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OPTIMAL NUCLEAR ROCKET ENGINE CONTROL

MASTER

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ABSTRACT

The problem of controlling a nuclear rocket engine so as to minimize propellant consumption is analyzed in this study. It is shown that the required control consists of a maximum allowable effort process. That is, reactivity and propellant flow rate must be varied so that one or both are at a maximum or minimum allowable value while the state of the system is inside some constraint boundary in the state space. To keep the trajectory on such a state-constraint boundary, however, it is required that reactivity and/or flow rate must vary appropriately. Though ostensibly a minimal-propellant consumption process, the control is also one of minimal time. Time-optimal neutronic control is discussed in Reference 1.

In addition to constraints in reactivity and flow rate, other necessary physical constraints (such as constraints in reactor power, rate of change of core temperature and pump performance) are considered. These constraints considerably complicate the analysis and synthesis of the optimal control. The analysis is made with Pontryagin's maximum principle (along with the extensions for state-constrained problems) and physical arguments.

In the analysis, the system dynamics are defined by a single-lump, direct-cycle, heat-exchange process and a mono-energetic neutron kinetics model. Such a system along with the hypothetical engine design configuration (which is used in this study) is discussed in Reference 3.

It should be recognized, however, that such a model only approximates the dynamic behavior of an actual engine. The optimal control, on the other hand, requires precise information in order that the state can be confined to some constraint boundary when necessary. Inaccuracies in the model and measurement (or estimation) of the state variables indicate that a suboptimal, closed-loop process may very well be in the best practical control. Such a closed-loop

system would also attenuate noise effects. With a knowledge of the optimal process, it is shown that conventional closed-loop control can be synthesized so that its performance nearly approximates the optimal performance, and such control may thereby be classified as suboptimal.

References

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INTRODUCTION

The problems of starting and controlling a simplified model of a nuclear rocket engine so as to minimize propellant consumption is analyzed in this study. Various constraints in the control and phase of the system are considered in the analysis. Utilizing the maximum principle, along with physical reasoning, shows that the optimal control is a maximum allowable effort process.

Though ostensibly a minimal-propellant process, the control derived here is also one of minimal time and requires a time-optimal neutronic control over part of the trajectory. Such optimal neutronic control is summarized in Reference 1 and more thoroughly analyzed in Reference 2.

Formulation of the Maximum Principle

Let us represent the system that is to be controlled by the vector differential equation,

$$\dot{\vec{x}} = \vec{f}(\vec{x}; \vec{u}), \quad (1)$$

where the dot represents differentiation with respect to time. Here $\vec{x}$ is the state vector with components $\vec{x}_1(t), \dots, \vec{x}_n(t)$ and $\vec{u}$ is the control vector with components $u_1(t), \dots, u_m(t)$. In this study it is assumed that $\partial f_i / \partial x_j$ are continuous in x_q and u_k for all i, j, k and q . Further, $\vec{u}(t)$ must be piecewise continuous and be in a closed region U , i.e., $\vec{u} \in U$.

An optimal control problem may be stated as follows: Given a process described by (1), determine an admissible control $\vec{u}(t)$ that transfers the process from some prescribed initial state $\vec{x}(t_0) = \vec{x}_0$ to some terminal state $\vec{x}(t_1) = \vec{x}_1$ in a manner designed to minimize an index of performance

$$J = \int_{t_0}^{t_1} C(\vec{x}; \vec{u}) dt. \quad (2)$$

For a solution trajectory $\vec{x}(t)$ to be optimal it is necessary that a costate vector $\vec{p}(t)$ is related to $\vec{x}(t)$ through Hamilton's equations. These equations are discussed below and relate the state vector to a costate vector

by means of a scalar function.

For convenience the system may be considered of order $n+1$ by letting C in equation (2) be f_{n+1} and adding the equation $\dot{x}_{n+1} = f_{n+1}$ to (1) with $x_{n+1}(t_0) = 0$ and $x_{n+1}(t_1) = J$. Then Hamilton's equations, which form the maximum principle, take the following form:

$$\dot{\bar{x}} = \bar{f}(\bar{x}; \bar{u}) = \frac{\partial \mathcal{R}}{\partial \bar{p}} \quad (3)*$$

[with $\bar{x}(t_0) = \bar{x}_0$ and $\bar{x}(t_1) = \bar{x}_1$]

and

$$\dot{\bar{p}} = - \frac{\partial \mathcal{R}}{\partial \bar{x}} \quad , \quad (4)$$

where $\bar{p}$ is a costate vector (not identically zero), $\bar{x}$ is a state vector and both are of order $n+1$. The scalar $\mathcal{R}$ in (3) and (4) is

$$\mathcal{R}(\bar{x}; \bar{p}; \bar{u}) = (\bar{p}, \bar{f}) = \sum_{i=1}^{n+1} p_i f_i \quad . \quad (5)$$

Then the Hamiltonian is

$$\mathcal{H}(\bar{x}; \bar{p}) = \sup_{\bar{u} \in U} \mathcal{R}(\bar{x}; \bar{p}; \bar{u}) \quad . \quad (6)$$

That is, the Hamiltonian is an absolute maximum with respect to all controls in the allowable set and the corresponding $\bar{u}(t)$ which maximizes $\mathcal{H}$ is the optimal control.

The presentation given here is only a summary of the excellent work given in Chapter 1 of Reference 3. The reader is referred to this work for precise theorems and explanations. It is shown in Chapter 1 of Reference 3

*For convenience, the partial derivative of a scalar with respect to a vector is considered to be a vector rather than a row matrix in this paper.

that the Hamiltonian is a constant and identically zero for the free end-time problem. Also p_{n+1} is a non-positive constant if equations (3) to (6) are satisfied. That is,

$$H [\bar{x}(t); \bar{p}(t)] \equiv 0 \quad (7)$$

and

$$p_{n+1} \leq 0 \quad (8)$$

From (4) and (5) it is seen that the costate system is adjoint to the original system (3). Hence

$$\dot{\bar{p}} = - \left[\frac{\partial \bar{f}}{\partial \bar{x}} \right]^T \bar{p} \quad (9)$$

where the superscript T refers to a matrix transpose.

