_{\nu}] = -ig_{\mu\nu} \quad (3.47)$$

This then gives the projective gauge operators [References 6-9].

$$L_0 = - \sum_{n=1}^{\infty} n a^{(n)\dagger} a^{(n)} - p^2 \quad (3.48)$$

$$L_1 = - \sum_{n=1}^{\infty} \sqrt{n(n+1)} a^{(n)\dagger} a^{(n+1)} + i\sqrt{2} p \cdot a^{(1)} \quad (3.49)$$

$$L_{-1} = (L_1)^{\dagger} \quad (3.50)$$

The fact that these satisfy the $SU(1,1)$ algebra follows from our construction. We can of course easily check explicitly by

$$[L_1, L_{-1}] = \sum_{n, n'=1}^{\infty} \sqrt{n(n+1) n'(n'+1)} [a^{(n)\dagger} a^{(n+1)}, a^{(n'+1)\dagger} a^{(n')}] \quad (3.51)$$

$$+ 2p_{\mu} p_{\nu} [a_{\mu}^{(1)}, a_{\nu}^{(1)\dagger}]$$

$$= \sum_{n, n'=1}^{\infty} n(n+1) n'(n'+1) (\delta_{nn'} a^{(n+1)\dagger} a^{(n+1)} - \delta_{n+1, n'+1} a^{(n)\dagger} a^{(n)}) \quad (3.52)$$

$$- 2p^2 = -2 \sum_{n=1}^{\infty} n a^{(n)\dagger} a^{(n)} - 2p^2 = 2L_0 \quad (3.53)$$

Similarly (we leave it as an exercise) one finds

$$[L_{\pm 1}, L_0] = \pm L_{\pm 1} \quad (3.54)$$

To proceed further recall that the expression for B_N , namely

$$B_N = \int \prod_{i=1}^N dz_i \left[\frac{dz_a dz_b dz_c}{(z_a - z_b)(z_b - z_c)(z_c - z_a)} \right]^{-1} \prod_{i < j} (z_i - z_j)^{-2p_i p_j} \prod_{i=1}^N (z_i - z_{i+1})^{-1+\alpha(0)} \quad (3.55)$$

is projective invariant ($SU(1,1)$ -invariant) for any intercept $\alpha(0)$.

When $\alpha(0)=1$, however, there is a higher symmetry-namely the integrand becomes totally symmetric under all $N!$ permutations of the $\{z_i, p_i\}$ argument pairs. It turns out that for the unit intercept case $\alpha(0)=1$ it is useful to introduce a generalized projective group with generators given by⁽¹⁰⁾

$$L_m = - \sum_{n=1}^{\infty} \sqrt{n(n+m)} a^{(n)\dagger} \cdot a^{(n+m)} + i\sqrt{2m} P \cdot a^{(m)} + \frac{1}{2} \sum_{n=1}^{m-1} \sqrt{n(m-n)} a^{(n)} \cdot a^{(m-n)} \quad (3.56)$$

$$L_{-m} = (L_m)^\dagger \quad (3.57)$$

These satisfy the algebra (of which $SU(1,1)$ is a subalgebra)

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{d}{12} m(m^2-1) \delta_{m+n,0} \quad (3.58)$$

where in the anomaly term⁽¹¹⁾ for $m+n=0$, d is the space-time dimension. We return to a detailed discussion of the anomaly term shortly.

It can be seen from our derivation of the representation matrices $D_{mn}^{(J^+)}(L_n)$ for the $SU(1,1)$ subgroup $n=0, \pm 1$ that the generators $|n| \geq 2$ correspond to transformations of the type

$$z'^n = \frac{az^n + b}{cz^n + d} \quad (3.59)$$

It is important to emphasize, however, that the B_N integrand is not invariant under these transformations, which contain branch cut singularities. The only vestige of the higher symmetry is the total

symmetry under permutations already mentioned.

Let us check the algebra for $m+n \neq 0$, where no anomaly term occurs. Then we have ($m, n > 0$, for example)

$$\begin{aligned}
 [L_m, L_n] = & \sum_{p,q=1}^{\infty} \sqrt{p(p+m)q(q+n)} [a^{(p)\dagger} a^{(p+m)}, a^{(q)\dagger} a^{(q+n)}] \\
 & - i \sqrt{2m} P_\mu [a_\mu^{(m)}, \sum_{q=1}^{\infty} \sqrt{q(q+n)} a^{(q)\dagger} a^{(q+n)}] \\
 & - i \sqrt{2n} P_\mu [\sum_{p=1}^{\infty} \sqrt{p(p+m)} a^{(p)\dagger} a^{(p+m)}, a_\mu^{(n)}] \\
 & - \frac{1}{2} \sum_{p=1}^{m-1} \sum_{q=1}^{\infty} [a^{(m-p)} a^{(p)}, a^{(q)\dagger} a^{(q+n)}] \sqrt{m(m-p)q(q+n)} \\
 & + \frac{1}{2} \sum_{q=1}^{n-1} \sum_{p=1}^{\infty} [a^{(n-q)} a^{(q)}, a^{(p)\dagger} a^{(p+m)}] \sqrt{q(n-q)p(p+m)} \quad (3.60)
 \end{aligned}$$

$$\begin{aligned}
 = & \sum_{p,q=1}^{\infty} \sqrt{p(p+m)q(q+n)} [\delta_{p,q+n} a^{(p+m)} a^{(q)\dagger} - \delta_{p+m,q} a^{(p)\dagger} a^{(q+n)}] \\
 & + i \sqrt{2m} a^{(m+n)} P \sqrt{m(m+n)} \\
 & - i \sqrt{2n} a^{(m+n)} P \sqrt{n(n+m)} \\
 & - \frac{1}{2} \sum_{p=1}^{m-1} \sum_{q=1}^{\infty} \sqrt{p(m-p)q(q+n)} [-\delta_{m-p,q} a^{(p)} a^{(q+n)} - \delta_{p,q} a^{(m-p)} a^{(q+n)}] \quad (3.61)
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2} \sum_{q=1}^{n-1} \sum_{p=1}^{\infty} \sqrt{q(n-q)p(p+m)} [-\delta_{n-q,p} a^{(q)} a^{(p+m)} - \delta_{q,p} a^{(n-q)} a^{(p+m)}] \quad (3.62)
 \end{aligned}$$

$$\begin{aligned}
 = & -(m-n) \sum_{p=1}^{\infty} \sqrt{p(p+m+n)} a^{(p)\dagger} a^{(p+m+n)} + i(m-n) \sqrt{2(m+n)} a^{(m+n)} P \\
 & + \frac{1}{2} (m-n) \sum_{p=1}^{m+n-1} \sqrt{p(m+n-p)} a^{(p)} a^{(m+n-p)} = (m-n) L_{m+n} \quad (3.63)
 \end{aligned}$$

as required.

When $m+n = 0$, an additional anomaly term arises from the commutation of the terms quadratic in a^+ and a ; after commutation the resultant terms have to be normal ordered to obtain L_0 and the normal ordering gives rise to a trace $g^{\mu\mu} = d$ of the metric tensor. Consider the term contained in $[L_m, L_{-m}]$ of the form

$$\begin{aligned} & \frac{1}{4} \sum_{p=1}^{m-1} \sum_{q=1}^{m-1} \sqrt{p(m-p)q(m-q)} [a^{(p)} a^{(m-p)}, a^{(q)+} a^{(m-q)+}] \\ &= \frac{1}{4} \sum_{p=1}^{m-1} \sum_{q=1}^{m-1} p(m-p) [a^{(m-p)+} a^{(m-p)} + a^{(p)+} a^{(p)} + 2g_{\mu\mu}] \end{aligned} \quad (3.64)$$

The c-number anomaly term is then

$$\frac{d}{2} \sum_{p=1}^{m-1} p(m-p) = \frac{d}{12} m(m^2-1) \quad (3.65)$$

as given before.

Since the presence of the anomaly term is so important let us show that its general structure follows from

i) the Jacobi identity

$$[[L_n, L_m], L_p] + [[L_m, L_p], L_n] + [[L_p, L_n], L_m] = 0 \quad (3.66)$$

ii) absence of an anomaly in the projective subalgebra.

We can see this by writing

$$[L_m, L_n] = (m-n) L_{m+n} + C_{m,n} \quad (3.67)$$

whereupon the anomaly $C_{m,n}$ satisfies

$$(n-m) C_{n+m,p} + (m-p) C_{m+p,n} + (p-n) C_{p+n,m} = 0 \quad (3.68)$$

Further we assume that

$$C_{11} = C_{00} = C_{10} = C_{-1,-1} = 0 \quad (3.69)$$

Then by putting $m = -1$, $p = 1-n$ for example we find

$$\frac{C_{n,-n}}{C_{n-1,-n+1}} = \frac{n+1}{n-2} \quad (3.70)$$

which implies that

$$C_{n,-n} = \frac{1}{6} n(n^2-1) C_{2,-2} \quad (3.71)$$

The fact that $C_{2,-2} = d/2$ follows from the explicit computation already given. Of course, these general considerations involving the Jacobi identity do not prove $C_{2,-2} \neq 0$; as we shall see later the nonvanishing of $C_{2,-2}$ and its dependence on d will lead to significant effects in the discussion of ghost elimination.

3b. Gauge Invariance

To study the linear dependences implied by the projective gauge invariance of the theory, it is convenient to introduce a field operator $Q_\mu(z)$ defined by

$$Q_\mu(z) = \frac{1}{\sqrt{2}} Q_\mu + i\sqrt{2} P_\mu \ln z + \sum_{n=1}^{\infty} \left(\frac{a^{(n)} z^n}{\sqrt{n!}} + \frac{a^{(n)+} z^{-n}}{\sqrt{n!}} \right) \quad (3.72)$$

and then to define a vertex operator by the normal-ordered form

$$V(p, z) = : \exp(\sqrt{2} i p \cdot Q(z)) : \quad (3.73)$$

Notice that this is related to the operator vertex defined earlier by

$$V(p, 1) = e^{i p \cdot Q} V(p) \quad (3.74)$$

$$V(p) = \exp\left(\sqrt{2} i p \cdot \sum_{n=1}^{\infty} \frac{a^{(n)+}}{\sqrt{n!}}\right) \exp\left(\sqrt{2} i p \cdot \sum_{n=1}^{\infty} \frac{a^{(n)}}{\sqrt{n!}}\right) \quad (3.75)$$

Further we define a conjugate to $Q_\mu(z)$ by

$$P_\mu(z) = -i z \frac{d}{dz} Q_\mu(z) \quad (3.76)$$

$$= \sqrt{2} P_\mu - i \sum_{n=1}^{\infty} \sqrt{n!} (a^{(n)} z^n - a^{(n)+} z^{-n}) \quad (3.77)$$

Defining

$$Q_{\mu}^{(0)}(z) = \frac{1}{\sqrt{2}} Q_{\mu} + i\sqrt{2} P_{\mu} \ln z \quad (3.78)$$

$$Q_{\mu}^{(+)}(z) = \sum_{n=1}^{\infty} \frac{a_{\mu}^{(n)}}{\sqrt{n}} z^n \quad (3.79)$$

$$Q_{\mu}^{(-)}(z) = \sum_{n=1}^{\infty} \frac{a_{\mu}^{(n)+}}{\sqrt{n}} z^{-n} \quad (3.80)$$

we see that

$$[Q_{\mu}^{(0)}(z), Q_{\nu}^{(0)}(z')] = -g_{\mu\nu} \ln(z/z') \quad (3.81)$$

$$= -g_{\mu\nu} \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \left\{ \frac{1}{\varepsilon} \left(\frac{z}{z'}\right)^{\varepsilon} + \frac{1}{(-\varepsilon)} \left(\frac{z}{z'}\right)^{-\varepsilon} \right\} \quad (3.82)$$

$$[Q_{\mu}^{(+)}(z), Q_{\nu}^{(-)}(z')] = -g_{\mu\nu} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{z}{z'}\right)^n \quad (3.83)$$

and hence

$$[Q_{\mu}(z), Q_{\nu}(z')] = -g_{\mu\nu} \left[\sum_{n=-\infty}^{\infty} \frac{1}{n} \left(\frac{z}{z'}\right)^n + \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \left\{ \frac{1}{\varepsilon} \left(\frac{z}{z'}\right)^{\varepsilon} + \frac{1}{-\varepsilon} \left(\frac{z}{z'}\right)^{-\varepsilon} \right\} \right] \quad (3.84)$$

$$= -2\pi i g_{\mu\nu} \varepsilon(\theta - \theta') \quad (3.85)$$

where we wrote $z = e^{i\theta}$, $z' = e^{i\theta'}$. Hence $Q_{\mu}(z)$ does not commute with itself at different z -values. By differentiating, however, we find that the commutator of $Q_{\mu}(z)$ with $P_{\nu}(z')$ and of $P_{\mu}(z)$ with $P_{\nu}(z')$ are local namely (using $iz \frac{d}{dz} = \frac{d}{d\theta}$)

$$[Q_{\mu}(z), P_{\nu}(z')] = -2\pi i g_{\mu\nu} \delta(\theta - \theta') \quad (3.86)$$

$$[P_{\mu}(z), P_{\nu}(z')] = 2\pi i g_{\mu\nu} \delta'(\theta - \theta') \quad (3.87)$$

