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**COMPONENTS OF ELECTROMAGNETIC STRESS
TENSOR IN PLASMA**

J. W. Shearer

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COMPONENTS OF ELECTROMAGNETIC STRESS TENSOR IN PLASMA

Abstract

This report is a supplement to a recent paper by the author and James Eddleman which derived the forces exerted by a transverse electromagnetic wave on a fully ionized plasma. Here the stress

tensor formalism is more fully examined for the case of an approximately plane parallel wave. An erroneous derivation of the "pondermotive" forces is compared to a correct derivation of them.

Introduction

In a recent paper,¹ James Eddleman and I have derived the forces exerted by a transverse electromagnetic wave on a fully ionized plasma through which it is propagating. One way of expressing these forces (sometimes called "pondermotive" forces) is in terms of the divergence of a

stress tensor similar to the Maxwell stress tensor in vacuum. In this report, we will work out this stress tensor in component form for the case where the wave can be approximated by a plane parallel formulation of the electromagnetic equations.

Plane Polarized Transverse Wave in Plasma

In this section we will review some of the elementary properties of the electromagnetic wave in a fully ionized plasma. Consider a plane-polarized travelling wave in Cartesian coordinates:

$$E = E_x = E_0 \exp [i (\omega t - Kz)] + cc, \quad (1)$$

$$B = B_y = B_0 \exp [i (\omega t - Kz)] + cc, \quad (2)$$

where E_0 and B_0 are the rms values of electric and magnetic field, respectively. We neglect absorption and relativistic ef-

fects, in which case the equation of motion of an individual electron is

$$m\ddot{x} = -eE_x. \quad (3)$$

The coherent motion of the electron in the field of the electromagnetic wave can be expressed as an electron current J_e ,

$$J_e = -eN_e \dot{x} = -eN_e u_e \quad (4)$$

where N_e is the electron density, and where u_e is the "quivering velocity" of the

electrons. We neglect the motion of the more massive ions. From Eqs. (1) and (2), we obtain

$$u_e = i \frac{eE_0}{m\omega} \exp[i(\omega t - Kz)] + cc, \quad (5)$$

$$J_e = -i \frac{e^2 N_e}{m\omega} E_0 \exp[i(\omega t - Kz)] + cc. \quad (6)$$

In a fully ionized plasma there are no bound electrons, and so it is convenient to treat the electron current explicitly in the Maxwell equations, thus avoiding the use of polarization concepts. In our plane polarized geometry, this implies the wave equation,

$$\frac{\partial^2 E_x}{\partial z^2} - \mu \frac{\partial J_e}{\partial t} = \frac{1}{c^2} \frac{\partial^2 E_x}{\partial t^2}. \quad (7)$$

Substitution of Eqs. (1) and (6) into this equation yields the dispersion relation for transverse waves in the plasma,

$$\omega^2 = (Kc)^2 + \omega_p^2, \quad (8)$$

where the plasma frequency ω_p is given by

$$\omega_p^2 = e^2 N_e / \epsilon_0 m. \quad (9)$$

The phase velocity v_p of the wave is then given by

$$v_p = \frac{\omega}{K} = \frac{c}{\left[1 - \left(\frac{\omega_p}{\omega}\right)^2\right]^{1/2}} = \frac{c}{n_i}, \quad (10)$$

where n_i is the index of refraction of the plasma.

We note that when the plasma frequency ω_p is less than the laser frequency ω , the

phase velocity v_p is greater than c and the index of refraction is less than one. These are characteristic properties of these "underdense" plasmas. The principle of relativity is not violated by this result, since the energy is transported at the group velocity v_g , given by

$$v_g = \frac{\partial \omega}{\partial K} = c \left[1 - \left(\frac{\omega_p}{\omega}\right)^2\right]^{1/2}. \quad (11)$$

When the plasma frequency ω_p is greater than the laser frequency ω , the plasma is "overdense," the index of refraction n_i becomes imaginary, and the wave will not propagate. It is also useful to define a "cutoff density," N_c , as that critical electron density at which $\omega_p = \omega$. In terms of plasma densities, the index of refraction can then be written as

$$n_i^2 = 1 - \frac{N_e}{N_c}. \quad (12)$$

To find the magnetic field B_0 , we substitute Eqs. (1) and (2) into the Maxwell equation in these coordinates,

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}. \quad (13)$$

We obtain

$$B_0 = \frac{K}{\omega} E_0 = \frac{n_i}{c} E_0. \quad (14)$$

One notes that as the electron density N_e (in an underdense plasma) approaches the cutoff density N_c , the refractive index n and the field ratio B_0/E_0 both approach zero.