The previous formulation has assumed that the terminal phase is fixed. Suppose the terminal phase is only confined to a specified smooth manifold. Then it is shown on pages 45-50 of Reference 3 that the costate vector must be orthogonal to the plane which is tangent to the manifold at the terminal end.

State-Variable Constraints

In addition to control constraints (i.e., $u \in U$), reactor processes must frequently be constrained in their state variables. Optimal control of processes with restricted phase co-ordinates is presented in Chapter 6 of Reference 3. This work is related to the classical calculus variations by L. D. Berkovitz⁴. More recent contributions in this area include publications by S. S. L. Chang⁵ and Bryson, Denham and Dreyfus⁶.

The object of this formal presentation is to summarize the necessary conditions (given in Chapter 6 of Reference 3) which are used in this study. The reader who is not interested in this complicated theory can proceed to

to the discussion section.

In addition to the restrictions placed on the allowable control region U above, assume that $\vec{u}(t)$ is composed of piecewise smooth components and furthermore, that the admissible control set satisfies certain regularity conditions in a neighborhood of its boundary points. These regularity conditions are defined below.

Let u_1 be an arbitrary boundary point of U which belongs to U and let $q_1(\vec{u})$ ($i=1, \dots, s$) be continuously differentiable scalar functions such that U is given by $q_1(\vec{u}_1) = 0$ ($i=1, \dots, s$) in the neighborhood of u_1 . Furthermore, the vectors $\partial q_1 / \partial \vec{u}$ ($i=1, \dots, s$) are linearly independent.

Let the region of allowed phase values be a smooth closed region B (in the n -dimensional phase space) defined by $S(\vec{x}) = 0$. $S(\vec{x})$, also, must have continuous second partial derivatives near the boundary $S(\vec{x}) = 0$ and the vector $\partial S(\vec{x}) / \partial \vec{x}$ must not vanish on the boundary.

As above the system is made to have order $n+1$ by letting $\dot{x}_{n+1} = f_{n+1}(\vec{x}; \vec{u}) = C(\vec{x}; \vec{u})$ with $x_{n+1}(t_0) = 0$ and $x_{n+1}(t_1) = J$. The optimal trajectory must connect $\vec{x}(t_0) = \begin{bmatrix} \vec{x}(t_0) \\ 0 \end{bmatrix}$ with $\vec{x}(t_1) = \begin{bmatrix} \vec{x}(t_1) \\ x_{n+1}(t_1) \end{bmatrix}$ so that $x_{n+1}(t_1)$ takes on the minimum possible value.

Every function which depends on $\vec{x}$ can be considered to be a function of $\vec{x}$; e.g., $S(\vec{x}) = S(\vec{x})$. Hence, let the closed region G in the $(n+1)$ -dimensional phase space be defined by $S(\vec{x}) \leq 0$.

For $S(\vec{x})$ negative, i.e., optimal motions not on the phase constraint boundary, the previous conditions stated by the maximum principle are valid. Necessary conditions for solutions to be optimal on the phase-constraint boundary,

$$S[\vec{x}(t)] \equiv S(\vec{x}) \equiv S(\bar{x}) \equiv 0, \quad (10)$$

are discussed below.

Equation (10) requires that all the time derivatives $S^{(k)}$ must vanish. Assume that the required $\vec{u}(t)$ may be computed from $\dot{S} = 0$, where

$$\dot{S}(\bar{x}; \vec{u}) = \frac{d}{dt} S(\bar{x}) = \left[\frac{\partial S(\bar{x})}{\partial \bar{x}} \right]^T \bar{f}(\bar{x}; \vec{u}) = 0. \quad (11)$$

Also, assume that the phase trajectory $\bar{x}$ is regular with respect to $\vec{u}_1$. That is, for each $\bar{x}$ on the trajectory $\bar{x}$ is regular, where regular means the following conditions are satisfied:

$$(1) \quad \dot{S}(\bar{x}; \vec{u}_1) = \dot{S}(\bar{x}; \vec{u}_1) = 0,$$

$$(2) \quad \frac{\partial \dot{S}(\bar{x}; \vec{u}_1)}{\partial \vec{u}_1} \neq 0$$

and

$$(3) \quad \frac{\partial \dot{S}(\bar{x}; \vec{u}_1)}{\partial \vec{u}_1}, \quad \frac{\partial q_i(\vec{u}_1)}{\partial \vec{u}_1} \quad (i = 1, \dots, s)$$

are linearly independent. If $\vec{u}$ [as determined by (11)] is an interior point, condition (3) doesn't appear. Let $\omega(\bar{x})$ designate the set of all admissible controls for which $\bar{x}$ is regular. If $\vec{u}_1$ is an interior point, $\omega(\bar{x})$ is a point.

With these assumptions, the maximum principle may be extended as follows (see pages 267 and 258 of Reference 3). If $\bar{x}(t)$ is an optimal trajectory of $\dot{\bar{x}} = f(\bar{x}; \vec{u})$ for corresponding $\vec{u}(t)$ and $\bar{x}$ is located entirely on the boundary of G , then a continuous $\bar{p}(t)$ and a piecewise smooth $\rho(t)$ exist such that

$$\dot{\bar{x}} = \bar{f}(\bar{x}; \vec{u}) = \frac{\partial R(\bar{x}; \bar{p}; \vec{u})}{\partial \bar{p}}, \quad (12)$$

$$\dot{\bar{p}} = - \frac{\partial R(\bar{x}; \bar{p}; \vec{u})}{\partial \bar{x}} + \rho \frac{\partial \dot{S}(\bar{x}; \vec{u})}{\partial \bar{x}} \quad (13)$$

and

$$H[\vec{x}(t); \vec{p}(t)] = \max_{\vec{u} \in \omega(\vec{x})} \mathcal{R}[\vec{x}(t); \vec{p}(t); \vec{u}(t)] = 0, \quad (14)$$

where as before $\omega(\vec{x}; \vec{p}; \vec{u}) = [\vec{p}, \vec{f}(\vec{x}; \vec{u})]$. $\rho(t)$ is determined from (14) as a Lagrange multiplier for $\partial \dot{S}(\vec{x}; \vec{u}) / \partial \vec{u}$ in

$$\frac{\partial \mathcal{R}(\vec{x}; \vec{p}; \vec{u})}{\partial \vec{u}} = \rho \frac{\partial \dot{S}(\vec{x}; \vec{u})}{\partial \vec{u}} + \sum_{j=1}^B \nu_j \frac{\partial g_j}{\partial \vec{u}}. \quad (15)$$