In order to find the commutators of L_n with $P_{\mu}(z)$, $Q_{\mu}(z)$ it is convenient to rewrite L_n as an expectation value.⁽¹²⁾ With the notation

$$\langle A(z) \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta A(e^{i\theta}) \quad (3.88)$$

we can check that

$$L_n = -\frac{1}{2} \langle z^{-n} : P^2(z) : \rangle \quad (3.89)$$

by using

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta e^{i(m-n)\theta} = \delta_{m-n,0} \quad (3.90)$$

as follows:

$$-\frac{1}{2} \langle : P^2(z) : \rangle = -\frac{1}{4\pi} \int_{-\pi}^{\pi} d\theta : \left[\sqrt{2} p_{\mu} - i \sum_{n=1}^{\infty} \sqrt{n} (a^{(n)} e^{in\theta} - a^{(n)\dagger} e^{-in\theta}) \right]^2 : \quad (3.91)$$

$$= -p^2 - \sum_{n=1}^{\infty} n a^{(n)\dagger} \cdot a^{(n)} = L_0 \quad (3.92)$$

$$-\frac{1}{2} \langle z^{-n} : P^2(z) : \rangle = -\frac{1}{4\pi} \int_{-\pi}^{\pi} d\theta e^{-in\theta} : \left[\sqrt{2} p_{\mu} - i \sum_{n=1}^{\infty} \sqrt{n} (a^{(n)} e^{in\theta} - a^{(n)\dagger} e^{-in\theta}) \right]^2 : \quad (3.93)$$

$$= i\sqrt{2n} p \cdot a^{(n)} - \sum_{m=1}^{\infty} \sqrt{m(m+n)} a^{(m)\dagger} \cdot a^{(m+n)} \\ + \frac{1}{2} \sum_{m=1}^{n-1} \sqrt{m(n-m)} a^{(m)} \cdot a^{(n-m)} \quad (3.94)$$

$$= L_n \quad (3.95)$$

as required. We can now easily compute

$$[L_n, Q_{\mu}(z)] = -\frac{1}{2} \langle z'^{-n} [P^2(z), Q_{\mu}(z)] \rangle \quad (3.96)$$

$$= -\frac{1}{2} \langle z'^{-n} 2p_{\mu} \delta(\theta-\theta') 2\pi i \rangle \quad (3.97)$$

$$= -i e^{-in\theta} p_{\mu}(z) \quad (3.98)$$

$$= -z^{-n} \left(z \frac{d}{dz} \right) Q_{\mu}(z) \quad (3.99)$$

and, similarly

$$[L_n, P_\mu(z)] = -z \frac{d}{dz} (z^{-n} P) \quad (3.100)$$

$$= -z \left(-n z^{-n-1} P + z^{-n} \frac{d}{dz} P \right) \quad (3.101)$$

$$= -z^{-n} \left(z \frac{d}{dz} - n \right) P_\mu(z) \quad (3.102)$$

Let us define generalized projective spin, S , of an operator field $O^S(z)$ by

$$[L_n, O^S(z)] = -z^{-n} \left(z \frac{d}{dz} + nS \right) O^S(z) \quad (3.103)$$

Then we see that $Q_\mu(z)$ is a generalized projective scalar ($s=0$) and $P_\mu(z)$ is a generalized projective vector ($s=-1$).

We can now study how the vertex operator $V(p, z)$ transforms under L_n . Here great care with normal ordering is essential (unless $p^2=0$ in which special case the normal ordering is trivial). It turns out to be convenient⁽¹³⁾ to define $V(p, z)$ as the limit

$$V(p, z) = : \exp(\sqrt{2} i p \cdot Q(z)) : \quad (3.104)$$

$$= \exp(\sqrt{2} i p \cdot Q^{(-)}(z)) \exp(\sqrt{2} i p \cdot Q^{(0)}(z)) \exp(\sqrt{2} i p \cdot Q^{(+)}(z)) \quad (3.105)$$

$$= \lim_{z' \rightarrow z} \left(1 - \frac{z'}{z} \right)^{p^2} \exp(\sqrt{2} i p \cdot Q(z)) \quad (3.106)$$

where in the last step we used the Baker-Hausdorff relation.

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A, B]} \quad (3.107)$$

and observed that

$$[Q_\mu^{(-)}(z), Q_\nu^{(+)}(z')] = -g_{\mu\nu} \ln \left(1 - \frac{z'}{z} \right) \quad (3.108)$$

To find $[L_n, V(p, z)]$ we make an infinitesimal transformation

$$V(p, z) + \varepsilon [L_n, V(p, z)] = \lim_{z' \rightarrow z} e^{\varepsilon L_n} \left(1 - \frac{z'}{z}\right)^{p^2} e^{-\varepsilon L_n} \exp\left(\sqrt{2}ip_\mu \left\{e^{\varepsilon L_n} Q_\mu(z) e^{-\varepsilon L_n}\right\}\right) \quad (3.109)$$

Now

$$e^{\varepsilon L_n} z e^{-\varepsilon L_n} = z + \varepsilon [L_n, z] \quad (3.110)$$

$$= z + \delta z \quad (3.111)$$

where $\delta z = -\varepsilon z^{-n+1}$

Hence

$$\begin{aligned} \lim_{z' \rightarrow z} e^{\varepsilon L_n} \left(1 - \frac{z'}{z}\right)^{p^2} e^{-\varepsilon L_n} \\ = \lim_{z' \rightarrow z} \left[\left(1 - \frac{z'}{z}\right) \left(1 - \frac{z' + \delta z'}{z + \delta z}\right) \right]^{p^2} \end{aligned} \quad (3.112)$$

$$= 1 - p^2 n \varepsilon z^{-n} + O(\varepsilon^2) \quad (3.113)$$

Using the fact that $Q_\mu(z)$ is a generalized projective scalar then gives for the full expression

$$[L_n, V(p, z)] = -z^{-n} \left(z \frac{d}{dz} - n p^2 \right) V(p, z) \quad (3.114)$$

Hence the result may be summarized by stating that $V(p, z)$ transforms as a generalized projective spin $S = p^2$. In particular it is a generalized projective vector, $s = -1$, if $p^2 = 1$ or equivalently $\alpha(0) = 1$.

(Note: we always take slope $\alpha' = 1$)

All of the technical apparatus needed to find the unit intercept gauge identities are now available. Recall that we wrote

$$B_N = \langle 0 | V(p_2, 1) \frac{1}{L_0 - 1} V(p_3, 1) \dots V(p_{N-1}, 1) | 0 \rangle \quad (3.115)$$

By using the generalized projective algebra it is immediate to derive that, for any $f(x)$

$$L_n f(L_0) = f(L_0 + n) L_n \quad (3.116)$$

and hence

$$(L_0 - L_n - 1) \frac{1}{L_0 - 1} = \frac{1}{L_0 + n - 1} (L_0 + n - 1 - L_n) \quad (3.117)$$

With $p^2 = -1$ we also know that

$$(L_0 + n - 1 - L_n) V(p, 1) = V(p, 1) (L_0 - L_n - 1) \quad (3.118)$$

By repeating this process and noting that for an on-shell ground state

$$(L_0 - L_n - 1) |0\rangle = 0 \quad (3.119)$$

we deduce that

$$(L_0 - L_n - 1) \left[\frac{1}{L_0 - 1} V(p_1, 1) \frac{1}{L_0 - 1} V(p_2, 1) \dots V(p_{n-1}, 1) |0\rangle \right] = 0 \quad (3.120)$$

for an arbitrary physical state, i.e., a state which can be made from some number of ground state particles.

On mass shell we have the additional condition

$$(L_0 - 1) |\phi\rangle = 0 \quad (3.121)$$

and hence

$$L_n |\phi\rangle = 0 \quad n = 1, 2, 3, \dots \quad (3.122)$$

are the on-shell gauge conditions.⁽¹⁴⁾ The infinite number of gauge conditions opens up the possibility of removing the ghost negative norm states with odd time occupancy and we shall prove that this does in fact happen shortly. The crucial point in the rigorous proof^(15,16) is the full recognition of the role played by the anomaly term in the generalized projective algebra. Once this point is

understood, the proof becomes beautiful and straightforward. There were many earlier papers which discussed the properties of the generalized projective algebra.....[References 13, 14, 17-22].

3c. Operatorial Duality...

By using the formalism now developed, combining both the projective invariance and the operator formalism we can now rewrite the N-point function in a form⁽³⁾ which exhibits explicitly both cyclic symmetry and factorizability as follows:

Recall that (for $\alpha(0)=1$)

$$B_N = \langle 0 | V(p_2, 1) \frac{1}{L_0-1} V(p_3, 1) \dots \frac{1}{L_0-1} V(p_{N-1}, 1) | 0 \rangle \quad (3.123)$$

$$= \int \prod_{i=2}^{N-2} du_{1i} \langle 0 | V(p_2, 1) u_{12}^{L_0-2} V(p_3, 1) \dots V(p_{N-1}, 1) | 0 \rangle \quad (3.124)$$

Now put

$$u_{1j} = \frac{z_j}{z_{j+1}} \quad j=2, 3, \dots, N-2 \quad (3.125)$$

with the corresponding Jacobian

$$\partial \begin{pmatrix} u_{12} & u_{13} & \dots & u_{1, N-2} \\ z_2 & z_3 & \dots & z_{N-2} \end{pmatrix} = (z_3 z_4 \dots z_{N-2})^{-1} \quad (3.126)$$

to obtain

$$B_N = \int \prod_{i=2}^{N-2} dz_i (z_3 z_4 \dots z_{N-2})^{-1} \langle 0 | V(p_2, 1) \left(\frac{z_2}{z_3} \right)^{L_0-2} V(p_3, 1) \dots \left(\frac{z_{N-2}}{z_{N-3}} \right)^{L_0-2} V(p_{N-1}, 1) | 0 \rangle \quad (3.127)$$

The Jacobian is obtained by noticing that

$$\delta \begin{pmatrix} u_{12} & u_{13} & \dots & u_{1,N-2} \\ z_2 & z_3 & \dots & z_{N-2} \end{pmatrix} = \left| \frac{\partial u_{1i}}{\partial z_j} \right| \quad (3.128)$$

$$= \begin{vmatrix} \frac{1}{z_3} & -\frac{z_2}{z_3^2} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{z_4} & -\frac{z_3}{z_4^2} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{z_5} & -\frac{z_4}{z_5^2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & \frac{1}{z_{N-2}} & -\frac{z_{N-2}}{z_{N-3}^2} \\ 0 & 0 & \dots & \dots & 0 & 1 \end{vmatrix} \quad (3.129)$$

$$= \left(z_3 z_4 z_5 \dots z_{N-2} \right)^{-1} \quad (3.130)$$

Now by noticing that

$$z^{p^2} \exp \left(\sqrt{2} i p \cdot Q^{(0)}(1) \right) z^{-p^2} =$$

$$= e^{ip \cdot Q} \exp \left[-i p_\mu \ln z [Q_\mu, p^2] \right] \quad (3.131)$$

$$= \exp \left[i \sqrt{2} p \cdot \left(\frac{Q}{\sqrt{2}} + i \sqrt{2} P \ln z \right) \right] e^{-p^2 \ln z} \quad (3.132)$$

$$= \exp \left(\sqrt{2} i p \cdot Q^{(0)}(z) \right) z^{-p^2} \quad (3.133)$$

we deduce that

$$\begin{aligned} z^{-L_0} V(p, 1) z^{L_0} &= \\ &= z^{-L_0} \exp(\sqrt{2} i p \cdot Q^{(-)}(1)) \exp(\sqrt{2} i p \cdot Q^{(0)}(1)) \exp(\sqrt{2} i p \cdot Q^{(+)}(1)) z^{L_0} \end{aligned} \quad (3.134)$$

$$= \exp(\sqrt{2} i p \cdot Q^{(-)}(z)) \exp(\sqrt{2} i p \cdot Q^{(0)}(z)) \exp(\sqrt{2} i p \cdot Q^{(+)}(z)) z^{-p^2} \quad (3.135)$$

$$= V(p, z) z \quad (3.136)$$

for $\alpha(0) = 1$, i.e., $p^2 = -1$.

Also we need to use the following limits for the bra and ket in the vacuum expectation value

$$\langle 0 | e^{i p \cdot Q} = \langle 0, p | = \lim_{z_1 \rightarrow 0} \langle 0 | V(p, z_1) \quad (3.137)$$

$$e^{i p_N \cdot Q} | 0 \rangle = | 0, p_N \rangle = \lim_{z_N \rightarrow \infty} V(p_N, z_N) | 0 \rangle \quad (3.138)$$

Collecting together the factors one deduces that

$$\begin{aligned} B_N &= \lim_{\substack{z_1 \rightarrow 0 \\ z_{N-1} \rightarrow 1 \\ z_N \rightarrow \infty}} \int \prod_{i=1}^N dz_i \left[\frac{dz_1 dz_2 \dots dz_N}{(z_1 - z_{N-1})(z_{N-1} - z_N)(z_N - z_1)} \right]^{-1} * \\ &\quad \langle 0 | V(p_1, z_1) V(p_2, z_2) \dots V(p_N, z_N) | 0 \rangle \end{aligned} \quad (3.139)$$

If we take $\alpha(0) \neq 1$ the same procedure goes through, except that in the integrand there is one new factor, namely

$$\begin{aligned} B_N &= \lim_{\substack{z_1 \rightarrow 0 \\ z_{N-1} \rightarrow 1 \\ z_N \rightarrow \infty}} \int \prod_{i=1}^N dz_i (dV_3)^{-1} \prod_{i=1}^N |z_i - z_{i+1}|^{\alpha(0)-1} * \\ &\quad * \langle 0 | V(p_1, z_1) V(p_2, z_2) \dots V(p_N, z_N) | 0 \rangle \end{aligned} \quad (3.140)$$

We can check that this is the appropriate Koba-Nielsen form by evaluating directly that

$$\langle 0 | V(p_1 z_1) V(p_2 z_2) \dots V(p_N z_N) | 0 \rangle = \prod_{i < j} |z_i - z_j|^{-2p_i p_j} \quad (3.141)$$

The form we have arrived at, for B_N , possesses operatorial duality in the following sense. Let us make a general projective transformation

$$z_i \rightarrow z'_i = \frac{az_i + b}{cz_i + d} \quad \text{or} \quad z'_i = \frac{dz'_i - b}{a - cz'_i} \quad (3.142)$$

so that

$$(z_i - z_{i+1}) = (z'_i - z'_{i+1}) (a - cz'_i)^{-1} (a - cz'_{i+1})^{-1} \quad (3.143)$$

Hence

$$\prod_{i=1}^N (z_i - z_{i+1})^{\alpha(b)} = \prod_{i=1}^N (z'_i - z'_{i+1})^{\alpha(b)} \prod_{i=1}^N (a - cz'_i)^{-2\alpha(b)} \quad (3.144)$$

The factor $\prod_{i=1}^N dz_i (dV_3)^{-1} \prod_{i=1}^N |z_i - z_{i+1}|$ is projective invariant by simple inspection of translation, dilation and inversion on z .

