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DUAL MODELS FOR FOUR MESON BORN TERMS

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ABSTRACT

A general formula for a dual four meson Born amplitude whose integrand is invariant under an S_3 permutation group is presented.

Recently, Frampton has proposed a class of dual models for mesonic Born amplitudes¹⁾ (1,2). It enjoys many of the symmetry properties of the unit intercept B functions while at the same time it has a realistic trajectory and no tachyons. In this note, we present a general formula for the four meson amplitudes of this class.

The four point functions of this class have the representation

$$A_4(-\alpha_s, -\alpha_t) = \lambda \int_0^1 dx x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (1-x+x^2)^{\frac{\Sigma+1}{2}} \phi_4(\alpha_s, \alpha_t, \alpha_u, x) \quad (1)$$

where $\alpha_x = \alpha(0) + \alpha'x$ is the Regge trajectory, $\Sigma = \alpha_s + \alpha_t + \alpha_u$ and λ is a constant. The function ϕ_4 is invariant under an action of the symmetric group S_3 as indicated by

$$\begin{aligned} \phi_4(\alpha_s, \alpha_t, \alpha_u, x) &= \phi_4(\alpha_t, \alpha_u, \alpha_s, \frac{1}{1-x}) \\ &= \phi_4(\alpha_u, \alpha_s, \alpha_t, \frac{x-1}{x}) \\ &= \phi_4(\alpha_s, \alpha_u, \alpha_t, \frac{1}{x}) \\ &= \phi_4(\alpha_t, \alpha_s, \alpha_u, 1-x) \\ &= \phi_4(\alpha_u, \alpha_t, \alpha_s, \frac{x}{x-1}) \end{aligned} \quad (2)$$

Further, it is an entire function of $\alpha_s, \alpha_t, \alpha_u$ while the only singularities in x that it is allowed to have are poles at $\exp(\pm i\frac{\pi}{3})$.

Our general solution for ϕ_4 is as follows: Let us define the auxiliary functions (cf.(2))

$$\begin{aligned}
 \beta_s &= x(x-e^{i\frac{\pi}{3}})^{-1} (x-e^{-i\frac{\pi}{3}})^{-1} \\
 \beta_t &= (1-x)(x-e^{i\frac{\pi}{3}})^{-1} (x-e^{-i\frac{\pi}{3}})^{-1} \\
 \beta_u &= -x(1-x)(x-e^{i\frac{\pi}{3}})^{-1} (x-e^{-i\frac{\pi}{3}})^{-1}
 \end{aligned} \tag{3}$$

which transform in exactly the same way as $\alpha_s, \alpha_t, \alpha_u$ under S_3 . Let us also define

$$\begin{aligned}
 \bar{\alpha} &= \alpha_s \alpha_t + \alpha_t \alpha_u + \alpha_u \alpha_s \\
 \underline{\alpha} &= \alpha_s \alpha_t \alpha_u \\
 z &= \beta_s \beta_t \beta_u = \frac{x^2(1-x)^2}{(1-x+x^2)^3}
 \end{aligned} \tag{4}$$

Then

$$\begin{aligned}
 \phi_4(\alpha_s, \alpha_t, \alpha_u, x) &= S_1 + \left[\frac{1}{3}(2\alpha_s - \alpha_t - \alpha_u)(2\beta_s - \beta_t - \beta_u) + (\alpha_t - \alpha_u)(\beta_t - \beta_u) \right] S_2 \\
 &+ \left[\frac{1}{3}(2\alpha_s^2 - \alpha_t^2 - \alpha_u^2)(2\beta_s - \beta_t - \beta_u) + (\alpha_t^2 - \alpha_u^2)(\beta_t - \beta_u) \right] S_3 \\
 &+ \left[\frac{1}{3}(2\alpha_s - \alpha_t - \alpha_u)(2\beta_s^2 - \beta_t^2 - \beta_u^2) + (\alpha_t - \alpha_u)(\beta_t^2 - \beta_u^2) \right] S_4 \\
 &+ \left[\frac{1}{3}(2\alpha_s^2 - \alpha_t^2 - \alpha_u^2)(2\beta_s^2 - \beta_t^2 - \beta_u^2) + (\alpha_t^2 - \alpha_u^2)(\beta_t^2 - \beta_u^2) \right] S_5 \\
 &+ (\alpha_s - \alpha_t)(\alpha_t - \alpha_u)(\alpha_u - \alpha_s)(\beta_s - \beta_t)(\beta_t - \beta_u)(\beta_u - \beta_s) S_6
 \end{aligned} \tag{5}$$

The functions S_i are power series in $\bar{\alpha}$, $\underline{\alpha}$ and z :

$$S_i \equiv S_i(\bar{\alpha}, \underline{\alpha}, z) = \sum_{p, q, r \geq 0} c_{pqr}^{(i)} (\bar{\alpha})^p (\underline{\alpha})^q z^r \tag{6}$$

Eq. (5) is our principal result. It is further restricted by requiring the absence of ancestors and tachyons. We shall consider these restrictions later.