Also, $p_{n+1}(t)$ is a nonpositive constant; $\vec{p}(t_a+)$ is a nonzero vector and is not collinear with $\partial S(\vec{x}) / \partial \vec{x}$ and $\dot{p}(t)$ is nonpositive. t_a+ is the time at which the trajectory $\vec{x}(t)$ initially is on a phase-constraint boundary.

We now have necessary conditions for optimal trajectories on phase-space constraint boundaries. For interior points the maximum principle as presented previously yields the appropriate necessary conditions. Next consider the necessary conditions such optimal trajectories must have at their junction points (i.e., points where phase trajectories meet phase-constraint boundaries). Consider a junction time t_a such that the trajectory is on a phase-constraint boundary for $t > t_a$ and is not on a phase-constraint boundary for $t < t_a$. Then either

$$\vec{p}(t_a-) = \vec{p}(t_a+) + \mu \left. \frac{\partial S(\vec{x})}{\partial \vec{x}} \right|_{t=t_a} \quad (16)$$

or

$$\vec{p}(t_a-) - \mu \frac{\partial S(\vec{x})}{\partial \vec{x}} = 0, \quad \mu \neq 0, \quad (17)$$

where μ is some real number (of opposite sign to that of Reference 3). Hence, $\vec{p}(t)$ may be discontinuous at the junction time.

In References 3 and 6 it is noted that an arbitrary jump [proportional to $\partial S(\vec{x}) / \partial \vec{x}$] can be added to $\vec{p}$ during (t_a, t_b) . This arbitrary jump can be utilized (as is done in Reference 6) so that if the trajectory leaves the

phase-constraint boundary at $t = t_b$, then $\bar{p}(t)$ can be assumed continuous at $t = t_b$. The necessary costate conditions at t_a and t_b could be interchanged but following the convention of Reference 6, assume this arbitrary jump is utilized to make $\bar{p}(t)$ continuous at the exit corner, $t = t_b$. Hence,

$$\bar{p}(t_b^+) = \bar{p}(t_b^-) . \quad (18)$$

It is shown in Reference 6, also, that $R(\bar{x}; \bar{p}; \bar{u})$ is continuous at both entrance corners and exit corners.

DISCUSSION

The optimal control problem analyzed in this study may be stated as follows: Given an initial engine steady-state, bring the system to a desired terminal steady-state so as to minimize the consumption of propellant. For a nuclear rocket engine, pounds of propellant saved will allow less bulky propellant tanks and more payload. Although nuclear rockets allow much higher specific impulse than chemical rockets, their engines are heavier and more complex than their chemical counterparts. Minimization of the nuclear system weight becomes necessary.

A description of the dynamic behavior and operation for a nuclear rocket engine is presented in Reference 7. In this study we will neglect dynamics in the propellant pumping system and control-rod drive mechanisms. These systems can be compensated so that their response is considerably faster than that of the neutronics heat-exchange process. The chief constraint of the pumping system seems to be the region of allowable phase of the pump. Further, a minimum amount of coolant (propellant) flow is necessary to keep certain necessary structural elements sufficiently cool.

The classical mono-energetic spatially-independent neutron kinetic equations are used to represent the power generation in the reactor. For

operation near criticality, the reactor power generation is then approximated by

$$\frac{dQ}{dt} = \frac{\delta k - \beta}{l} Q + \sum_{i=1}^6 \lambda_i C_i \quad (19)$$

and

$$\frac{dC_i}{dt} = \frac{\beta_i}{l} Q - \lambda_i C_i \quad (i = 1, \dots, 6), \quad (20)$$

where Q is the rate of heat generation from fission, λ_i is the decay constant of the i th precursor group C_i (note units of C_i must be that of power), β_i is the portion of neutrons delayed due to the i th precursor group, δk is reactivity and l is the mean generation time of neutrons.

A qualitative description of the direct cycle heat-exchange dynamics may be represented by a one-stage model of the reactor (no axial or radial subdivisions)⁷. Such a one-stage model is represented below by equations (21) to (23). These equations may be formulated heuristically by writing an energy balance as follows. The heat rate generated from the fission process is equal to the rate at which heat is stored in the core (which results in a core wall temperature change) plus the rate at which heat is transferred from the core wall to the coolant. That is,

$$Q = MC \frac{dT}{dt} + hA(T - T_g), \quad (21)$$

where Q = heat rate generated from fission process, proportional to n ,
 MC = reactor core effective mass heat capacity, T = average core temperature,
 T_g = average coolant temperature, A = heat-transfer area, h = heat-transfer coefficient and t = time.

The rate of heat transfer to the coolant is equal to the rate of heat storage in the coolant plus the rate of increase in enthalpy of the coolant. That is,

$$hA(T - T_g) = w c_p \frac{dT_g}{dt} + \dot{w} c_p (T_0 - T_i) , \quad (22)$$

where w = weight of coolant in the core, c_p = specific heat of coolant, $\dot{w}$ = coolant weight flow rate, T_0 = coolant temperature at the core exit and T_i = coolant temperature at core entrance.

The average coolant temperature is a weighted average of the inlet and exit temperatures,

$$T_g = \frac{T_i + \theta T_0}{1 + \theta} , \quad (23)$$

where θ is a positive weighting factor dependent on the axial power distribution. $\theta = 2$ is found to be a good estimate for many reactors.