The vertex transformation property is most easily seen by using the limiting procedure introduced earlier, namely

$$V(p, z) = \lim_{\tilde{z} \rightarrow z} \left[\exp \left(\sqrt{2} i p \left\{ Q^-(z) + Q^0(z) + Q^+(z) \right\} \right) \left(1 - \frac{\tilde{z}}{z} \right)^{p^2} \right] \quad (3.145)$$

and hence

$$\Lambda(\tau) V(p, z) \Lambda^{-1}(\tau) = V(p, z') \lim_{\substack{\tilde{z} \rightarrow z \\ \tilde{z}' \rightarrow z'}} \left(\frac{z - \tilde{z}}{z' - \tilde{z}'} \right)^{p^2} \quad (3.146)$$

$$= (a - cz')^{-2p^2} V(p, z') \quad (3.147)$$

$$= (a - cz')^{2\alpha(b)} V(p, z') \quad (3.148)$$

Hence projective invariance of the full integrand follows, because the factor

$$\prod_{i=1}^N (a - cz'_i)^{-2\alpha(b)} \quad (3.149)$$

arising from the nearest-neighbor factor is precisely compensated by a similar factor in the vertex transformation.

To see the cyclic symmetry we simply make a transformation which takes Z_{N-1} to $Z'_{N-1} = \infty$, Z_{N-2} to $Z'_{N-2} = 1$ and Z_N to $Z'_N = 0$. (These three constraints completely specify the transformation.) We then arrive at

$$B_N = \int \prod_{i=1}^N dz'_i (dV_3)^{-1} \prod_{i=1}^N |z'_i - z'_{i+1}|^{\alpha(b)} (-1)^{2\alpha(b)} \quad (3.150)$$

$$\langle 0 | V(p_1, z'_N) V(p_2, z'_1) \dots V(p_N, z'_1) | 0 \rangle$$

where we exhibit a phase arising from the rearrangement of two factors $(Z_2 - Z_1)$ and $(Z_1 - Z_N)$. The vertex $V(p_1, z'_N)$ may now be commuted by using the Baker-Hausdorf relation in

$$V(p, z) V(p', z') = V(p', z') V(p, z) X \quad (3.151)$$

with

$$X = \exp \left[-2 p_\mu p'_\nu \left\{ [Q_\mu^{(+)}(z), Q_\nu^{(-)}(z')] + [Q_\mu^{(-)}(z), Q_\nu^{(+)}(z')] + [Q_\mu^{(0)}(z), Q_\nu^{(0)}(z')] \right\} \right] \quad (3.152)$$

$$= \exp \left[-2 p_\mu p'_\nu \left\{ g_{\mu\nu} \ln(1 - z/z') - g_{\mu\nu} \ln(1 - z'/z) - \ln(z/z') \right\} \right] \quad (3.153)$$

$$= (-1)^{-2p \cdot p'} \quad (3.154)$$

The commutation picks up an additional factor

$$(-1)^{-2p_1 \sum_{j=2}^N p_j} = (-1)^{2p_1^2} = (-1)^{-2\alpha(0)} \quad (3.155)$$

giving

$$B_N = \int \prod_{i=1}^N dz'_i (dV_3)^{-1} \prod_{i=1}^N (z'_i - z'_{i+1})^{\alpha(0)} \quad (3.156)$$

$$\langle 0 | V(p_1, z'_1) V(p_2, z'_2) \dots V(p_N, z'_N) | 0 \rangle$$

which we may now refactorize (in a new multiperipheral configuration) by the use of

$$V(p, z) = z^{L_0} V(p, 1) z^{-L_0} z^{p^2} \quad (3.157)$$

To make more clear what we have succeeded in doing, compare Figs. 1(a) and 1(b). We have obtained an expression for B_N which enables us to go from factorization of the first configuration to factorization in the second very directly by exploiting the cyclic symmetry.

This operatorially dual form for B_N thus combines the advantages of all earlier forms, where there was a choice of which one of the two principal properties (cyclic symmetry and factorizability) to exhibit explicitly.

3d. Unit Intercept

We have derived the on-shell gauge conditions

$$L_n |\phi\rangle = 0 \quad n = 1, 2, \dots \quad (3.158)$$

only for the case $\alpha(0) = 1$. Actually the unit intercept condition (which is unphysical since it gives rise to a tachyon at $\alpha_s = 0$ on the leading trajectory) is essential in order to obtain the full set of generalized projective gauges, and only in this special case are there enough linear dependences to remove the ghosts which arise from the indefinite metric.

Because the unit intercept condition is so important, it is instructive to examine⁽²³⁾ why the rather unimportant-seeming factor

$$\prod_{i=1}^N |z_i - z_{i+1}|^{-1+\alpha(0)} \quad (3.159)$$

is enough to spoil the generalized projective invariance. Note that this factor is the only factor in the B_N integrand which does not have the property of total symmetry under all $N!$ permutations made on the N momenta p_i and their corresponding Koba-Nielsen z_i variables.

The N -point function B_N may be written for general $\alpha(0)$ as

$$B_N = \int \prod_{i=2}^{N-2} du_{1i} (1-u_{1i})^{-1+\alpha(0)} u_{1i}^{-1-\alpha(0)} \langle 0 | V(p_2) u_{12}^{L_0} \dots V(p_{N-1}) | 0 \rangle \quad (3.160)$$

One simple way to factorize this expression was given earlier, namely to define a propagator by

$$D(s) = \int_0^1 dx x^{L_0-1-\alpha(0)} (1-x)^{\alpha(0)-1} = B(L_0-\alpha(0), \alpha(0)) \quad (3.161)$$

and then write

$$B_N = \langle 0 | V(p_2) D(s_{12}) V(p_3) \dots D(s_{1,N-2}) V(p_{N-1}) | 0 \rangle \quad (3.162)$$

This method of factorization is really too trivial in order to study the gauge problem. It is better to introduce a fifth dimension⁽²⁴⁾ in the oscillator operators $a_5^{(n)}, a_5^{(n)\dagger}$ ($n=1,2,3,\dots$) satisfying

$$[a_5^{(m)}, a_5^{(n)\dagger}] = +\delta_{mn} \quad (3.163)$$

and then define

$$L_0 = - \sum_{r=1}^{\infty} r a_5^{(r)\dagger} a_5^{(r)} - \hat{p}^2 \quad (3.164)$$

$$L_n = i\sqrt{2n} \hat{p} \cdot \hat{a}^{(n)} - \sum_{r=1}^{\infty} \sqrt{r(r+n)} \hat{a}^{(r)\dagger} a_5^{(r+n)} + \frac{1}{2} \sum_{r=1}^{n-1} \sqrt{r(n-r)} \hat{a}^{(r)} \cdot \hat{a}^{(n-r)} \quad (3.165)$$

where the scalar product is now $\hat{A} \cdot \hat{B} = A_0 B_0 - \underline{A} \cdot \underline{B} - A_5 B_5$ and we define $P_5 = \lambda/2$ with $\lambda^2 = (1-\alpha(0))$.

We now define a propagator and vertex by

$$\hat{D} = (L_0 + \frac{1}{2}\lambda^2 - 1)^{-1} \quad (3.166)$$

$$\hat{V}(p) = V(p) V_5 \quad (3.167)$$

$$V_5 = \exp\left(\lambda \sum_{n=1}^{\infty} \frac{a_5^{(n)\dagger}}{\sqrt{n}}\right) |0\rangle_5 \langle 0|_5 \exp\left(\lambda \sum_{n=1}^{\infty} \frac{a_5^{(n)}}{\sqrt{n}}\right) \quad (3.168)$$

to find

$$B_N = \langle 0 | \hat{V}(p_2) \frac{1}{L_0 + \frac{1}{2}\lambda^2 - 1} \hat{V}(p_3) \dots \hat{V}(p_{N-1}) |0\rangle \quad (3.169)$$

as is easily proved by using the identity

$$\begin{aligned} \langle 0 | \exp\left(\lambda \sum_{n=1}^{\infty} \frac{a_5^{(n)}}{\sqrt{n}}\right) \times \sum_{n=1}^{\infty} n a_5^{(n)\dagger} a_5^{(n)} \exp\left(\lambda \sum_{n=1}^{\infty} \frac{a_5^{(n)\dagger}}{\sqrt{n}}\right) |0\rangle_5 \\ = (1-\chi)^{-1+\alpha(0)} \end{aligned} \quad (3.170)$$

Notice that the vacuum projection of the fifth dimension in V_5 is essential to avoid factors of the form

$$(1 - u_{1i} u_{1j})^{-1 + \alpha(b)} \quad (3.171)$$

in B_N . This ugly vacuum projection is, therefore, linked to the absence of total symmetry in the integrand, and we will see that it is precisely what leads to loss of generalized projective invariance.

Consider now the gauge operator $(\hat{L}_0 = \frac{\lambda^2}{2} - 1 - \hat{L}_n)$ acting as the new propagator. We have

$$\begin{aligned} & \left(\hat{L}_0 + \frac{\lambda^2}{2} - 1 - \hat{L}_n \right) \frac{1}{\left(\hat{L}_0 + \frac{\lambda^2}{2} - 1 \right)} = \\ & = \frac{1}{\left(\hat{L}_0 + n + \frac{\lambda^2}{2} - 1 \right)} \left(\hat{L}_0 + n + \frac{\lambda^2}{2} - 1 - \hat{L}_n \right) \end{aligned} \quad (3.172)$$

To commute with the full vertex $\hat{V}(p)$ note that, by writing in an obvious notation

$$\hat{L}_n = L_n + L_n^5 \quad (3.173)$$

$$\hat{L}_0 = L_0 + L_0^5 \quad (3.174)$$

then

$$[\hat{L}_0 - \hat{L}_n, \hat{V}(p)] = [L_0 - L_n, V(p)] V_5 + [L_0^5 - L_n^5, V_5] V(p) \quad (3.175)$$

$$= n p^2 \hat{V}(p) + [L_0^5 - L_n^5, V_5] V(p) \quad (3.176)$$

$$= -n \hat{V}(p) + V(p) [V_5 n \lambda^2 + [L_0^5 - L_n^5, V_5]] \quad (3.177)$$

where we used our previous result for $[L_0 - L_n, V(p)]$.

To calculate the fifth dimension commutator write

$$[L_0^5, V_5] = \sum_{n=1}^{\infty} n [a_5^{n\dagger} a_5^n, V_5] \quad (3.178)$$

$$= \lambda \sum_{n=1}^{\infty} \sqrt{n} (a_5^{(n)\dagger} V_5 - V_5 a_5^{(n)}) \quad (3.179)$$

and similarly

$$[L_1^5, V_5] = \sum_{n=1}^{\infty} \sqrt{n(n+1)} [a_5^{(n)\dagger} a_5^{(n+1)}, V_5] + [\lambda a_5^{(1)}, V_5] \quad (3.180)$$

$$= \lambda \sum_{n=1}^{\infty} \sqrt{n(n+1)} \left(\frac{1}{\sqrt{n+1}} a_5^{(n)\dagger} V_5 - \frac{1}{\sqrt{n}} V_5 a_5^{(n+1)} \right) + \lambda^2 V_5 + \lambda V_5 a_5^{(1)} \quad (3.181)$$

$$= \lambda \sum_{n=1}^{\infty} \sqrt{n} (a_5^{(n)\dagger} V_5 - V_5 a_5^{(n)}) + \lambda^2 V_5 \quad (3.182)$$

Hence

$$[L_0^5 - L_1^5, V_5] = -\lambda^2 V_5 \quad (3.183)$$

In these equations we have used the general identity

$$[a_5, f(a_5^\dagger)] = f'(a_5^\dagger) \quad (3.184)$$

and the related forms. Now consider, by the same method

$$[L_2^5, V_5] = \sum_{n=1}^{\infty} \sqrt{n(n+2)} [a_5^{(n)\dagger} a_5^{(n+2)}, V_5] + [\sqrt{2} \lambda a_5^{(2)} + a_5^{(1)} a_5^{(1)}, V_5] \quad (3.185)$$

$$= \lambda \sum_{n=1}^{\infty} \sqrt{n(n+2)} \left(\frac{1}{\sqrt{n+2}} a_5^{(n)\dagger} V_5 - \frac{1}{\sqrt{n}} V_5 a_5^{(n+2)} \right) + 2\lambda^2 V_5 - \sqrt{2} V_5 a_5^{(2)} - V_5 a_5^{(1)} a_5^{(1)} \quad (3.186)$$

$$= \lambda \sum_{n=1}^{\infty} \sqrt{n} (a_5^{(n)\dagger} V_5 - V_5 a_5^{(n)}) + 2\lambda^2 V_5 + \lambda V_5 a_5^{(1)} - V_5 a_5^{(1)} a_5^{(1)} \quad (3.187)$$

and hence

$$[L_0^5 - L_2^5, V_5] = -2\lambda^2 V_5 + (\text{anomaly terms}) \quad (3.188)$$

The anomaly terms arise precisely from the presence of the fifth dimension vacuum projection.