To prove (5), we note that the S_3 invariance of ϕ_4 implies that it is a superposition of functions of the form ϕ where

$$\begin{aligned} \phi(\alpha_s, \alpha_t, \alpha_u, x) = & \psi^{(S)}(\alpha_s, \alpha_t, \alpha_u) \chi^{(S)}(x) + \psi^{(A)}(\alpha_s, \alpha_t, \alpha_u) \chi^{(A)}(x) \\ & + \sum_{\rho, \sigma=1}^2 \psi_{\rho}^{(M)}(\alpha_s, \alpha_t, \alpha_u) g_{\rho\sigma} \chi_{\sigma}^{(M)}(x) \end{aligned} \quad (7)$$

Here $\psi^{(S)}$ is fully symmetric in $\alpha_s, \alpha_t, \alpha_u$ while $\chi^{(S)}$ is invariant under $x \rightarrow (1-x)^{-1} \rightarrow (x-1)x^{-1} \rightarrow x^{-1} \rightarrow (1-x) \rightarrow x(x-1)^{-1}$. Similarly, $\psi^{(A)}$ and $\chi^{(A)}$ carry the antisymmetric representation of S_3 while $\psi_{\rho}^{(M)}, \chi_{\rho}^{(M)}$ are the basis for the two-dimensional irreducible representation with the invariant metric g . [Both $\psi_{\rho}^{(M)}$ and $\chi_{\rho}^{(M)}$ transform in the same way under S_3 .]

The method for constructing the $\psi^{(i)}$ is known from work on two variable expansions of scattering amplitudes (3,4). Thus $\psi^{(S)}$ and $\psi^{(A)}$ are of the form

$$\begin{aligned} \psi^{(S)}(\alpha_s, \alpha_t, \alpha_u) &= \sum_{p, q \geq 0} E_{pq}^{(S)} \bar{\alpha}^p \underline{\alpha}^q \\ \psi^{(A)}(\alpha_s, \alpha_t, \alpha_u) &= (\alpha_s - \alpha_t)(\alpha_t - \alpha_u)(\alpha_u - \alpha_s) \sum_{p, q \geq 0} E_{pq}^{(A)} \bar{\alpha}^p \underline{\alpha}^q \end{aligned} \quad (8)$$

while the two dimensional representations are carried by functions either of the form

$$(\psi_1^{(M)}(\alpha_s, \alpha_t, \alpha_u), \psi_2^{(M)}(\alpha_s, \alpha_t, \alpha_u)) = (2\alpha_s - \alpha_t - \alpha_u, \alpha_t - \alpha_u) \sum_{p, q \geq 0} G_{pq}^{(1)} \bar{\alpha}^p \underline{\alpha}^q \quad (9)$$

or of the form

$$(\psi_1^{(M)}(\alpha_s, \alpha_t, \alpha_u), \psi_2^{(M)}(\alpha_s, \alpha_t, \alpha_u)) = (2\alpha_s^2 - \alpha_t^2 - \alpha_u^2, \alpha_t^2 - \alpha_u^2) \sum_{p, q \geq 0} G_{pq}^{(2)} \bar{\alpha}^p \underline{\alpha}^q \quad (10)$$

(or suitable linear combinations thereof).

From (9) or (10), the metric is checked to be given by

$$g = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \quad (11)$$

The form of $\chi^{(S)}$ may be found as follows. By hypothesis, ϕ_4 and hence $\chi^{(S)}$ is regular in x except for possible poles at $\exp(\pm i\frac{\pi}{3})$. The equations

$$\begin{aligned} \chi^{(S)}(x) - \chi^{(S)}(0) &= \chi^{(S)}\left(\frac{x}{x-1}\right) - \chi^{(S)}(0), \\ \chi^{(S)}(x) - \chi^{(S)}(0) &= \chi^{(S)}\left(\frac{1}{1-x}\right) - \chi^{(S)}(0), \\ \chi^{(S)}(x) - \chi^{(S)}(0) &= \chi^{(S)}\left(\frac{1}{x}\right) - \chi^{(S)}(0) \end{aligned} \quad (12)$$

and their derivatives at $x = 0$ show that $\chi^{(S)}(x) - \chi^{(S)}(0)$ has zeros of order at least two at $x = 0, 1$ and ∞ .