Equations (21) and (22) are generally nonlinear since the coefficients are functions of the corresponding temperature. For gaseous coolants the mass heat capacity term $w c_p \dot{T}_g$ is negligible and at high temperature the coefficients are nearly constant with the exception of the heat-transfer coefficient. Further, for nuclear reactors the resistance to heat transfer due to conduction through the core is usually negligible compared to the resistance due to convection through the fluid film. If the coolant is a gas and the flow is turbulent, the convective heat-transfer coefficient is approximately proportional to $\dot{w}^{0.8}$. With these assumptions and with the coolant weight flow rate specified as a function of time, the heat exchanger is a linear nonautonomous system.

From (22) and (23), with the coolant mass heat capacity and inlet temperature negligible, it is seen that

$$T \approx \left(\frac{c_p \dot{w}}{hA} + \frac{\theta}{1 + \theta} \right) T_0 . \quad (24)$$

Substitution of (24) into (21) with T_g related to T_0 by (23) shows that

$$\frac{dT}{dt} \approx \frac{Q}{MC} - \frac{T}{\tau_h} , \quad (25)$$

where (since $h \propto \dot{w}^{0.8} \approx \dot{w}$)

$$\tau_h = MC \left[\frac{\theta}{c_p \dot{w} (\theta + 1)} + \frac{1}{hA} \right] \approx \frac{1}{a \dot{w}} \quad (26)$$

and

$a =$ a heat-transfer constant.

Notice that (25) is approximately bilinear in $\dot{w}$ and T . This simple model is found to be a good approximation of the actual process if the effective mass heat capacity is selected according to the axial position of the temperature of interest.

The heat-exchange process is coupled to the neutron kinetics by temperature reactivity which may be approximated by

$$\delta k_t \approx c_t T, \quad (27)$$

where c_t is a temperature coefficient of reactivity (usually negative).

This coupling has a stabilizing effect on the power reactor. It is important to realize that the coupled neutronics heat-exchange process is a nonlinear system even if both processes are linear.

Along with the above variables (which are of interest for most reactors) thrust and specific impulse are of prime significance for a nuclear rocket engine. Thrust F is approximately derived from the stagnation pressure at the nozzle entrance P_c and the nozzle throat area A_t . That is,

$$F \approx c_f A_t P_c, \quad (28)$$

where c_f is a nozzle constant. Coolant or propellant temperature at the nozzle entrance (reactor exit) determines specific impulse I_{sp} as shown by

$$I_{sp} = \frac{F}{\dot{w}} = c_s (T_o / MW)^{\frac{1}{2}}, \quad (29)$$

where $c_s =$ a constant for the nozzle and $MW =$ propellant molecular weight⁸.

Equation (29) shows the importance of specific impulse to rocket propulsion. The promise of nuclear rockets and hydrogen propellant for space travel is shown by (29). Hydrogen is a good moderator and hence a density reactivity further couples the heat exchanger to the neutron kinetics. Assuming the propellant is gaseous in the core, the density may be computed from the gas equation:

$$\rho = P/RT_g, \quad (30)$$

where ρ = average propellant density in the core, R = gas constant and P = average propellant core pressure.

The density reactivity contribution is computed by

$$\delta k_\rho = k_\rho V \rho, \quad (31)$$

where k_ρ = propellant density coefficient of reactivity (positive) and V = reactor core void volume.

The flow is assumed choked through an exit nozzle and static conditions are approximately equal to stagnation conditions due to the low core exit mach number. Then pressure at the core exit is

$$P_c \approx c_n \dot{w} T_o^{\frac{1}{2}}, \quad (32)$$

where c_n = a nozzle constant⁸. With the approximations: $P \propto P_c$ and

$T \propto T_g$, the propellant density reactivity is

$$\delta k_\rho \approx \frac{c_\rho \dot{w}}{T^{\frac{1}{2}}}, \quad (33)$$

where c_ρ = a constant. The magnitude of propellant reactivity may very well be greater than that of the classical temperature reactivity. Although the reactivity of the propellant is positive, its derivative with respect to temperature is negative. This fact further enhances reactor stability.

A hypothetical nuclear rocket engine configuration (at rated design conditions) is presented in Table 1.

The computer studies which are to follow utilize the data given by this table unless otherwise specified.

Table 1 Hypothetical nuclear rocket rated design conditions

Thrust, F	100,000 lb
Specific impulse (with losses), I_{sp}	760 sec
Reactor power, Q	2260 megawatts
Propellant flow rate, $\dot{w}$	130 lb/sec
Reactor exit propellant temperature, T_0	4500°R
Nozzle throat area, A_t	61 in ²
Heat exchanger thermal time constant, τ_h	1.5 sec
Inlet Propellant temperature, T_i	120°R
Neutron mean-effective lifetime, λ	3 (10) ⁻⁵ sec
Propellant reactivity, δk_p	0.0065
Temperature reactivity, δk_t	-0.0065
Effective core mass heat capacity, MC	1140 $\frac{\text{Btu}}{^\circ\text{R}}$

Optimal Heat-Exchange Process

Consider the optimal start program of (25) between steady-state end points so as to minimize propellant consumption with flow rate constrained as follows:

$$u_a \leq \dot{w} \leq u_b \quad , \quad (34)$$

with $\dot{w}(t_0) = u_a$. u_a is a minimum flow rate necessary to cool sufficiently certain necessary structural elements. Although this constraint actually varies with temperature, it is not too well defined at this time and will be introduced as shown by (34). u_b is the maximum flow rate the coolant system can develop.