Now, collecting results we have

$$[L_0^\lambda - L_1^\lambda, \hat{V}(p)] = -\hat{V}(p) \quad (3.189)$$

which implies that

$$\begin{aligned} (L_0^\lambda + \frac{\lambda^2}{2} - 1 - L_1^\lambda) \frac{1}{L_0^\lambda + \frac{\lambda^2}{2} - 1} \hat{V}(p) = \\ \frac{1}{(L_0^\lambda + \frac{\lambda^2}{2})} \hat{V}(p) (L_0^\lambda + \frac{\lambda^2}{2} - 1 - L_1^\lambda) \end{aligned} \quad (3.190)$$

which together with the fact that for an on-shell ground state

$$(L_0^\lambda + \frac{\lambda^2}{2} - 1 - L_1^\lambda) |0\rangle = 0 \quad (3.191)$$

enables us to derive that physical states satisfy (on-shell)

$$(L_0^\lambda + \frac{\lambda^2}{2} - 1) |\phi\rangle = 0 \quad (3.192)$$

$$L_1^\lambda |\phi\rangle = 0 \quad (3.193)$$

The higher gauges fail to work because of the anomaly terms, that is, in general

$$L_n^\lambda |\phi\rangle \neq 0 \quad n=2, 3, 4, \dots \quad (3.194)$$

The L_1 gauges give linear dependences which reflect that underlying projective invariance. They are not sufficient in number to allow ghost elimination.

The only case in which the generalized projective gauge invariance is possible is therefore when $\alpha(0) = 1$; thus we are led to associate total symmetry of the integrand under all $N!$ permutations of the $\{z_i, p_i\}$ variable pairs (i.e., absence of a preferred status for nearest neighbour z_i differences) with the full set of subsidiary conditions

$$L_n |\phi\rangle = 0 \quad n = 1, 2, 3, 4, \dots \quad (3.195)$$

3e. Null States

To discuss the way in which null states decouple⁽¹⁴⁾ in the dual resonance model, it is convenient first to recall some features of quantum electrodynamics, in particular the ways in which the formalism ensures that only the two transverse components of the massless photon survive, with non-zero couplings, on mass shell.

To make calculations in quantum electrodynamics, we may take one of two quite different techniques. Either we may use a non-manifestly-covariant method where we define a three-vector field $\underline{A}$, in, for example, the Coulomb gauge $\underline{\nabla} \cdot \underline{A} = 0$; or we may use a fully covariant method with a four vector A_μ and the Lorentz gauge condition.

In the Coulomb gauge method, we may make a Fourier decomposition, assuming periodic boundary conditions inside a large cubic box of volume $V = L^3$

$$\underline{A} = \frac{1}{\sqrt{V}} \sum_{\underline{k}} \sum_{\alpha=1,2} \frac{1}{\sqrt{2E_{\underline{k}}}} \left[a_{\underline{k},\alpha} \underline{\epsilon}^\alpha e^{i\underline{k} \cdot \underline{x} - iE_{\underline{k}}t} + a_{\underline{k},\alpha}^\dagger \underline{\epsilon}^\alpha e^{-i\underline{k} \cdot \underline{x} + iE_{\underline{k}}t} \right] \quad (3.196)$$

where $\underline{\epsilon}^\alpha$ ($\alpha=1,2$) are the transverse polarization vectors with $\underline{\epsilon}^1, \underline{\epsilon}^2$ and $\underline{k}/|\underline{k}|$ forming an orthogonal triad of unit vectors. The Coulomb gauge $\underline{\nabla} \cdot \underline{A} = 0$ has been imposed and the presence of only transverse components is manifest. Going to an infinitely large box one obtains

$$\underline{A} = \sum_{\alpha=1,2} \frac{1}{(2\pi)^3} \int \frac{d^3k}{2E_k} \left\{ a_\alpha(\underline{k}) \underline{\epsilon}^\alpha e^{i\underline{k} \cdot \underline{x} - iE_k t} + a_\alpha^\dagger(\underline{k}) \underline{\epsilon}^\alpha e^{-i\underline{k} \cdot \underline{x} + iE_k t} \right\} \quad (3.197)$$

and then, to quantize, one imposes the canonical commutation relations

$$[a_\alpha(\underline{k}), a_\beta^\dagger(\underline{k}')] = (2\pi)^3 \delta_{\alpha\beta} \delta(\underline{k} - \underline{k}') \quad (3.198)$$

In such a theory of quantized transverse electromagnetic fields, the transversality or Coulomb gauge condition is noncovariant and, in general, after any Lorentz transformation we must adopt a new gauge. This is the method developed by Dirac and Fermi in the late 1920's and which is discussed in many text books [25,26,27].

Some twenty years later Feynman, Schwinger and Tomonaga used a fully covariant approach, in which the transversality property while not very explicitly displayed is not too difficult to extract. [See the original papers in Reference 27.] It is this last point which we wish to explain in detail here since exactly the same mechanism is operative in the (fully covariant) dual resonance model.

Consider the diagram for a single photon exchange indicated in Figure 2. In terms of photon creation and annihilation operators this diagram corresponds to an amplitude

$$T_{fi} = - \langle f | a_\mu^\dagger | i \rangle \frac{g_{\mu\nu}}{q^2} \langle 0 | a_\nu | i \rangle \quad (3.199)$$

where $|i\rangle, |f\rangle$ are the initial, final multiparticle states respectively and

$$q_\mu = \sum_{i=1}^j p_{i\mu} = - \sum_{i=j+1}^N p_{i\mu} \quad (3.200)$$

In the photon propagator $-g_{\mu\nu}(q^2)^{-1}$ there appear four components of which the time component would give a negative (ghost) contribution. Gauge invariance of the first kind i.e., charge conservation is expressed by the Lorentz condition that

$$\langle f | q_\mu a_\mu^\dagger | 0 \rangle = 0 \quad (3.201)$$

for any physical state $|f\rangle$. Thus the state

$$q_\mu a_\mu^\dagger | 0 \rangle = L^\dagger | 0 \rangle \quad (3.202)$$

is spurious and does not couple. Here we defined $L = q_\mu a_\mu$. Alternatively we may represent the Lorentz gauge condition by stating that any physical state must satisfy

$$L | \phi \rangle = 0 \quad (3.203)$$

Let us take $q_\mu = (1; 0, 0, 1)$ along the 3-axis. Then the most general physical state becomes

$$| \phi \rangle = \alpha_1 | \lambda_1 \rangle + \alpha_2 | \lambda_2 \rangle + \alpha_3 | \mu \rangle \quad (3.204)$$

where we have chosen a linearly independent basis

$$| \lambda_1 \rangle = a_1^\dagger | 0 \rangle \quad (3.205)$$

$$| \lambda_2 \rangle = a_2^\dagger | 0 \rangle \quad (3.206)$$

$$| \mu \rangle = \frac{1}{\sqrt{2}} (a_0^\dagger - a_3^\dagger) | 0 \rangle \quad (3.207)$$

$$| \mu_2 \rangle = \frac{1}{\sqrt{2}} (a_0^\dagger + a_3^\dagger) | 0 \rangle \quad (3.208)$$

and $|\mu_2\rangle$ is omitted because it is spurious ($L|\mu_2\rangle \neq 0$). If we take the norms of these four states we find that $\langle\lambda_1|\lambda_1\rangle = \langle\lambda_2|\lambda_2\rangle = 1$ but that $\langle\mu_1|\mu_1\rangle = \langle\mu_2|\mu_2\rangle = 0$. We may therefore say that $|\mu_1\rangle$ is a null physical state.

We seem to have three components remaining after applying the Lorentz gauge condition whereas we know there are only the two transverse ones. How does a second state decouple?

The answer is very simple when we recognize that $|\mu_1\rangle$ and $|\mu_2\rangle$ are linked together as conjugate null states and that the Hilbert space of on-photon states has the unit operator (using $\langle\mu_1|\mu_2\rangle = 1$)

$$1 = |\lambda_1\rangle\langle\lambda_1| + |\lambda_2\rangle\langle\lambda_2| + |\mu_1\rangle\langle\mu_2| + |\mu_2\rangle\langle\mu_1| \quad (3.209)$$

This means that both the null spurious state $|\mu_2\rangle$ and the null physical state $|\mu_1\rangle$ decouple, leaving just the two transverse states $|\lambda_1\rangle, |\lambda_2\rangle$ as required.

Before proceeding let us make the following definitions:

- i) A physical state is a state which satisfies the gauge constraints.
- ii) A spurious state is a state which is orthogonal to all physical states, and which does not satisfy the gauge conditions.
- iii) A null state is a state with zero norm. The conjugate of a null physical state will be a null spurious state.

With this terminology the four photon states comprise three physical states and one spurious state. The spurious state is null, as is one of the physical states; this leaves the two non-null (transverse) physical states which couple.

In the dual resonance model the on-shell physical states satisfy (for unit intercept)

$$(L_0 - 1) |\phi\rangle = 0 \quad (3.210)$$

$$L_n |\phi\rangle = 0 \quad (3.211)$$

At a given mass shell we have a completeness relation

$$1 = \sum_{\lambda_+} |\lambda_+\rangle \langle \lambda_+| - \sum_{\lambda_-} |\lambda_-\rangle \langle \lambda_-| + \sum_{\mu} [|\mu\rangle \langle \mu_c| + |\mu_c\rangle \langle \mu|] \quad (3.212)$$

where the $|\mu\rangle$ are null states and the subscript c means conjugate.

The task is then to prove that all the $|\lambda_-\rangle$ are spurious.

Consider the state $L_1^\dagger(p) |0\rangle$ for $p_\mu = p(1; 0, 0, 1)$. This state is a null spurious state. Its conjugate is a null physical state. There remain two transverse positive-norm states. Explicitly we have

$$L_1^\dagger(p) |0\rangle = (a_0^{(1)\dagger} - a_3^{(1)\dagger}) |0\rangle \quad \text{null spurious state} \quad (3.213)$$

$$(a_0^{(1)} + a_3^{(1)\dagger}) |0\rangle \quad \text{null physical state} \quad (3.214)$$

$$a_1^{(1)\dagger} |0\rangle \quad (3.215)$$

$$\left. \begin{array}{l} a_1^{(1)\dagger} |0\rangle \\ a_2^{(1)\dagger} |0\rangle \end{array} \right\} \quad \text{physical states} \quad (3.216)$$

in precise analogy to the photon states of quantum electrodynamics.

More generally, in the dual resonance model note that any state of the form

$$L_n^\dagger |f, N-n\rangle \quad (3.217)$$

is spurious, as a direct consequence of the physical gauge condition.

Here $N-n$ is the eigenvalue of $R = (L_0 + p^2)$.

In particular, any state made by L_1^+ acting on a physical state is a null spurious state since

$$\| L_1^+ |\phi, N-1\rangle \|^2 = \langle \phi, N-1 | [L_1, L_{-1}] |\phi, N-1\rangle \quad (3.218)$$

$$= 0. \quad (3.219)$$

Thus the null states proliferate at least as fast as the number of physical states.

3f. Critical Dimension

We have already stated that the physical state conditions, on mass shell are

$$L_n |\phi, N\rangle = 0 \quad (3.220)$$

$$(L_0 - 1) |\phi, N\rangle = 0 \quad (3.221)$$

and that the gauge operators satisfy

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{d}{12} m(m^2-1) \delta_{m+n,0} \quad (3.222)$$

with d = space-time dimension. Here N is the eigenvalue of $R = L_0 + p^2$. We will now see how the anomaly term, proportional to d , enables us to prove the required absence of ghosts, provided $d \leq 26$. In particular, for the realistic value $d = 4$ there is such a theoretical consistency.

We first set up a basis for the complete Fock space, $\mathcal{F}$, spanned by $a_\mu^{(n)}, a_\mu^{(n)\dagger}$ by defining

$$K_n = k \cdot a^{(n)} / \sqrt{n} \quad n = 1, 2, \dots \quad (3.223)$$

$$K_{-n} = K_n^\dagger \quad (3.224)$$

where k_μ is some standard vector. For the moment let us take k_μ

to be a light-like vector, $k_\mu = (1; 1, 0, 0, \dots)$ so that $k_n = (a_0^{(n)} - a_1^{(n)})/\sqrt{n}$, and we have chosen the 1-axis to be longitudinal. There are a further (d-2) spatial transverse directions 2, 3, 4, ..., d-1.

Now we may set up a linearly-independent basis in the form

$$|\{\lambda, \mu\}, N\rangle = \prod_{m,n} (L_m^\dagger)^{\lambda_m} (K_n^\dagger)^{\mu_n} |\lambda=\mu=0, N'\rangle \quad (3.225)$$

where $N' = (N - \sum_m m\lambda_m - \sum_n n\mu_n)$ is the eigenvalue of $R = L_0 + p^2$.