Also if $\chi^{(S)}(x) - \chi^{(S)}(0)$ is not identically zero, then $[\chi^{(S)}(x) - \chi^{(S)}(0)]^{-1}$ vanishes either at $\exp(i\frac{\pi}{3})$ or at $\exp(-i\frac{\pi}{3})$. Let us assume that $\chi^{(S)}(x) - \chi^{(S)}(0) \neq 0$. Then, invariance under $x \rightarrow x^{-1}$ implies that $[\chi^{(S)}(x) - \chi^{(S)}(0)]^{-1}$ vanishes at both these points. The

first two derivatives of

$$[\chi^{(S)}(x) - \chi^{(S)}(0)]^{-1} = [\chi^{(S)}(\frac{1}{1-x}) - \chi^{(S)}(0)]^{-1} \quad (13)$$

at $x = \exp(\pm i\frac{\pi}{3})$ show that they too vanish here.

In sum, if $\chi^{(S)}(x)$ is not a constant, $\chi^{(S)}(x) - \chi^{(S)}(0)$ has zeros of order at least two at $x = 0, 1$ and ∞ and poles of order at least three at $x = \exp(\pm i\frac{\pi}{3})$. We may therefore write

$$\chi^{(S)}(x) = \chi^{(S)}(0) + z \chi_1^{(S)}(x) \quad (14)$$

where the only singularities $\chi_1^{(S)}$ is allowed to have are poles at $x = \exp(\pm i\frac{\pi}{3})$. Since $z = \beta_s \beta_t \beta_u$ is invariant under S_3 , $\chi_1^{(S)}$ is invariant under S_3 as well. So we can repeat the argument we used for $\chi^{(S)}$. In this way, we find that $\chi^{(S)}$ is a polynomial in z :

$$\chi^{(S)}(x) = \sum_{v=0}^N k_v z^v \quad (15)$$

The function $\chi^{(A)}$ changes sign under $x \rightarrow x^{-1}$, $x \rightarrow 1-x$ and $x \rightarrow x(x-1)^{-1}$ and therefore has zeros at the points left fixed by these permutations, that is at $-1, 0, 1/2, 1, 2$ and ∞ . The argument used for $\chi^{(S)}$ also shows that unless $\chi^{(A)}(x) \equiv 0$, it has poles of order at least three at $\exp(\pm i\frac{\pi}{3})$. Therefore

$$\begin{aligned} \chi^{(A)} &= - \frac{2(x+1)x(x-\frac{1}{2})(x-1)(x-2)}{(1-x+x^2)^3} \bar{\chi}^{(S)}(x) \\ &= (\beta_s - \beta_t)(\beta_t - \beta_u)(\beta_u - \beta_s) \bar{\chi}^{(S)}(x) \end{aligned} \quad (16)$$

where $\bar{\chi}^{(S)}$ has the usual singularity structure. Further, from (16), we see that it is invariant under S_3 . Thus it is a polynomial in z .

The deductive method we have till now used is rather tedious when it is applied to find the general form of $\chi^{(M)}$. However, in

analogy to (9) and (10), we may guess that $\chi^{(M)}$ is either of the form

$$\begin{aligned} & (2\beta_s - \beta_t - \beta_u, \beta_t - \beta_u) \hat{\chi}^{(S)}(x) \\ & = \left(\frac{1-4x+x^2}{1-x+x^2}, \frac{1-x^2}{1-x+x^2} \right) \hat{\chi}^{(S)}(x) \end{aligned} \quad (17)$$

or of the form

$$\begin{aligned} & (2\beta_s^2 - \beta_t^2 - \beta_u^2, \beta_t^2 - \beta_u^2) \tilde{\chi}^{(S)}(x) \\ & = \left(\frac{2x^2 - (1-x)^2(1+x^2)}{(1-x+x^2)^2}, \frac{(1-x^2)(1-x)^2}{(1-x+x^2)^2} \right) \tilde{\chi}^{(S)}(x) \end{aligned} \quad (18)$$

where $\hat{\chi}^{(S)}$ and $\tilde{\chi}^{(S)}$ are polynomials in z . The proof in the appendix of (3) can now be easily adapted to show that (17) and (18) indeed exhaust all such functions $\chi^{(M)}$ of interest.