For the time being, assume that the reactor power Q can be changed

instantly with the following constraint:

$$\begin{aligned} 0 \leq Q \leq Q_m \\ \text{and} \quad Q(t_1) = Q_1, \end{aligned} \quad (35)$$

where Q_m is the rated design power of the reactor. The cost function for this problem is coolant flow rate. Then, with reference to equation (5), $(\bar{p}, \bar{f})$ is

$$\mathcal{R}(T; \bar{p}; Q, \dot{w}) = \dot{T}p_1 + \dot{w}p_2 = \left(\frac{1}{MC}Q - a\dot{w}T\right)p_1 + \dot{w}p_2 \quad (36)$$

and the Hamiltonian is

$$H(T; \bar{p}) = \sup_{\bar{u} \in U} \mathcal{R}(T; \bar{p}; Q, \dot{w}). \quad (37)$$

Equation (36) is made a maximum by

$$Q^0 = \begin{cases} Q_m & \text{for } p_1 > 0 \\ 0 & \text{for } p_1 < 0 \end{cases} \quad (38)$$

and

$$\dot{w}^0 = \begin{cases} u_a & \text{for } p_2 - aTp_1 < 0 \\ u_b & \text{for } p_2 - aTp_1 > 0 \end{cases}. \quad (39)$$

The costate is defined by the adjoint equation:

$$\dot{p}_1 = a\dot{w}p_1 \quad (40)$$

and p_2 is a non-positive constant. The solution to (40) is

$$p_1 = p_{10} \exp\left(a \int_{t_0}^t \dot{w} d\sigma\right), \quad (41)$$

where $p_{10} = p_1(t_0)$. p_1 cannot change sign since $a\dot{w}$ is always positive and there is no switching for $t_0 < t < t_1$. Hence, to increase temperature it is required that

$$Q^0 = Q_m \quad (42)$$

for $t_0 < t < t_1$ and p_{10} is positive.

Since the system (25) is initially at steady-state with the minimum coolant flow rate [i.e., $\dot{w}(t_0) = u_a$], the required initial power is

$$Q_0 = MC_a u_a T_0 \quad (43)$$

At the terminal time it is desired that $\dot{T}(t_1) = 0$, $T(t_1) = T_1$ and $Q(t_1) = Q_1$; then from (25)

$$\dot{w}(t_1) = \frac{Q_1}{MC_a T_1} \quad (44)$$

Since temperature is increased most rapidly by low coolant flow rate $\dot{w}$ and high reactor power Q [see (25) and (26)], it is obvious that the minimal coolant trajectory utilizes $\dot{w} = u_a$ between end points. Thus from (41)

$$p_1 = p_{10} \exp [a u_a (t - t_0)] \quad (45)$$

on the interval (t_0, t_1) .

For a step change in power $\Delta Q = Q_m - Q_0$ at $t = t_0$, the solution to (25) for $\dot{w} = u_a$ is

$$T = \left[T(t_0) - \frac{Q_m}{MC_a u_a} \right] e^{-a u_a (t - t_0)} + \frac{Q_m}{MC_a u_a} \quad (46)$$

For very small flow rate the later term in (25) is negligible and

$$T \approx T(t_0) + \frac{\Delta Q}{MC} (t - t_0) \quad (47)$$

Substituting (45) and (46) into the switching function, $p_2 - aTp_1$, shows that $\dot{w} = u_a$ for $t_0 \leq t < t_1$ (or the switching function has no zeros) if p_{10} is positive. Since $w \equiv 0$ [see (7)], it is required by (36) that $p_{20} = p_{10} [aT(t_0) - \{Q(t_1) - Q(t_0)\}/MCu_a]$. Thus the optimal process involves constant maximum power and constant minimum propellant flow rate between end points. Switching occurs at the end points to leave or reach the steady-state conditions. Such optimal start trajectories are presented

in Fig. 1.

An ideal optimal decrease in state with the above initial and terminal conditions reversed is caused by the above variation in power and flow rate with time reversed. Here, after heating may require u_g to be increased.

It is obvious from the system dynamics that the above minimal-propellant process is also one of minimal heat energy and minimal time. Although the power and coolant flow rate cannot be changed instantly, their change can be very rapid compared to the temperature response. Core temperature reactivity and propellant density reactivity coupling, along with neutron kinetics, are shown below to complicate the process.

Optimal Neutronics Heat-Exchange Process

The following analysis depends on an understanding of the system dynamics, as well as the maximum principle, in order to obtain an optimal trajectory. Total reactivity and coolant weight flow rate are treated as control variables with magnitude constraints. The end points of these control variables are specified by the end states in question. Further, power is constrained to have no overshoot and propellant consumption is the performance index minimized. Let us represent the system by equations (19), (20), (25) and (26) with

$$\delta k = u_1 = u_c + \delta k_t + \delta k_p \quad (48)$$

and

$$u_2 = \dot{w}, \quad (49)$$

where

$$\delta k_t \approx c_t T, \quad (50)$$

$$\delta k_p \approx c_p u_2 / T^{1/2} \quad (51)$$

FIGURE 1. IDEAL OPTIMAL STARTUP
OF A HEAT-EXCHANGE PROCESS

and u_c = control reactivity.

Suppose it is desired to start the system from steady-state Q_0, T_0 with $u_2 = u_a$ to some desired terminal steady-state Q_1, T_1 with power constrained such that $Q(t) < Q_1$. The cost function is u_2 and $(\bar{p}, \bar{f})$ is

$$R(\bar{x}; \bar{p}) = I(\bar{x}; \bar{p}) + p_1 Q u_1 / l + (p_9 - a p_8 T) u_2, \quad (52)$$

where the components of $\bar{x}$ are $Q, C_1, \dots, C_6, T, u_2$ and $I(\bar{x}; \bar{p})$ is that part of R not involving the control explicitly. Again the flow rate is constrained by (34). Suppose total reactivity is constrained for safety reasons such that

$$u_1 \leq \gamma_1 \beta$$

and by the finite amount available such that

$$-\gamma_2 \beta \leq u_1.$$

Although γ_2 may be larger than γ_1 , we will assume symmetrical limits for convenience.

$$|u_1| \leq \gamma \beta. \quad (53)$$

Then it is seen from (52) that the optimal control is

$$u_1^o = \gamma \beta \operatorname{sgn} p_1 \quad (54)$$

(where $\operatorname{sgn} p_1 = p_1 / |p_1|$) and

$$u_2^o = \begin{cases} u_a & \text{for } F < 0 \\ u_b & \text{for } F > 0 \end{cases}, \quad (55)$$

where

$$F = p_9 - a T p_8. \quad (56)$$

The optimal control reactivity is

$$u_c^o = \gamma \beta \operatorname{sgn} p_1 - (\delta k_t + \delta k_p). \quad (57)$$

Meanwhile

$$H(\bar{x}; \bar{p}) = \max_{\bar{u} \in U} R(\bar{x}; \bar{p}; \bar{u}) = 0. \quad (58)$$

With reactivity coupling the heat-exchange process to the neutron kinetics, equations (19), (48) and (51) show that either of the control variables could be used to change power level. Since there is a constraint on total reactivity, however, no increase in flow rate u_2 would permit a faster change of state than does $u_1 = \gamma\beta$ for startup and $u_1 = -\gamma\beta$ for shutdowns. In other words, the power is changed in minimum time if $u_1 = \gamma\beta$ and any increase in flow rate u_2 would require the process to use more propellant.