We set up such a basis iteratively as follows: denote a state with $\lambda=\mu=0$ by $|\psi, N'\rangle$. Then we start from

$$|\psi, 0\rangle = |0\rangle \quad (3.226)$$

then set up

$$|\{\lambda_m = \delta_{m1}, \mu_n = 0\}, 1\rangle = L_1^\dagger |0\rangle \quad (3.227)$$

$$|\{\lambda_m = 0, \mu_n = \delta_{n1}\}, 1\rangle = K_1^\dagger |0\rangle \quad (3.228)$$

To complete the basis at $R=1$ we find two states $|\psi, 1\rangle$ orthogonal to these two states. Then we proceed to $R=2$, and so on. This iteratively defines the ψ -states which have the properties

$$L_n |\psi, N\rangle = 0 \quad n=1, 2, \dots \quad (3.229)$$

$$K_n |\psi, N\rangle = 0 \quad n=1, 2, \dots \quad (3.230)$$

and also are of positive norm

$$\langle \psi | \psi \rangle > 0 \quad (3.231)$$

This norm property follows because combinations of excitations of the type $(a^{(n)\dagger} + a^{(n)})$ are absent (otherwise K_n would not annihilate $|\psi, N\rangle$), as are the combinations $(a^{(n)\dagger} - a^{(n)\dagger}) = nK_n^\dagger$ since by definition $\mu_n = 0$ for a ψ -state.

Now all the states with $\lambda_m \neq 0$ for at least one m are spurious states and let us define by $\mathcal{S}_0$ the spurious subspace with eigenvalue of L_0 equal to ℓ_0 . That is, $\mathcal{S}_1$ is the on-shell spurious subapce. $\mathcal{S}_0$ is the spurious subspace one unit below mass-shell, $\mathcal{S}_{-1}$ is two units below, and so on.

Incidentally, note that the spurious states with $\lambda_i = \delta_{i1}$ are null states, as proved at the end of the previous section.

Now take a general state $|f, N\rangle \in \mathcal{F}$. It can be decomposed into (we take $|f, N\rangle$ on-shell, i.e., $\ell_0 = 1$)

$$|f, N\rangle = |s, N\rangle + |\phi, N\rangle \quad (3.232)$$

where $|s, N\rangle \in \mathcal{S}_1$ and $|k, N\rangle$ is a state, in the complementary subspace to $\mathcal{S}_1$, which we may call $\mathcal{K}_1$.

Since $|f, N\rangle$ is on mass-shell it satisfies

$$(L_0 - 1) |f, N\rangle = 0 \quad (3.233)$$

Let us now consider the most general physical state, that is let $|f, N\rangle$ satisfy in addition the gauge conditions

$$L_n |f, N\rangle = L_n (|s, N\rangle + |k, N\rangle) = 0 \quad (3.234)$$

It is not true for arbitrary space-time dimension, that we can deduce from this that L_n annihilates separately $|s, N\rangle$ and $|f, N\rangle$. We shall show, however, that such a deduction is possible if and only if $d=d_1=26$ and this will provide a shortcut to the proof of absence of ghosts.

First we must prove that L_1 maps $\mathcal{S}_1 \rightarrow \mathcal{S}_0$, for arbitrary d . The most general state belonging to $\mathcal{S}_1$ may be written

$$|s, N\rangle = L_1^\dagger |f, N-1\rangle + L_2^\dagger |f, N-2\rangle \quad (3.235)$$

To check this assertion, observe that the generalized projective algebra may be used to put all spurious states into this form, for example

$$L_3^\dagger |f, N-3\rangle = [L_1^\dagger, L_2^\dagger] |f, N-3\rangle \quad (3.236)$$

We act with L_1 on $|s, N\rangle$ to find

$$L_1 |s, N\rangle = L_1^\dagger L_1 |f, N-1\rangle + (L_2^\dagger L_1 + 3L_1^\dagger) |f, N-2\rangle \quad (3.237)$$

$$= |s, N-1\rangle \in \mathcal{S}_0 \quad (3.238)$$

This is a spurious state belonging to $\mathcal{S}_0$, as required. Thus we may deduce that the expressions

$$L_1 |f, N\rangle = L_1 |s, N\rangle = L_1 |k, N\rangle = 0 \quad (3.239)$$

separately vanish.

We now wish to find a special operator

$$\tilde{L}_2 = \alpha L_1 L_1 + \beta L_2 \quad (3.240)$$

that will map directly $\mathcal{S}_1 \rightarrow \mathcal{S}_{-1}$. Once this has been achieved the proof of absence of ghosts will follow almost immediately. We need the commutators

$$\begin{aligned} [L_1^\dagger L_1^\dagger, L_1 L_1] &= L_1^\dagger [L_1^\dagger, L_1] L_1 + L_1 [L_1^\dagger, L_1] L_1^\dagger \\ &\quad + [L_1^\dagger, L_1] L_1^\dagger L_1 + L_1 L_1^\dagger [L_1^\dagger, L_1] \end{aligned} \quad (3.241)$$

$$= -2(L_1^\dagger L_0 L_1 + L_0 L_1^\dagger L_1 + L_1 L_1^\dagger L_0 + L_1 L_0 L_1^\dagger) \quad (3.242)$$

$$= 4(2L_0 - 1)L_0 - 8L_0 L_1 L_1^\dagger \quad (3.243)$$

Further we can find that

$$[L_1^\dagger L_1^\dagger, L_2] = [L_2^\dagger, L_1 L_1] = -6 L_1 L_1^\dagger + 6 L_0 \quad (3.244)$$

$$[L_2^\dagger, L_2] = -4 L_0 - \frac{1}{2} d \quad (3.245)$$

and hence

$$[\tilde{L}_2, L_1^\dagger] = 4\alpha L_1 L_0 + (3\beta - 2\alpha) L_1 \quad (3.246)$$

$$\begin{aligned} [\tilde{L}_2, \tilde{L}_2^\dagger] = & \alpha^2 (8 L_0 L_1 L_1^\dagger - 4(2L_0 - 1)L_0) \\ & + \alpha\beta (6 L_1 L_1^\dagger - 6 L_0) \\ & + \beta^2 (4 L_0 + \frac{1}{2} d) \end{aligned} \quad (3.247)$$

Let us rewrite the most general spurious state in $\mathcal{L}_1$ as

$$|s, N\rangle = L_1^\dagger |f, N-1\rangle + \tilde{L}_2^\dagger |f, N-2\rangle \quad (3.248)$$

then we see that

$$\begin{aligned} \tilde{L}_2 |s, N\rangle = & (3\beta - 2\alpha) L_1 |f, N-1\rangle + L_1^\dagger \tilde{L}_2 |f, N-1\rangle \\ & + [(-8\alpha^2 + 12\alpha\beta) L_1^\dagger L_1 \\ & + (4\alpha^2 - 12\alpha\beta - 4\beta^2 + \frac{1}{2} d \beta^2)] |f, N-2\rangle \\ & + \tilde{L}_2^\dagger \tilde{L}_2 |f, N-2\rangle \end{aligned} \quad (3.249)$$

In order to achieve the desired result we put $3\beta - 2\alpha = 0$ e.g. by putting $\beta=1, \alpha=3/2$ such that

$$\tilde{L}_2 = L_2 + \frac{3}{2} L_1 L_1 \quad (3.250)$$

whereupon

$$\begin{aligned} \tilde{L}_2 |s, N\rangle = & \tilde{L}_1^\dagger \tilde{L}_2 |f, N-1\rangle + \tilde{L}_2^\dagger \tilde{L}_2 |f, N-2\rangle \\ & + \frac{1}{2}(d-26) |f, N-2\rangle \end{aligned} \quad (3.251)$$

If, now, we put $d = 26$ then

$$\tilde{L}_2 |s, N\rangle = |s, N-2\rangle \in \mathcal{S}_{-1} \quad (3.252)$$

and we have proved that L_2 maps $\mathcal{S}_1 \rightarrow \mathcal{S}_{-1}$, as required. Note the peculiar way in which d enters through the anomaly term occurring in $[L_2, L_{-2}]$. If the anomaly term were absent, we could never obtain such a simple result. The result, for $d = 26$, is that the two components, spurious and non-spurious, in our general physical state satisfy

$$L_n |s, N\rangle = L_n |k, N\rangle = 0 \quad n=1, 2, \dots \quad (3.253)$$

The fact that $|s, N\rangle$ is annihilated by the L_n implies that we have a state which is simultaneously spurious and physical; it is therefore a null state, that is

$$\langle s, N | s, N \rangle = 0 \quad (3.254)$$

Concerning the state $|k, N\rangle$ we may write its most general form as

$$|k, N\rangle = \sum_{\{\mu_m\}} C_{\{\mu_m\}} \prod_m (k_m^+)^{\mu_m} |\psi, N - \sum_m \mu_m\rangle \quad (3.255)$$

Now acting with L_n it is not difficult to show that $L_n |k, N\rangle = 0$ implies that all $C_{\{\mu_m\}} = 0$ ($\mu_m \neq 0$) and therefore

$$|k, N\rangle = |\psi, N\rangle \quad (3.256)$$

Thus we have the result that any physical state may be written (for $d=d_c=26$)

$$|f, N\rangle = |\psi, N\rangle + |\text{null state}\rangle \quad (3.257)$$

The null state may be absorbed by redefinition since it does not alter the properties. Since $|\psi, N\rangle$ has been shown to have positive norm, this proves that all physical states have positive norm.

To discuss a space-time dimension $d \leq 26$, note that we may simply add $(26-d)$ extra spacelike dimensions, with zero momentum components and treat the smaller d as a subspace of the $d = 26$ case. Hence, an immediate corollary of our proof is that all physical states have positive norm for $d \leq 26$.

Let us write down the two necessary and sufficient conditions for absence of imaginary coupling constants in the generalized Veneziano model:

- (i) The intercept must be $\alpha(0) = 1$.
- (ii) The space-time dimension must be an integer satisfying $d \leq d_c = 26$.

We should immediately add that the choice of k_μ as a light-like vector, has no deep significance. We can equally choose k_μ to have only a timelike component ($k_\mu = (1, 0, 0, 0, \dots)$). We can then see that the physical states, all of positive norm, may be taken as pure angular momentum eigenstates in the rest system, because of the explicit rotational invariance of the construction procedure.

This last point is important because to prove the absence of ghosts it is not sufficient simply to prove the decoupling of the time excitations. We must further prove (as we now have done) that the partial wave analysis into irreducible representations of $O(d-1)$, in the rest frame, leads to positive partial-wave coefficients.

The above proof of absence of ghosts can be reinterpreted in terms of null states. We have already noted that any state

$$|s, N\rangle = L_1^\dagger |\phi, N-1\rangle \quad (3.258)$$

is null, where $|\phi, N-1\rangle$ is physical. For the case $d = 26$ it is straightforward to prove that any state of the form

$$|s, N\rangle = L_1^+ |\phi, N-1\rangle + \tilde{L}_2^+ |\phi, N-2\rangle \quad (3.259)$$

is a null state, where $|\phi, N-1\rangle$ and $|\phi, N-2\rangle$ are both physical as follows:

$$\begin{aligned} & \| L_1^+ |\phi, N-1\rangle + \tilde{L}_2^+ |\phi, N-2\rangle \|^2 = \\ & = \langle \phi, N-1 | [L_1, L_1^+] | \phi, N-1 \rangle + 2 \operatorname{Re} \langle \phi, N-1 | [L_1, \tilde{L}_2^+] | \phi, N-2 \rangle + \end{aligned} \quad (3.260)$$

$$+ \langle \phi, N-2 | [\tilde{L}_2, \tilde{L}_2^+] | \phi, N-2 \rangle \quad (3.261)$$

$$\begin{aligned} & = \frac{1}{2} (d-26) \langle \phi, N-2 | \phi, N-2 \rangle \\ & = 0 \text{ for } d = d_c = 26, \end{aligned} \quad (3.262)$$

where we have used the expressions for the commutators worked out earlier.

Note that for $d < d_c$ these new null states do not occur, and it can be shown that the null states generated by L_1^+ acting on a physical state are the only ones.

Let us finally indicate how to count the numbers of physical, spurious and null states both for $d = d_c$ and for $d < d_c$. Define $p(x)$ and $T^{(n)}(N)$ by

$$(p(x))^n = \left[\prod_{r=1}^{\infty} (1-x^r)^{-1} \right]^n = \sum_{N=0}^{\infty} T^{(n)}(N) x^N \quad (3.263)$$

then in the full Fock space the total number of states at $R = N$ is given by $T^{(d)}(N)$ where d is the space-time dimension.

The number of physical states at $d = d_c$ is equal to the number of our ψ -states and this is straightforwardly seen to be $T^{(d_c-2)}(N)$.

On the other hand, the number of states satisfying the gauge conditions is $T^{(d_c-1)}(N)$. This then gives the breakdown at the level $R = N$:

$$T^{(d_c-2)}(N) \quad \text{non-null (positive-norm) physical states} \quad (3.264)$$

$$T^{(d_c-1)}(N) - T^{(d_c-2)}(N) \quad \text{null physical states} \quad (3.265)$$

$$T^{(d_c)}(N) - 2T^{(d_c-1)}(N) + T^{(d_c-2)}(N) \quad \text{non-null spurious states} \quad (3.266)$$

$$T^{(d_c-1)}(N) - T^{(d_c-2)}(N) \quad \text{null spurious states} \quad (3.267)$$

$$T^{(d_c)}(N) \quad \text{total number of states in the Fock space.} \quad (3.268)$$

For $d \neq d_c$ the breakdown of the total number $T^{(d)}(N)$ of states at $R=N$ (on mass shell) becomes

$$T^{(d-1)}(N) - T^{(d-1)}(N-1) \quad \text{non-null physical states} \quad (3.269)$$

$$T^{(d-1)}(N-1) \quad \text{null physical states} \quad (3.270)$$

$$T^{(d)}(N) - T^{(d-1)} - T^{(d-1)}(N-1) \quad \text{non-null spurious states} \quad (3.271)$$

$$T^{(d-1)}(N-1) \quad \text{null spurious states} \quad (3.272)$$

If $d < d_c$ the non-null physical states all have positive norm.