We have completed the demonstration that ϕ_4 has the representation (5).

The absence of ancestors is ensured by the condition

$$\left. \frac{\partial^n \phi_4}{\partial x^n} (\alpha_s, \alpha_t, \alpha_u, x) \right|_{\substack{x=0 \\ \alpha_t=n}} = \begin{cases} \text{Polynomial of degree } \leq n \\ \text{in } \alpha_s, n=1,2,3,\dots \end{cases} \quad (19)$$

We shall assume that $\alpha(0) > 0$. For a tachyon free model, we therefore require that

$$\phi_4(\alpha_s, \alpha_t, \alpha_u, x) = 0 \quad \text{at } \alpha_t = x = 0 \quad (20)$$

In the analysis of (19) and (20), for simplicity, we will only partially consider the possibility of cancellations between the different terms in (5).

From (19) and (5), we see that the sum in (6) must be restricted to $p + q \leq r$ for S_1 . If $\hat{S}_1(\bar{\alpha}z, \underline{\alpha}z, z)$ is a power series in its arguments, it follows that

$$S_1(\bar{\alpha}, \underline{\alpha}, z) = \hat{S}_1(\bar{\alpha}z, \underline{\alpha}z, z) \quad (21)$$

As the coefficient of S_6 carries angular momentum three at $\alpha_t = n$, (19) implies that

$$S_6(\bar{\alpha}, \underline{\alpha}, z) = z \hat{S}_6(\bar{\alpha}z, \underline{\alpha}z, z) \quad (22)$$

The coefficients of S_2 and S_4 at $x = 0$ are $2\alpha_t - 2\Sigma/3$. These coefficients thus do not generate ancestors. Therefore

$$S_i(\bar{\alpha}, \underline{\alpha}, z) = \hat{S}_i(\bar{\alpha}z, \underline{\alpha}z, z), \quad i = 2, 4 \quad (23)$$

The coefficient of S_3 at $x = 0$ is $2\alpha_t^2 - \frac{2}{3}(\alpha_s^2 + \alpha_t^2 + \alpha_u^2)$ while its linear term in x has angular momentum unity. The coefficient of S_5 at $x = 0$ is also $2\alpha_t^2 - \frac{2}{3}(\alpha_s^2 + \alpha_t^2 + \alpha_u^2)$ while its linear term in x is zero. So we add the term

$$\frac{2}{3}(\alpha_s^2 + \alpha_t^2 + \alpha_u^2) \quad (24)$$

to these coefficients and set

$$S_i(\bar{\alpha}, \underline{\alpha}, z) = \hat{S}_i(\bar{\alpha}z, \underline{\alpha}z, z), \quad i = 3, 5 \quad (25)$$

to eliminate ancestors. The addition of (24) preserves the S_3 invariance of the coefficients and makes them vanish at $\alpha_t = x = 0$. [Cf. (20)].

With the modifications (21-25) of the formula (5), the condition (20) for the absence of tachyons implies that the function

$$\hat{S}_1(\bar{\alpha}z, \underline{\alpha}z, z) - \frac{2}{3} \Sigma \hat{S}_2(\bar{\alpha}z, \underline{\alpha}z, z) - \frac{2}{3} \Sigma \hat{S}_4(\bar{\alpha}z, \underline{\alpha}z, z) \quad (26)$$

vanishes at $\bar{\alpha}z = \underline{\alpha}z = z = 0$ and may be written as $\tilde{S}_1(\bar{\alpha}z, \underline{\alpha}z, z) - \tilde{S}_1(0, 0, 0)$.

Eliminating $\hat{S}_1$ in favour of $\tilde{S}_1$ and calling $\tilde{S}_1$ once more as $\hat{S}_1$, we find finally