At the constraint surface $Q(t) = Q_1$, it is necessary to keep steady-state power. From (19) and (20), this requires that

$$u_1^o = \lambda \sum_{i=1}^6 \dot{C}_i / a_i. \quad (59)$$

Such open-loop type of control is found to be unstable in practice².

Conventional type of fast acting closed-loop control can be used effectively.

A closed-loop type of dither control can be synthesized which is even more accurate. An optimal engine start which utilizes a dither (multiple bang-bang) reactivity process is presented in Fig. 2. Here, reactivity is rapidly varied about the theoretical value defined by (59) according to an allowed error in $Q(t)$ from Q_1 ². At the desired terminal temperature, flow rate is increased to the required terminal value (to obtain designed thrust and hold temperature): $u_2(t_1) = Q_1 / MCaT_1 = 130$ lb/sec.

For the trajectories given in Fig. 2, temperature increases nearly linearly for (t_0, t_1) with $u_2 = u_a = 20$ lb/sec. Since core thermal stresses are roughly minimized by a linear variation in average core temperature, a power magnitude constraint may also be interpreted roughly as a $\dot{T}$ or core thermal-stress constraint for a minimal-coolant process. Such constraint might be less than the desired terminal power Q_1 .

FIGURE 2. MINIMAL PROPELLANT CONSUMPTION
STARTUP WITH DITHER TERMINAL CONTROL

The above optimal process is hypothesized by the maximum principle along with physical reasoning. It is necessary however that the proper costate equations have a solution which satisfies the maximum principle as extended for phase-constrained problems. These costate equations are analyzed below.

The costate is defined by the adjoint system for an interval of no phase constraint (i.e., $t < t_a$),

$$\dot{p}_1 = \frac{\beta - u_1}{\ell} p_1 - \sum_{\ell=1}^6 \frac{\beta_{\ell}}{\ell} p_{\ell+1} - \frac{p_8}{MC} \quad , \quad (60)$$

$$\dot{p}_j = \lambda_{j-1} (p_j - p_1) \quad (j = 2, \dots, 7), \quad (61)$$

$$\dot{p}_8 = a u_2 p_8 \quad (62)$$

and

$$\dot{p}_9 = 0 \quad . \quad (63)$$

Also, p_9 is non-positive.

Equations (62) and (63) show that F as given by (56) cannot change sign if $p_8(t_0)$ is non-negative since aT is positive. It is shown in Chapter II of Reference 2 that $p_1(t)$ cannot change sign (for the proper terminal conditions) for the minimal-time neutronic process with p_8 missing in (60). The behavior of p_1 is only changed slightly by the addition of a particular solution of slow transient due to $p_8(t)$, if $p_8(t_0)$ is selected small enough. Hence, one would expect that $\vec{p}(t_0)$ may again be chosen so that $p_1(t)$ does not change sign.

It is shown below in (46) that $p_j(t_a) = 0$ for $j = 2, \dots, 7$. This is also the terminal condition that yields no zeros for $p_1(t)$ in Chapter II of Reference 2.

While $Q(t) = Q_1$, it is seen from (13) that with $t_a < t < t_1$,

$$\dot{\bar{p}} = - \frac{\partial \mathcal{R}}{\partial \bar{x}} + \rho \frac{\partial \dot{S}}{\partial \bar{x}} , \quad (64)$$

where $S = Q - Q_1$. Since $\partial \dot{S} / \partial \bar{x} =$
$$\begin{bmatrix} (u_1 - \beta)/l \\ \lambda_1 \\ \vdots \\ \lambda_6 \\ 0 \end{bmatrix} ,$$

$$\dot{p}_1 = \frac{\beta - u_1}{l} p_1 - \sum_{i=1}^6 \frac{\beta_i}{l} p_{i+1} - \frac{p_8}{MC} + \frac{(u_1 - \beta)}{l} \rho , \quad (65)$$

$$\dot{p}_j = \lambda_{j-1} (p_j - p_1) + \lambda_{j-1} \rho , \quad (j = 2, \dots, 7) \quad (66)$$

$$\dot{p}_8 = a u_2 p_8 \quad (67)$$

and

$$\dot{p}_9 = 0 \quad (68)$$

As hypothesized above, let $u_1 = \frac{6}{l} \sum_{i=1}^6 \dot{C}_i / Q_1$ and $u_2 = \dot{w} = u_a$ for the interval (t_a, t_b) . Then from (15)

$$\frac{\partial \mathcal{R}}{\partial \bar{u}} = \rho \frac{\partial \dot{S}}{\partial \bar{u}} + v \frac{\partial q}{\partial \bar{u}} , \quad (69)$$

where

$$q = u_a - \dot{w} .$$

Substituting (52) into (69) one obtains:

$$\frac{p_1 Q}{l} = \frac{\rho Q}{l} \quad (70)$$

and

$$F = (p_9 - a p_8 T) = -v .$$

Hence, from (70)

$$\rho = p_1 \quad (71)$$

and from (67)

$$p_8 = p_{8a} \exp \left[a \int_{t_a}^t u_2(\sigma) d\sigma \right],$$

where $p_{8a} = p_8(t_a+)$ is the value of p_8 at the point where the trajectory enters the phase constraint boundary. Therefore,

$$F = p_9 - a p_{8a} T(t) \exp \left[a \int_{t_a}^t u_2(\sigma) d\sigma \right]. \quad (72)$$

Then since T and u_2 are positive, the switching function (72) cannot change sign if p_{8a} is non-negative.