If $d > d_c$ there are physical states of both positive and negative norms, as can be seen, most easily by considering the level $\alpha_s = 2$ in

$$B(-\alpha_s, -\alpha_t) = \sum_{N=0}^{\infty} R_N(\alpha_t) (N-\alpha_s)^{-1} \quad (3.273)$$

$$R_2(\alpha_t) = \frac{25}{8} \left(z^2 - \frac{1}{25} \right) \quad (3.274)$$

which implies a scalar ghost for all $d \geq 27$ as we found much earlier in discussing the four-particle function.

3g. Physical State Construction

We have amply demonstrated the usefulness of the operators $a^{(n)}$, $a^{(n)+}$ in discussing the factorization and duality properties. Further, even the absence of ghosts could be demonstrated. In studying the spectrum, however, we see that these covariant operators create an enormous number of redundant spurious states. We shall now find new operators⁽²⁰⁾ $A_i^{(n)}, A_i^{(n)+}$ ($i = 2, 3, 4, \dots, (d-1)$) which create all the physical states in the critical dimension, and then briefly discuss the situation for a sub-critical case. The new operators are not manifestly covariant and their use is strictly analogous to the use of the Coulomb gauge in quantum electrodynamics.

Let us first construct the physical states coupled to two ground states. Recall that

$$V(p, z) = \exp(\sqrt{2} i p \cdot Q^{(-)}(z)) \exp(\sqrt{2} i p \cdot Q^{(0)}(z)) \exp(\sqrt{2} i p \cdot Q^{(+)}(z)) \quad (3.275)$$

$$\exp(\sqrt{2} i p \cdot Q^{(0)}(z)) = e^{i p \cdot Q} z^{-2p \cdot P} z^{p^2} \quad (3.276)$$

$$\langle 0, p_1 | = \lim_{z_1 \rightarrow 0} \langle 0 | V(p_1, z_1) \quad (3.277)$$

Take $p_{1\mu}$ and a second momentum $p_{2\mu}$ such that

$$1 + (p_1 + p_2)^2 = n, \text{ an integer} \quad (3.278)$$

$$2p_1 \cdot p_2 = n + 1 \quad (3.279)$$

and consider

$$\lim_{z_1 \rightarrow 0} \langle 0 | V(p_1, z_1) \oint \frac{dz_2}{z_2} V(p_2, z_2) \quad (3.280)$$

where we have taken a contour which encircles the origin $z_2 = 0$.

This becomes

$$\langle 0, (p_1 + p_2) | \oint \frac{dz_2}{z_2^{n+1}} \exp(\sqrt{2} i p_2 \cdot Q^{(+)}(z_2)) \quad (3.281)$$

where we used

$$z^{-2p_2 \cdot (p_1 + p_2)} z^{-1} = z^{-2p_1 \cdot p_2 + 1} = z^{-n} \quad (3.282)$$

It is clear that the integral is single-valued (because we are precisely on mass shell) and that we are creating a physical state of momentum $(p_1 + p_2)_\mu$ and eigenvalue of R equal to n .

By using the commutator

$$[L_n, V(p, z)] = -z^{-n} \left(z \frac{d}{dz} + n p^2 \right) V(p, z) \quad (3.283)$$

which implies for $p^2 = -1$,

$$\left[L_n, \frac{V(p, z)}{z} \right] = -\frac{d}{dz} \left(z^{-n} V(p, z) \right) \quad (3.284)$$

we can check that the gauge conditions are satisfied. We write

$$\langle 0, p_1 | \oint \frac{dz_2}{z_2} V(p_2, z_2) L_n^\dagger \quad (3.285)$$

$$= \langle 0, p_1 | L_n^\dagger \left(\oint \frac{dz_2}{z_2} V(p_2, z_2) + \oint dz_2 \frac{d}{dz_2} \left(z_2^n V(p_2, z_2) \right) \right) \quad (3.286)$$

$$= 0 \quad \text{if the integral is single-valued.} \quad (3.287)$$

To make further physical states consider the more general N -particle expression.

$$\lim_{z_1 \rightarrow 0} \langle 0 | V(p_1, z_1) \oint \frac{dz_2}{z_2} V(p_2, z_2) \oint_{z_2} \frac{dz_3}{z_3} V(p_3, z_3) \dots \oint_{z_{N-1}} \frac{dz_N}{z_N} V(p_N, z_N) \quad (3.288)$$

with

$$\begin{aligned} 1 + (p_1 + p_2)^2 &= \eta_{12} \\ 1 + (p_1 + p_2 + p_3)^2 &= \eta_{13} \\ &\vdots \\ 1 + (p_1 + p_2 + \dots + p_N)^2 &= \eta_{1N} \end{aligned} \quad (3.289)$$

The integration contour for dz_k encircles the point $z_k = z_{k-1}$, as indicated. That this multiparticle state satisfies the gauge conditions may also be checked explicitly.

It turns out to be very useful to consider the vertex for emission of the massless vector state of the unit intercept theory. The reason is that because $p^2 = 0$ the normal ordering difficulties completely disappear, as we have seen earlier in discussing the commutator $[L_n, V(p, z)]$.

It is straightforward to derive the vertex for massless vector emission. Putting

$$(p_1 + p_2)^2 = 0 \quad (3.290)$$

$$\therefore p_1 \cdot p_2 = 1 \quad (3.291)$$

we consider forming the massless vector from two tachyons by

$$\frac{V(p_1, z_1)}{z_1} \oint_{z_1} dz_2 \frac{V(p_2, z_2)}{z_2} \quad (3.292)$$

Use the commutators derived earlier to find

$$\begin{aligned} & \exp(\sqrt{2} i p_1 \cdot Q^{(+)}(z_1)) \exp(\sqrt{2} i p_2 \cdot Q^{(-)}(z_1)) = \\ & \exp(\sqrt{2} i p_2 \cdot Q^{(-)}(z_2)) \exp(\sqrt{2} i p_1 \cdot Q^{(+)}(z_1)) \left(1 - \frac{z_1}{z_2}\right)^{-2} \end{aligned} \quad (3.293)$$

Hence the z_2 integral has a double pole at $z_2 = z_1$.

Explicitly:

$$\begin{aligned}
 & \frac{V(p_1, z_1)}{z_1} \oint_{z_1} dz_2 \frac{V(p_2, z_2)}{z_2} = \\
 & = \oint_{z_1} dz_2 \frac{z_2}{z_1} \frac{1}{(z_2 - z_1)^2} \exp \left(i\sqrt{2} p_1 \cdot Q^{(-)}(z_1) + i\sqrt{2} p_2 \cdot Q^{(-)}(z_2) \right) * \\
 & * \exp \left(i\sqrt{2} p_1 \cdot Q^{(0)}(z_1) \right) \exp \left(i\sqrt{2} p_2 \cdot Q^{(0)}(z_2) \right) \exp \left(i\sqrt{2} p_1 \cdot Q^{(+)}(z_1) + \right. \\
 & \quad \left. + i\sqrt{2} p_2 \cdot Q^{(+)}(z_2) \right) \quad (3.294)
 \end{aligned}$$

$$\begin{aligned}
 & = \frac{\partial}{\partial z_2} \left[\frac{z_2}{z_1} \exp \left[i\sqrt{2} (p_1 \cdot Q^{(-)}(z_1) + p_2 \cdot Q^{(-)}(z_2)) \right] * \right. \\
 & * \exp \left(i\sqrt{2} [p_1 \cdot Q^{(0)}(z_1) + p_2 \cdot Q^{(0)}(z_2)] \right) \left(\frac{z_2}{z_1} \right)^{-p_1 \cdot p_2} \\
 & \left. \exp \left(i\sqrt{2} [p_1 \cdot Q^{(+)}(z_1) + p_2 \cdot Q^{(+)}(z_2)] \right) \right]_{z_1 = z_2} \\
 & \quad (3.295)
 \end{aligned}$$

$$= i\sqrt{2} p_2 \cdot \frac{dQ(z)}{dz} V(p_1 + p_2, z) \quad (3.296)$$

Since $p_2 \cdot (p_1 + p_2) = 0$ we may regard p_2 as the polarization vector and write, within an overall normalization,

$$V_\varepsilon(p, z) = \varepsilon \cdot \frac{dQ(z)}{dz} V(p, z) \quad (3.297)$$

Note that the factor $(z_1/z_2)^{-p_1 \cdot p_2}$ arises from the contraction

$$\left(\frac{z_1}{z_2}\right)^{-p_1 \cdot p_2} = \exp \left[- p_{1\mu} p_{2\nu} [Q_\mu^{(0)}(z_1), Q_\nu^{(0)}(z_2)] \right] \quad (3.298)$$

Now we are ready to construct positive-norm physical states. Let us put $p_{1\mu} = (0; 1, 0, 0, \dots)$ and introduce auxiliary light-like vectors

$$k_\mu = \left(\frac{1}{2}; -\frac{1}{2}, 0, 0, \dots \right) \quad (3.299)$$

$$k_{L\mu} = \left(\frac{1}{2}; \frac{1}{2}, 0, 0, \dots \right) \quad (3.300)$$

and let the vector momentum be $p_{2\mu} = n k_\mu$. Then

$$1 + (p_1 + p_2)^2 = 1 + (p_1 + nk)^2 = n \quad (3.301)$$

Now define

$$A_n^i = \frac{1}{\sqrt{2} \pi i} \oint \frac{P^i(z)}{z} V(nk, z) \quad (3.302)$$

In which $i = 2, 3, 4, \dots (d-1)$ and

$$P_\mu(z) = -iz \frac{d}{dz} Q_\mu(z) \quad (3.303)$$

$$= \sqrt{2} P_\mu + i \sum_{n=1}^{\infty} n (a^{(n)\dagger} z^{-n} - a^{(n)} z^n) \quad (3.304)$$

From the properties

$$(P^i(z))^\dagger = P^i(1/z) \quad (3.305)$$

$$(V(nk, z))^\dagger = V(-nk, 1/z) \quad (3.306)$$

$$d(1/z) = -dz/z^2 \quad (3.307)$$

we deduce that

$$A_n^{i\dagger} = A_{-n}^i \quad (3.308)$$

Define, in addition,

$$k.A_n = \frac{1}{\sqrt{2}\pi i} \oint \frac{k.P(z) V(nk, z) dz}{z} \quad (3.309)$$

Note that

$$\frac{d}{dz} (V(nk, z)) = -\sqrt{2} n \frac{k.P(z) V(nk, z)}{z} \quad (3.310)$$

and hence

$$k.A_n = \delta_{n0} \frac{1}{\sqrt{2}\pi i} \oint \frac{dz}{z} \cdot \sqrt{2} k.p_1 = \delta_{n0} \quad (3.311)$$

Where we have substituted $V(0, z) = 1$, and the explicit form for $p_1(z)$ written earlier.

For $p^2 = 0$, $V(p, z)$ transforms, like $Q_\mu(z)$, as a generalized projective scalar, namely

$$[L_n, V(p, z)] = -z^{-n+1} \frac{d}{dz} V(p, z) \text{ for } p^2=0 \quad (3.312)$$

$$[L_n, Q_\mu(z)] = -z^{-n+1} \frac{d}{dz} Q_\mu(z) \quad (3.313)$$

It follows that

$$[L_n, P_\mu(z) V(p, z)] = -z \frac{d}{dz} (z^{-n} P_\mu(z) V(p, z)) \quad (3.314)$$

$$\text{where we used the definition } P_\mu(z) = -iz \frac{d}{dz} Q_\mu(z). \quad (3.315)$$

In particular

$$[L_n, \frac{P_\mu(z) V(p, z)}{z}] = -\frac{d}{dz} (z^{-n} P_\mu(z) V(p, z)) \quad (3.316)$$

and it follows immediately that

$$[L_n, A_n^i] = [L_n, A_n^{i\dagger}] = 0 \quad (3.317)$$

which confirms that states created by A_n^i satisfy the physical gauge conditions.

Now consider the important commutator⁽²¹⁾

$$[A_m^i, A_n^j] = -\frac{1}{2\pi^2} \oint \frac{dz_1}{z_1} \frac{dz_2}{z_2} [P^i(z_1) V(nk, z_1), P^j(z_2) V(nk, z_2)] \quad (3.318)$$

$$= -\frac{1}{2\pi^2} \oint \frac{dz_1}{z_1} \frac{dz_2}{z_2} [P^i(z_1), P^j(z_2)] V(nk, z_1) * V(nk, z_2). \quad (3.319)$$

where we have used the fact that the $P^i(z_1)$, $P^j(z_2)$ commute with the $V(mk, z_1)$, $V(nk, z_2)$ and that the two V factors commute with each other. [These simplifications are the reason for considering the massless case].

Put $z_i = e^{i\theta_i}$ then

$$[Q_\mu(z_1 = e^{i\theta_1}), Q_\nu(e^{i\theta_2})] = i\pi g_{\mu\nu} \varepsilon(\theta_1 - \theta_2) \quad (3.320)$$

and it follows that

$$[P^i(z_1), P^j(z_2)] = -i\pi \delta_{ij} \frac{d}{d\theta_2} S(\theta_1 - \theta_2) \quad (3.321)$$

Noting that $dz_1/z_1 = i d\theta_1$, and that

$$\frac{d}{d\theta_2} V(nk, z_2) = -\sqrt{2} i k \cdot P(z_2) V(nk, z_2) \quad (3.322)$$

we find eventually

$$[A_m^i, A_n^j] = -\delta_{ij} \eta k \cdot A_{m+n} \quad (3.323)$$

$$= \eta \delta_{ij} \delta_{m+n,0} \quad (3.324)$$

To collect the results together, we now have transverse operators A_n^i satisfying

$$[L_n, A_n^i] = 0 \quad (3.325)$$

$$[A_n^i, A_m^{j\dagger}] = \eta \delta_{ij} \delta_{mn} \quad (3.326)$$

and hence these operators can be used to construct directly linearly-independent positive norm physical states. In space-time dimension d we can so construct $T^{(d-2)}(N)$ states at the level $R = N$. In particular, for $d = d_c$ we can make a complete basis for physical states in this way.