$$\begin{aligned} \phi_4(\alpha_s, \alpha_t, \alpha_u, x) = & [\hat{S}_1(\bar{\alpha}z, \underline{\alpha}z, z) - \hat{S}_1(0, 0, 0)] \\ & + \left[\frac{1}{3}(2\alpha_s - \alpha_t - \alpha_u)(2\beta_s - \beta_t - \beta_u) + (\alpha_t - \alpha_u)(\beta_t - \beta_u) + \frac{2}{3}\Sigma \right] \hat{S}_2(\bar{\alpha}z, \underline{\alpha}z, z) \\ & + \left[\frac{1}{3}(2\alpha_s^2 - \alpha_t^2 - \alpha_u^2)(2\beta_s - \beta_t - \beta_u) + (\alpha_t^2 - \alpha_u^2)(\beta_t - \beta_u) + \frac{2}{3}(\alpha_s^2 + \alpha_t^2 + \alpha_u^2) \right] \hat{S}_3(\bar{\alpha}z, \underline{\alpha}z, z) \\ & + \left[\frac{1}{3}(2\alpha_s - \alpha_t - \alpha_u)(2\beta_s^2 - \beta_t^2 - \beta_u^2) + (\alpha_t - \alpha_u)(\beta_t^2 - \beta_u^2) + \frac{2}{3}\Sigma \right] \hat{S}_4(\bar{\alpha}z, \underline{\alpha}z, z) \\ & + \left[\frac{1}{3}(2\alpha_s^2 - \alpha_t^2 - \alpha_u^2)(2\beta_s^2 - \beta_t^2 - \beta_u^2) + (\alpha_t^2 - \alpha_u^2)(\beta_t^2 - \beta_u^2) + \frac{2}{3}(\alpha_s^2 + \alpha_t^2 + \alpha_u^2) \right] \hat{S}_5(\bar{\alpha}z, \underline{\alpha}z, z) \\ & + (\alpha_s - \alpha_t)(\alpha_t - \alpha_u)(\alpha_u - \alpha_s)(\beta_s - \beta_t)(\beta_t - \beta_u)(\beta_u - \beta_s)z \hat{S}_6(\bar{\alpha}z, \underline{\alpha}z, z) \end{aligned} \quad (27)$$

It is instructive to rewrite (27) in terms of the functions

$$\begin{aligned} \phi_4^{(1)} &= \alpha_s \beta_s + \alpha_t \beta_t + \alpha_u \beta_u \\ \phi_4^{(2)} &= \alpha_s \beta_t \beta_u + \alpha_t \beta_u \beta_s + \alpha_u \beta_s \beta_t \\ \phi_4^{(3)} &= \alpha_s \alpha_t \beta_s \beta_t + \alpha_t \alpha_u \beta_t \beta_u + \alpha_u \alpha_s \beta_u \beta_s \\ \phi_4^{(4)} &= \alpha_s^2 \beta_s + \alpha_t^2 \beta_t + \alpha_u^2 \beta_u \\ \phi_4^{(5)} &= (\alpha_s - \alpha_t)(\alpha_t - \alpha_u)(\alpha_u - \alpha_s)(\beta_s - \beta_t)(\beta_t - \beta_u)(\beta_u - \beta_s) \end{aligned} \quad (28)$$

where $\phi_4^{(i)}$ ($i=1,2,3,4$) are taken from ref. 2. [The latter did not explicitly consider $\phi_4^{(5)}$ due to its triple poles at $\exp(\pm i\frac{\pi}{3})$.]

We find

$$\begin{aligned} \phi_4(\alpha_s, \alpha_t, \alpha_u, x) = & [\hat{S}_1(\bar{\alpha}z, \underline{\alpha}z, z) - \hat{S}_1(0, 0, 0)] \\ & + 2\phi_4^{(1)} \hat{S}_2(\bar{\alpha}z, \underline{\alpha}z, z) + 2\phi_4^{(4)} \hat{S}_3(\bar{\alpha}z, \underline{\alpha}z, z) \\ & + 2[\phi_4^{(1)} + \phi_4^{(2)}] \hat{S}_4(\bar{\alpha}z, \underline{\alpha}z, z) \\ & + 2[\Sigma\phi_4^{(2)} + \phi_4^{(3)} + \phi_4^{(4)}] \hat{S}_5(\bar{\alpha}z, \underline{\alpha}z, z) \\ & + \phi_4^{(5)} \hat{S}_6(\bar{\alpha}z, \underline{\alpha}z, z) \end{aligned} \quad (29)$$

We may also note that a recent ghost free, intercept zero model of Gervais and Neveu (5) may be found from (27) by setting $\hat{S}_i$ equal to zero for $i \geq 2$ and $\hat{S}_1$ equal to a certain (non-polynomial) function of z alone.

We have proved that for acceptable ranges of parameters, the model where $\phi_4 = \phi_4^{(1)}$ fulfills the positivity conditions studied first by Gribov and Pomeranchuk and later extensively by Martin and others (6). These are necessary conditions for the absence of ghosts. The proof tends to confirm the numerical work on residues in refs. (1,2). The details of this research will be presented elsewhere.

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FOOTNOTES

1. This class is the outgrowth of much previous work appropriate references to which may be found in (1) and (2).

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