The other switching function p_1 is described by

$$\dot{p}_1 = - \sum_{i=1}^6 \frac{\beta_i}{L} p_{i+1} - \frac{p_8}{MC}, \quad (73)$$

where

$$\dot{p}_j = \lambda_{j-1} p_j, \quad (j = 2, \dots, 7) \quad (74)$$

or

$$\dot{p}_1 = - \sum_{i=1}^6 \frac{\beta_i}{L} p_{i+1}(t_a) e^{\lambda_i(t-t_a)} - \frac{p_{8a}}{MC} \exp \left[a \int_{t_a}^t u_2(\sigma) d\sigma \right]. \quad (75)$$

Since the terminal manifold is defined by $T(t_1) = T_1$ and $Q(t_1) = Q_1$, $p_j(t_1) = 0$ (for $j = 2, \dots, 7$). Then

$$p_j(t) = 0 \quad (76)$$

and

$$\dot{p}_1 = - \frac{p_8}{MC}$$

or

$$p_1 = p_{1a} - \frac{1}{MC} \int_{t_a}^t p_8(\sigma) d\sigma, \quad (77)$$

where $t_a < t < t_1$ and $p_{1a} = p_1(t_a+)$. Substituting (71) into (77),

$$p_1 = p_{1a} - \frac{p_{8a}}{MC} \int_{t_a}^t \exp \left[a \int_{t_a}^{\sigma} u_2(\lambda) d\lambda \right] d\sigma. \quad (78)$$

From (16),

$$p_j(t_a-) = p_j(t_a+) \quad (79)$$

$$(j = 2, \dots, 9)$$

and

$$p_1(t_a-) = p_1(t_a+) + \mu,$$

where, since there is no constraint exit corner, μ may be set to zero. Hence, by selecting p_{1a} large compared to p_{8a} or p_{10} large compared to p_{80} , $p_1(t)$ remains positive. Furthermore, $\dot{p}(t) = \dot{p}_1(t)$ is nonpositive for $t_a < t < t_1$ and $\bar{p}(t_a)$ is not collinear with $\partial S(\bar{x})/\partial \bar{x}$ (as required).

In summary then the costate system of equations is satisfied by the optimal startup control if: 1) $p_8(t_0)$ is a small non-negative number; 2) p_9 is a negative constant; 3) $p_1(t_0)$ is selected positive and considerably larger than $p_9(t_0)$ so that $p_1(t)$ doesn't change sign; 4) $p_j(t_1) = 0$, $j = 2, \dots, 9$; and 5) the actual initial values of p_1 , p_8 , and p_9 should be selected so that $K = 0$.

The actual solution of the costate equation should again progress from the terminal end. By selecting the above conditions (1), (3) and (5) [along with (4) at the terminal end], the costate equations can be solved and there are no switchings in control between end points.

An example startup trajectory with steady-state power and temperature at the end points is presented in Table 2. Recognize that the switching functions $p_1(t)$ and $F(t) = p_9 - a T p_8$ are positive and negative respectively, between end points and hence the hypothesized constant control satisfies the maximum principle. These data were obtained from an IBM 7090 computer.

In addition to power constraint the coolant pumping system may be constrained as discussed next.

Table 2 Optimal Reactor Startup Trajectory with $Q(t) \leq Q_1$

Input Data:

$$u_1 = 0, t = 0.1 \text{ sec} = t_0$$

$$u_1 = 0.9 \beta; t = 0.1, p_1 = 0 \text{ and } Q \leq Q_1 = 2.142(10)^6 \text{ Btu/sec}$$

$$u_1 = \ell \sum_{i=1}^6 \dot{C}_i / Q_1, Q = Q_1$$

$$u_2 = u_g = 2 \text{ lb/sec}, T \leq T_1 \text{ and } F \leq 0$$

$$u_2 = Q_1 / MC_s T_1, T = T_1 = 4501.1^\circ \text{R}$$

$$K = 0, t_0 \leq t \leq t_1$$

$$a = 1.153(10)^{-2}, T(0) = 500^\circ \text{R}, \dot{Q}(0) = \dot{T}(0) = 0 = \dot{C}_i(0) \quad (i=1, \dots, 6).$$

$$\dot{T}(t_1) = \dot{Q}(t_1) = 0, \text{ other data given by Table 1.}$$

Time sec	Power 10^{-4} Btu/sec	Temperature 10^{-2}°R	$p_1(t)$ $10^6 \text{ sec}^2/\text{Btu}$	$p_8(t)$ $10^4 \text{ sec}/^\circ \text{R}$
0	1.3144	5.0000		
$t_0 = 0.1$	1.3144	5.0000	11.946	7.2139
0.2	13.253	5.0651	9.2838	7.2306
0.4	24.889	5.3720	5.6093	7.2640
0.6	42.642	5.9257	3.3932	7.2976
0.8	71.793	6.8790	2.0589	7.3313
1.0	119.91	8.4903	1.2663	7.3652
$t_g = 1.24$	214.20	11.911	5.0718	7.4101
1.4	214.20	14.868	4.9676	7.4375
1.6	214.20	18.549	4.8368	7.4718
1.8	214.20	22.213	4.7054	7.5064
2.0	214.20	25.860	4.5734	7.5411
2.2	214.20	29.490	4.4408	7.5759
2.4	214.20	33.104	4.3076	7.6110
2.6	214.20	36.701	4.1737	7.6461
2.8	214.20	40.281	4.0393	7.6815
$t_1 = 3.08$	214.20	45.011	3.8500	7.7751

Flow System Constraints

The coolant may be supplied from a pressurized tank or may be pumped from a low pressure supply tank. With regards to nuclear rockets, the latter is more desirable due to the saving in tank weight and the pump is driven by a gas turbine. The output flow rate of the system is constrained by the velocity of the flow control valve (which is very fast), mechanical stresses and certain regions of performance on the pump map. The following discussion assumes the first two constraints are negligible.