It is instructive to consider the infinite momentum limit of our A_n^i . That is, we act on a ground-state with momentum

$$p_\mu = (E; \sqrt{E^2 - 1}, 0, 0, \dots) \quad (3.327)$$

and let $E \rightarrow \infty$. Now fix

$$1 + (p + \lambda k)^2 = n \quad (3.328)$$

$$2\lambda p \cdot k = n \quad (3.329)$$

and hence

$$\frac{1}{\sqrt{2}\pi i} \oint \frac{dz}{z} P^i(z) V(\lambda k, z) \xrightarrow{E \rightarrow \infty} \quad (3.330)$$

$$\begin{aligned} &\rightarrow \frac{1}{\sqrt{2}\pi i} \oint \frac{dz}{z^{n+1}} \left(\sqrt{2} P^i + i \sum_{r=1}^{\infty} \sqrt{r} (a^{(r)\dagger} z^{-r} - a^{(r)} z^r) \right) \\ &= \sqrt{2n} i a_i^{(n)}. \end{aligned} \quad (3.331)$$

Thus we obtain just the transverse states created by the original $a^{(n)}$, $\bar{a}^{(n)+}$. We may say that the $d = d_c$ model appears as a genuine nonrelativistic oscillator in the infinite-momentum limit.

For $d < d_c$, we need further operators involving the longitudinal dimension in order to create all physical states; for full details of the construction we refer the reader to the literature [see Brower, Reference 15]. We may understand intuitively what is happening for $d < d_c$ by considering a simple example. The physical states at the level $R = 2$ are contained in a tensor (in the rest frame)

$$\left[a_i^{(0)+} a_j^{(0)+} - \frac{1}{25} \delta_{ij} (a^{(0)+} \cdot a^{(0)+}) \right] |0\rangle \quad (3.332)$$

where $i, j = 1, 2, 3, \dots, (d-1)$. For $d = d_c = 26$ this is a single irreducible representation of $O(25)$. For $d < d_c$ it is a linear combination of two irreducible representations of $O(d-1)$; for example for $d = 4$ it is a linear combination of a spin two and a spin zero. More generally, we may distinguish⁽²⁸⁾ between the genuine daughters which are present in the critical dimension and trace daughters which occur only in the sub-critical cases. These trace daughters arise simply because one is starting from the $O(25)$ irreducible representations of the model, reducing to a smaller representation space (smaller d) and reducing with respect to the new group $O(d-1)$.

This completes our study of the spectrum of the model, and we now turn to some other useful constructs of the operator formalism.

3h. Twisting Operator

It is useful to find an operator⁽²⁹⁾ $\Omega(\pi)$ which performs the function indicated in Figure 3, namely

$$\begin{aligned} & \Omega(\pi) V(p_1 z_1) V(p_2 z_2) \dots V(p_N z_N) |0\rangle \\ &= V(p_N z_1) V(p_{N-1} z_2) \dots V(p_1 z_N) |0\rangle \end{aligned} \quad (3.333)$$

with $\pi_\mu = - \sum_{i=1}^N p_{i\mu}$

The appropriate operator is^(6,29)

$$\Omega(\pi) = (-1)^R e^{-L_-(\pi)} \quad (3.334)$$

as can be checked by using the projective group properties

$$\begin{aligned} & \Omega(\pi) V(p_1 z_1) V(p_2 z_2) \dots V(p_N z_N) |0\rangle \\ &= (-1)^{L_0 + \pi^2} V(p_1, z_1 - 1) V(p_2, z_2 - 1) \dots V(p_N, z_N - 1) |0\rangle \end{aligned} \quad (3.335)$$

$$= (-1)^{\pi^2} V(p_1, 1 - z_1) V(p_2, 1 - z_2) \dots V(p_N, 1 - z_N) |0\rangle \quad (3.336)$$

$$= (-1)^N V(p_N, 1 - z_N) V(p_{N-1}, 1 - z_{N-1}) \dots V(p_1, 1 - z_1) |0\rangle \quad (3.337)$$

$$= V(p_N, z'_1) V(p_{N-1}, z'_2) \dots V(p_1, z_N) |0\rangle \quad (3.338)$$

as required. In the final step we simply changed variables to

$z'_i = 1 - z_{N-i+1}$, (which has Jacobian $(-1)^N$).

For the earlier steps we have recalled that L_- corresponds to

$$L_- \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (3.339)$$

and hence

$$e^{-L_-} V(p, z) e^{L_-} = V(p, z - 1) \quad (3.340)$$

Next we used

$$z^{-L_0} V(p, z) z^{L_0} = V(p, z) \quad (3.341)$$

to see that

$$(-1)^{L_0} V(p, z) = V(p, -z) (-1)^{L_0} \quad (3.342)$$

Finally we used the relationship, derived earlier, that

$$V(p, z) V(p', z') = V(p', z') V(p, z) (-1)^{-2p \cdot p'} \quad (3.343)$$

which in the present case gives rise to a phase

$$(-1)^{-2 \sum_j p_i \cdot p_j} = (-1)^{-\pi^2 + N} \quad (3.344)$$

An important property⁽⁶⁾ of the twisting operator is the fact that physical states are eigenstates, since

$$\Omega |\phi\rangle = (-1)^R e^{-L_0} |\phi\rangle \quad (3.345)$$

$$= (-1)^R |\phi\rangle \quad (3.346)$$

By combining with the propagator, we can arrive at a twisted propagator

$$\tilde{D} = D \Omega \quad (3.347)$$

which, however, is hermitian only if restricted to the physical subspace. That is, between physical states

$$\langle \phi | D \Omega | \phi' \rangle = \langle \phi | \Omega^\dagger D | \phi' \rangle \quad (3.348)$$

To obtain^(30,31,32) a hermitian twisted propagator in the full Fock space, it turns out to be useful to define

$$\Theta(x) = \Omega (1-x)^{L_0 - L_1} \quad (3.349)$$

whereupon the hermitian twisted propagator is

$$\tilde{D} = \int_0^1 dx \quad x^{-1-\alpha(s)+R} (1-x)^{-1+\alpha(b)} \Theta(x) \quad (3.350)$$

We refer to the literature for details. This hermitian twisted propagator now has the property that it does not propagate spurious states.

It is worth pointing out that the underlying projective group may be embodied into the operator formalism by writing the so-called canonical forms [References 33, 34]. [Reference 34 is also an operator-approach report which, unlike these lectures, concentrates almost exclusively on the $O(2,1)$ subalgebra of the generalized projective algebra.] Notice that

$$x^{L_0} = e^{-p^2 \ln x} x^R \quad (3.351)$$

$$= e^{-p^2 \ln x} : \exp \left[- \sum_{n=1}^{\infty} n a^{(n) \dagger} a^{(n)} (x^n - 1) \right] : \quad (3.352)$$

where we have used the general formula

$$x^{a^\dagger a} = : \exp [a^\dagger (x-1) a] : \quad (3.353)$$

which is easily checked by taking matrix elements between arbitrary coherent states,

$$\langle \beta | : \sum_{m=0}^{\infty} \frac{1}{m!} (a^\dagger (x-1) a)^m : | \alpha \rangle \quad (3.354)$$

$$= \langle \beta | \alpha \rangle e^{\beta^* \alpha (x-1)} = e^{\beta^* \alpha x} \quad (3.355)$$

$$= \langle \beta | x^{a^\dagger a} | \alpha \rangle \quad (3.356)$$

and then noting that the coherent states form an overcomplete basis.

More generally we may write the fundamental operators of the theory in the form (canonical form)

$$\sigma_\Lambda = \exp \left[- \sum_{n=1}^{\infty} a^{(n)†} A_n \right] : \exp \left[- \sum_{m,n=1}^{\infty} a^{(n)†} (C_{nm} - \delta_{nm}) a^{(m)} \right] : \exp \left[- \sum_{n=1}^{\infty} a^{(n)} B_n \right] e^{-\phi} \quad (3.357)$$

The matrix C_{mn} may be discovered from the relevant projective transformation

$$z \rightarrow z' = \Lambda z = \frac{az+b}{cz+d} \quad (3.358)$$

and it is given by

$$\sum_{m=1}^{\infty} C_{nm} \frac{z^m}{\sqrt{m!}} = \left[\left(\frac{az+b}{cz+d} \right)^n - \left(\frac{b}{d} \right)^n \right] \frac{1}{\sqrt{n!}} \quad (3.359)$$

This implies that

$$C_{nm} = \frac{1}{m!} \sqrt{\frac{m!}{n}} \left[\frac{\partial^m}{\partial z^m} \left(\frac{az+b}{cz+d} \right)^n \right] \Big|_{z=0} \quad (3.360)$$

For example, for the propagator

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \quad (3.361)$$

and

$$C_{nm} = \frac{1}{m!} \sqrt{\frac{m!}{n}} \frac{\partial^m}{\partial x^m} x^n \Big|_{x=0} \quad (3.362)$$

Of course, this is nothing more than a convenient prescription, but the group property becomes particularly evident when we multiply two operators

$$\sigma_\Lambda = \sigma_{\Lambda_1} \sigma_{\Lambda_2} \quad (3.363)$$

since the product operator is related to the two factors by

$$\phi = \phi_1 + \phi_2 + \sum_{n=1}^{\infty} B_{1n} A_{2n} \quad (3.364)$$

$$A = A_1 + C_1 A_2 \quad (3.365)$$

$$B = B_2 + B_1 C_2 \quad (3.366)$$

$$C = C_1 C_2 \quad (3.367)$$

As an application, consider the twisting operator Ω for which

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \quad (3.368)$$

Hence for this case

$$C_{nm} = (-1)^m \binom{n}{m} \sqrt{\frac{m}{n}} \quad (3.369)$$

and we may rewrite

$$\Omega(\pi) = (-1)^R e^{-L_-} \quad (3.370)$$

$$= \exp \left(\pi \cdot \sum_{n=1}^{\infty} \frac{a(n)^+}{\sqrt{n}} \right) * \quad (3.371)$$

$$* \exp \left(\sum_{m,n=1}^{\infty} \left[\binom{n}{m} \sqrt{\frac{m}{n}} (-1)^m - \delta_{mn} \right] a_n^+ \cdot a_m \right) \quad (3.372)$$

which is the form of $\Omega(\pi)$ originally proposed by Caneschi, Schwimmer and Veneziano. (29)

3i. Multiregion Vertex

Implicit in the set of multiparticle amplitudes A_N for external ground state particles are the amplitudes for excited external particles. The latter may be obtained from the former by factorization. It is convenient, therefore, to obtain an explicit operator expression from which the amplitudes for arbitrary particles may be directly obtained. The expression we shall use was discovered [Lovelace, 35; see also the modification by Olive, 36] as the culmination of a long series of paper (References 29, 33, 37-44).

The canonical formalism mentioned in the last section may be extended^(33,34) to canonical N-vertices of the general type (Fig. 4(a)).

$$V_N = \langle 0_1 0_2 \dots 0_p | \exp \left\{ \sum_{m,n=0}^{\infty} \sum_{i \neq j} \alpha_{im} D_{mn}(A_{ij}) \alpha_{jn} \right\} | 0_{p+1} 0_{p+2} \dots 0_N \rangle \quad (3.373)$$

Here $\alpha_i = a_i$ if i is a right-pointing leg and $\alpha_i = -a_i^+$ if i is left-pointing. We have introduced separately for each leg i the operators $a_i^{(n)}$, $a_i^{(n)+}$ satisfying

$$[a_{i\mu}^{(n)}, a_{j\nu}^{(m)+}] = -\delta_{ij} \delta_{mn} g_{\mu\nu} \quad (3.374)$$

Finally $D_{mn}(A_{ij})$ is the representation matrix, of a projective transformation A_{ij} , which is realized on the Fock space spanned by the $a_{i\mu}^{(n)+}$, $a_{i\nu}^{(n)}$.

Factorization of V_N into V_{p+1} and V_{N-p+1} according to Fig. 4(b) is easily seen to follow at once from the group properties. Inserting a complete set of intermediate states $|\lambda_e\rangle$ we arrive at

$$\begin{aligned}
 & \langle 0, 0_2, \dots, 0_p | \exp \left\{ \sum_{m,n=0}^{\infty} \sum_{i \neq j} \alpha_{im} D_{mn}(A_{ij}) \alpha_{jn} \right\} \\
 & \quad | 0_{p+1} 0_{p+2} \dots 0_N \rangle = \\
 & = \langle 0, 0_2, \dots, 0_p | \langle 0_e | \exp \left\{ \sum_{m,n=0}^{\infty} \left[\sum_{i \neq j=p+1}^N a_{im}^{\dagger} D_{mn}(A_{ij}^L) a_{jn}^{\dagger} - \right. \right. \\
 & \quad \left. \left. - \sum_{i=p+1}^N a_{im}^{\dagger} D_{mn}(A_{ij}^L) a_{en} \right] \right\} \sum_{\lambda_e} |\lambda_e\rangle \langle \lambda_e| \\
 & \quad \exp \left\{ \sum_{m,n=0}^{\infty} \left[\sum_{i \neq j=1}^p a_{im} D_{mn}(A_{ij}^R) a_{jn} - \sum_{i=1}^p a_{en}^{\dagger} D_{mn}(A_{ij}^R) a_{in} \right] \right\} \\
 & \quad | 0_e \rangle | 0_{p+1} 0_{p+2} \dots 0_N \rangle \quad (3.375)
 \end{aligned}$$

with the identifications

$$A_{ij} = A_{ij}^L \quad (p+1) \leq i, j \leq N \quad (3.376)$$

$$= A_{ij}^R \quad 1 \leq i, j \leq p \quad (3.377)$$

$$= \sum_k A_{ik}^L A_{kj}^R \quad \text{otherwise} \quad (3.378)$$

For the particular case of the multireggeon vertex, it can be shown^(3.5) that the transformation A_{ij} factorizes into

$$A_{ij} = X_i Y_j \quad (3.379)$$

where in terms of Koba-Nielsen z -variables

$$Y_j = \frac{1}{\sqrt{(z_{j-1} - z_{j+1})(z_j - z_{j+1})(z_j - z_{j-1})}} \begin{pmatrix} -z_{j-1}(z_j - z_{j+1}) & z_j(z_{j-1} - z_{j+1}) \\ -(z_j - z_{j+1}) & (z_{j-1} - z_{j+1}) \end{pmatrix} \quad (3.380)$$

and

$$X_i = \Gamma Y_i^{-1} \quad (3.381)$$

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with

$$\Gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3.382)$$

Since

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} (ad - bc)^{-1} \quad (3.383)$$

it follows that

$$X_i = \frac{1}{\sqrt{(z_{i-1} - z_{i+1})(z_i - z_{i+1})(z_i - z_{i-1})}} \begin{pmatrix} (z_i - z_{i+1}) & -z_{i-1}(z_i - z_{i+1}) \\ (z_{i-1} - z_{i+1}) & -z_i(z_{i-1} - z_{i+1}) \end{pmatrix} \quad (3.384)$$

and that

$$A_{ij} = \frac{a_{ij}}{\sqrt{(z_{i-1} - z_{i+1})(z_i - z_{i+1})(z_i - z_{i-1})(z_{j-1} - z_{j+1})(z_j - z_{j+1})(z_j - z_{j-1})}} \quad (3.385)$$

where

$$a_{ij} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (3.386)$$

$$A = (z_i - z_{i+1})(z_j - z_{j+1})(z_{i-1} - z_{j+1}) \quad (3.387)$$

$$B = (z_j - z_{i-1})(z_i - z_{i+1})(z_{j-1} - z_{j+1}) \quad (3.388)$$

$$C = (z_i - z_{j-1})(z_{i-1} - z_{i+1})(z_j - z_{j+1}) \quad (3.389)$$

$$D = (z_{j-1} - z_{j+1})(z_{i-1} - z_{i+1})(z_j - z_i) \quad (3.390)$$

Note, incidentally, that A_{ij} is manifestly projective invariant (consider, as usual, translations, dilations and inversions).

The N-reggeon vertex comprises the appropriate canonical N-vertex together with certain integration measure factors. The full expression is

$$V_N = \int \prod_{i=1}^N dz_i \left[\frac{dz_a dz_b dz_c}{(z_a - z_b)(z_b - z_c)(z_c - z_a)} \right]^{-1} \\ \prod_{i=1}^N (z_i - z_{i+1})^{\alpha(0)} (z_i - z_{i+1})^{-\alpha(0)-1}$$

$$\langle 0_1 0_2 \dots 0_N | \exp \left\{ \sum_{m,n=0}^{\infty} \sum_{i \neq j} a_{im} D_{mn}(X_i Y_j) a_{jn} \right\} \rangle \quad (3.391)$$

It is instructive to take the projection on the ground state particles, because we shall see how important is the role of the projective-spin-zero representation.

We take, as discussed earlier, the $D_{mn}^{(-\varepsilon, +)}$ matrices ($-J = \varepsilon \rightarrow 0+$) and calculate the expression

$$V_N \prod_{i=1}^N \left\{ e^{ip_i \cdot Q_i} |0_i\rangle \right\} \quad (3.392)$$

in which p_i is the momentum (incoming) of the i particle.

In terms of zeroth-mode operators $a^{(s)}$, $a^{(s)+}$ recall that

$$Q_\mu = \frac{1}{\sqrt{2\varepsilon}} (a_\mu^{(\varepsilon)} + a_\mu^{(\varepsilon)+}) \quad (3.393)$$

$$P_\mu = -i\sqrt{\frac{\varepsilon}{2}} (a_\mu^{(\varepsilon)} - a_\mu^{(\varepsilon)+}) \quad (3.394)$$

$$a_\mu^{(\varepsilon)} = \frac{i}{\sqrt{2\varepsilon}} (P_\mu - i\varepsilon Q_\mu) \quad (3.395)$$

$$a_\mu^{(\varepsilon)+} = -\frac{i}{\sqrt{2\varepsilon}} (P_\mu + i\varepsilon Q_\mu) \quad (3.396)$$

$$[a_\mu^{(\varepsilon)}, a_\nu^{(\varepsilon)+}] = -g_{\mu\nu} \quad (3.397)$$

$$[Q_\mu, P_\nu] = -ig_{\mu\nu} \quad (3.398)$$

Therefore only $D_{00}^{(\varepsilon,+)}(A_{ij})$ gives a contribution to the ground-state projection. Recall from our discussion of the projective group that for a transformation

$$z \rightarrow z' = \lambda z = \frac{az+b}{cz+d} \quad (3.399)$$

the representation matrix is calculable from

$$\begin{aligned} \sqrt{\frac{\Gamma(m+2\varepsilon)}{m!}} (az+b)^m (cz+d)^{-2\varepsilon-m} |\Delta|^\varepsilon \\ = \sum_{n=0}^{\infty} D_{mn}^{(-\varepsilon,+)}(\lambda) \sqrt{\frac{\Gamma(m+2\varepsilon)}{n!}} z^n \end{aligned} \quad (3.400)$$

with $\Delta = (ad-bc)$. It follows that

$$D_{00}^{(-\varepsilon,+)} = (cz+d)^{-2\varepsilon} |\Delta|^\varepsilon \quad (3.401)$$

$$= \exp \left[-2\varepsilon \ln(cz+d) + \varepsilon \ln |\Delta| \right] \quad (3.402)$$

$$= 1 - 2\varepsilon \ln \frac{d}{\sqrt{ad-bc}} + o(\varepsilon^2) \quad (3.403)$$

$$= 1 - \varepsilon \ln \frac{d^2}{(ad-bc)} + o(\varepsilon^2) \quad (3.404)$$

Hence, from our earlier expression for A_{ij} we deduce that

$$\begin{aligned} D_{00}^{(-\varepsilon,+)}(A_{ij}) = 1 - \varepsilon \ln \left[\frac{(z_{j-1} - z_{j+1})(z_{i-1} - z_{i+1})(z_j - z_i)(z_i - z_j)}{(z_i - z_{i-1})(z_j - z_{j-1})(z_i - z_{i-1})(z_j - z_{j-1})} \right] \\ + o(\varepsilon^2) \end{aligned} \quad (3.405)$$

Now use

$$\overline{X} = \prod_{i=1}^N \left\{ e^{ip_i Q_i} |0_i\rangle \right\} = \prod_{i=1}^N \left\{ \exp \left(\frac{p_i \cdot a_i^{(\varepsilon)+}}{\sqrt{2\varepsilon}} \right) |0_i\rangle \right\} \quad (3.406)$$

Using coherent state techniques it follows that

$$V_N \prod_{i=1}^N \left\{ e^{ip_i \cdot Q_i} |0_i\rangle \right\} =$$

$$\int \prod_{i=1}^N dz_i \left[\frac{dz_a dz_b dz_c}{(z_a - z_b)(z_b - z_c)(z_c - z_a)} \right]^{-1} \prod_{i=1}^N (z_i - z_{i+2})^{\alpha(0)}$$

$$\prod_{i=1}^N (z_i - z_{i+1})^{-\alpha(0)-1} M_N(p_i, z_i) \quad (3.407)$$

where

$$M_N(p_i, z_i) = \lim_{\varepsilon \rightarrow 0} \langle 0, 0_2, \dots, 0_N | \exp \left(- \sum_{i \neq j} a_i^{(\varepsilon)} \cdot a_j^{(\varepsilon)} \right) * \exp \left(\sum_{i \neq j} \varepsilon a_i^{(2)} \cdot a_j^{(\varepsilon)} \ln \left[\frac{(z_{j-1} - z_{j+1})(z_{i-1} - z_{i+1})(z_i - z_j)(z_j - z_i)}{(z_i - z_{i+1})(z_j - z_{j+1})(z_i - z_{i-1})(z_j - z_{j-1})} \right] \right) \rangle \quad (3.408)$$

Therefore, dropping an infinite constant

$$\exp \left(\sum_{i \neq j} \frac{p_i \cdot p_j}{2\varepsilon} \right) = \exp \left(\frac{N\alpha(0)}{4\varepsilon} \right) \quad (3.409)$$

we find

$$M_N(p_i, z_i) = \prod_{i \neq j} \left[\frac{(z_{j-1} - z_{j+1})(z_{i-1} - z_{i+1})(z_j - z_i)(z_i - z_j)}{(z_i - z_{i+1})(z_j - z_{j+1})(z_i - z_{i-1})(z_j - z_{j-1})} \right]^{-\frac{p_i \cdot p_j}{2}} \quad (3.410)$$

$$= \prod_{i=1}^N \left[\frac{z_i - z_{i+2}}{(z_i - z_{i+1})^2} \right]^{-\alpha(0)} \prod_{i < j} |z_i - z_j|^{-2p_i \cdot p_j} \quad (3.411)$$

Substituting back this expression for $M_N(p_i, z_i)$ we obtain for the projection of our N-reggeon vertex on to N ground states

$$\begin{aligned}
V_N & \prod_{i=1}^N \{ e^{i p_i \cdot Q_i} |0_i\rangle \} \\
&= \int \prod_{i=1}^N dz_i \left[\frac{dz_a dz_b dz_c}{(z_a - z_b)(z_b - z_c)(z_c - z_a)} \right] \prod_{i=1}^N (z_i - z_{i+1})^{-1+\alpha(0)} \\
& \quad \prod_{i < j} |z_i - z_j|^{-2p_i \cdot p_j}
\end{aligned} \tag{3.412}$$

which is the familiar N-point function.

In this calculation we should notice how critical it is to have the $J \rightarrow 0$ scalar representation in the multireggeon vertex, since it gives rise to the factors $[p_i \cdot p_j \ln(z_i - z_j)]$ in the exponent and hence to the basic pole-producing mechanism.

Finally we should note an alternative way⁽³⁵⁾ to write the matrix element in the N-reggeon vertex. We introduce

$$[c_{m\mu}, c_{nr}^\dagger] = -g_{\mu\nu} \delta_{mn} \tag{3.413}$$

$$Q_{n\mu}^{(i)} = i \sum_{m=0}^{\infty} \{ c_{m\mu}^\dagger D_{mn}(Y_i) + D_{mn}(X_i) c_{m\mu} \} \tag{3.414}$$

whereupon we can rewrite

$$\begin{aligned}
& \langle 0_1 0_2 \dots 0_N | \exp \left(\sum_{m,n=0}^{\infty} \sum_{i \neq j} a_i^m D_{mn}(X_i Y_j) a_j^n \right) \\
&= \langle 0_1 0_2 \dots 0_N | \langle 0_c | \prod_{i=1}^N : \exp \left(\sum_{k=0}^{\infty} Q_k^{(i)} a_{ik} \right) : | 0_c \rangle
\end{aligned} \tag{3.415}$$

where the normal ordering is with respect to the c-oscillators. This is merely a shorthand which takes advantage of the factorizability in $A_{ij} = X_i Y_j$. This final form was introduced in the original paper of

Lovelace.⁽³⁵⁾ It is easily checked by using coherent states and the Baker-Hausdorff formula, to find that

$$\begin{aligned} \langle 0_c | \prod_{i=1}^N : \exp i \left\{ \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \left[c_{mk}^{\dagger} a_{ip}^k D_{mk}(y_i) + \right. \right. \\ \left. \left. D_{km}(x_i) c_{mp} a_{ip}^k \right] \right\} : | 0_c \rangle \\ = \exp \left(\sum_{m,n=0}^{\infty} \sum_{i \neq j=1}^N a_i^m D_{mn}(x_i y_j) a_{jn} \right) \end{aligned} \quad (3.415)$$

as required.

To summarize, the multiregion vertex provides one useful construct in which to study the group properties. Both the factorization and cyclic symmetry of the theory are manifest. One rather strong result, as we have discussed above, is that the scalar representation of the projective group must be present in order to generate the usual pole producing factor

$$\prod_{i < j} |z_i - z_j|^{-2p_i p_j} \quad (3.416)$$

of the multiparticle dual amplitude.

Summary

By realizing the projective group on the Fock space spanned by harmonic-oscillator operators we are able to write the N-particle amplitudes in a form which exhibits manifestly the two basic properties of cyclic symmetry and factorizability.

An extension to the generalized projective group provides an infinite set of gauge conditions for the physical states in the case of unit intercept. The mechanism by which the gauge operators create

null states is similar to that occurring in quantum electrodynamics. The gauge relations remove sufficient states from the Fock space that the remainder have real coupling constants; the proof of this fact involves consideration that depend on the space-time dimensionality.

It is possible to find operators which are written in terms of a Cauchy integral over the vertex function and which create linearly independent positive-norm physical states.

In constructing the twisting operator and the multiregion vertex the projective group properties are again the guiding principle. In the general vertex the necessity of the projective-scalar representation, in order to produce resonance poles, becomes evident.

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(a)

(b)

FIG. 1

FIG. 2

FIG. 3

(a)

(b)

FIG. 4