A hypothetical pump performance map is presented in Fig. 3. This is a plot of pressure rise across the pump (approximately proportional to nozzle entrance pressure) vs pump flow rate. Constant specific speed lines are parabolic on this map⁹. Temperature lines refer to the nozzle entrance and are approximately linear due to the nozzle equation (32). The quantity, specific speed, is defined according to

$$N_s = NQ^{1/2}/H^{3/4} \quad , \quad (80)$$

where N = pump speed (rpm), Q = flow rate (gpm) and H = pump pressure head (ft.)⁹.

The importance of specific speed may best be explained by a pump map (see Fig. 3). Here the flow system is constrained by regions of stall and cavitation. Above some specific speed N_{ss} the pump stalls and below some specific speed N_{sc} cavitation exists. Operation in either of the regions is not tolerable for any finite time. A typical optimal trajectory o-d-e-c in Fig. 3, with a bang-bang flow variation, shows a pump constraint violation. For such a case the temperature could only be allowed to increase to T_s . At point s mass flow rate and temperature would have to be increased so as to maintain constant specific speed N_{ss} . If flow rate were increased at some lower temperature or power, more coolant would be consumed upon reaching the

N_s = SPECIFIC-SPEED

T_1 = TERMINAL TEMPERATURE

T_0 = INITIAL TEMPERATURE

FIGURE 3. HYPOTHETICAL PUMP PERFORMANCE MAP

desired terminal temperature. At point e mass flow rate could be stepped to point c.

Suppose the pressure during stall specific speed operation is approximated by

$$P = b_s \dot{W}^2, \quad (81)$$

where b_s is a constant. Substitution of the nozzle equation (32) into (81) (with $T = T_0$) yields

$$\dot{W} = \frac{c'_n T^{1/2}}{b_s}, \quad (82)$$

where c'_n is a constant. Then for $T > T_s$, the optimal trajectory (as discussed above) is altered as follows. At $T(t_s) = T_s$ (where $t_a < t_s < t_b$)

$$u_2^o = u_a = \frac{c'_n T_s^{1/2}}{b_s}. \quad (83)$$

Assume $u_2(t)$ remains so small that temperature increases nearly linearly, i.e., $T = \alpha_m (t - t_s) + T_s$, where $\alpha_m = Q_1/MC$. Then it is seen from (82) that

$$u_2(t) \approx \frac{c'_n}{b_s} [\alpha_m (t - t_s) + T_s]^{1/2} \quad (84)$$

for $t_s < t < t_1$. At the terminal end, $T = T_1$ and $Q = Q_1$; then from (21) with $\dot{T} = 0$ recognize that flow rate must be increased to

$$u_2(t) = \frac{Q_1}{MCaT_1}. \quad (85)$$

In summary then, the optimal-control process as defined above should be pieced together continuously with (8) to (84) if the pump stall constraint must be considered.

It is possible to bypass some of the coolant back to the tank and only send the minimum mass flow rate to the reactor, but increased energy is required to pump this bypass flow. In a bleed-cycle or topping-cycle nuclear

rocket, the turbine gets this energy from the reactor⁷. In practice such a process requires more coolant than the previous one due to energy losses.

The arguments presented in this section are not rigorous, but the constraint problem is so complicated (involving vector control with a constraint explicit including both control and phase) that physical arguments are much simpler. Furthermore, the theory developed in Reference 3 and discussed above does not apply to optimization problems with constraints which involve both phase and control jointly.

Again in practice, a simple closed-loop suboptimal process would be desirable for most applications. Such closed-loop systems for a nuclear rocket engine are presented in Fig. 4 and 5. There the inputs are limited to meet the necessary constraints that were discussed above. Though the latter control system is simplified by the coupling from propellant reactivity (i.e., an increase in flow rate causes an increase in power), the more complicated system allows the necessary constraints to be applied in a simpler manner. Notice that control reactivity still appears in Fig. 4 and may be used to bias appropriately the total reactivity.

As mentioned previously, specific impulse (or temperature) and thrust (or pressure) are the variables of most interest to a rocket engine. Hence, these are the variables controlled directly in Fig. 4. Temperature and pressure determine flow rate from the nozzle equation (32). Therefore, flow rate can be held approximately constant by programming pressure demand from temperature according to the nozzle equation for constant flow rate. An alternate suboptimal scheme could use a closed $\dot{w}$ loop.

Regarding system malfunctions it might be desirable to derive one demand from another measured variable. Since the thrust loop can be considerably faster than the temperature loop, it seems better to derive thrust demand from temperature measured according to the necessary system constraint.

FIGURE 4. CLOSED-LOOP NUCLEAR ROCKET ENGINE
CONTROL SYSTEM WITH CONSTRAINED DEMANDS

FIGURE 5. A SIMPLE TYPE OF CLOSED-LOOP NUCLEAR
ROCKET ENGINE CONTROL SYSTEM
WITH CONSTRAINED INPUTS

CONCLUSIONS

Although safety considerations of many commercial reactors require more stringent constraints than those given here, the techniques may still be applied to the optimal-control design.

From the above study, it is seen that the minimal propellant control of a neutron-dynamics direct-cycle heat-exchange process, such as that of the nuclear rocket engine, is at least partly a bang-bang process. Though such control is one of minimal propellant, the process is also one of minimal time.

Physical constraints in maximum power level (approximately rate of change of core temperature or core thermal stress) and the propellant flow system cause the optimal control trajectories to be a connection of those due to bang-bang and continuous types of control.

Although the control has been assumed to be inertialess in this study the conclusions are readily extended to the actual system and the control is a maximum allowable effort process. In other words, the process is always operating on a control-constraint boundary or a phase-constraint boundary.

In practice, the state equations and the constraints are only approximations. The optimal process, on the other hand, requires precise information about the system. Hence, in actual applications suboptimal closed-loop control may be utilized advantageously to approximate the optimal process in a simple manner. Such control may be used for all operations (with the necessary constraints built into the control system) or merely while the trajectories must stay on a phase constraint or hold a desired terminal phase. A closed-loop dither control also may be utilized to maintain essentially constant power while precursor levels and temperature are not at steady state.

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