The average intensity I of a plane transverse wave is useful because it is an invariant in the absence of absorption or divergence. It is given by the equation for the time-averaged Poynting vector,

$$I = \frac{E_0 E_0}{\mu} = \epsilon n_i \epsilon_0 E_0^2, \quad (15)$$

where we have used Eqs. (1), (2) and (14), and where ϵ_0 is the dielectric constant in vacuum.

Finally, we wish to express the average energy densities of the fields and the electrons in terms of the average intensity I . The average electric field energy density U_E is found from Eq. (15),

$$U_E = \frac{\epsilon_0 E^2}{2} = \frac{I}{2n_i c}. \quad (16)$$

The average magnetic field energy density U_B is found from Eqs. (14) and (15):

$$U_B = \frac{B_0^2}{2\mu} = \frac{n_i I}{2c}. \quad (17)$$

The sum of these average energy densities is the average energy in the electromagnetic wave U_w ,

$$U_w = \left(\frac{1}{n_i} + n_i \right) \frac{I}{2c} \quad (18)$$

We are not yet finished, however. We must also consider the average kinetic energy density of the coherent electron current U_J which can be written

$$U_J = \frac{1}{2} N_e m u_0^2, \quad (19)$$

where u_0 is the rms value of the "quivering velocity," which is found from Eq. (A5) to be

$$u_0 = \frac{eE_0}{m\omega}. \quad (20)$$

Substituting Eq. (20) into Eq. (19) and making use of Eqs. (9), (12) and (16), one obtains

$$U_J = \left(\frac{1}{n_i} - n_i \right) \frac{I}{2c} \quad (21)$$

which expresses the average coherent electron energy density in the same parameters as for the electromagnetic field.

Thus, we can now write down the total energy density U_T (electromagnetic and kinetic) for the transverse wave in the plasma as the sum of Eqs. (18) and (21),

$$U_T = \frac{I}{n_i c}. \quad (22)$$

A plot of some of these ratios is given in Fig. 1.

We note that most other authors define a dielectric constant ϵ for a fully ionized plasma. Their formulas should be consistent with these results through the relation,

$$\epsilon = \epsilon_0 n_i^2. \quad (23)$$

Fig. 1. Energy density versus plasma density.

Component Form of the Transverse Wave Stress Tensor of a Plane Polarized Wave

Using Eq. (38) of our recent paper,¹ we write the equation of motion of the plasma in a slowly varying electromagnetic field where the electromagnetic momentum term can be neglected,

$$\rho \frac{d\mathbf{u}}{dt} \approx \langle \nabla \cdot \underline{\underline{S}} \rangle - \nabla P, \quad (24)$$

where ρ is the plasma density, $\mathbf{u}$ is its velocity, and P is its thermal pressure. The transverse wave stress tensor $\underline{\underline{S}}$ is given by

$$\underline{\underline{S}} = \epsilon \underline{\underline{E}} \underline{\underline{E}} + \frac{1}{\mu} \underline{\underline{B}} \underline{\underline{B}} - \frac{1}{2} \underline{\underline{I}} \times \left(\epsilon_0 E^2 + \frac{1}{\mu} B^2 \right), \quad (25)$$

as shown in Eq. (27) of our paper.¹

Consider the situation where the electromagnetic wave varies "slowly" in space. More precisely, we set up a local orthogonal coordinate system such that one can approximate the electromagnetic field by the plane transverse wave equations developed in the previous section (see Fig. 2). The conditions for this approximation are

$$E_x \gg E_y, E_z, \quad (26)$$

$$B_y \gg B_x, B_z. \quad (27)$$

Let us now evaluate the components S_{ij} of the stress tensor $\underline{\underline{S}}$, as given by

Fig. 2. Local coordinate system.

$$S_{ij} = \epsilon E_i E_j + \frac{1}{\mu} B_i B_j - \frac{\delta_{ij}}{2} \times \left(\epsilon_0 E^2 + \frac{1}{\mu} B^2 \right), \quad (28)$$

where δ_{ij} is unity when $i = j$, and zero otherwise. Using the approximations of Eqs. (26) and (27), we obtain:

$$S_{xx} \approx \left(\epsilon - \frac{\epsilon_0}{2} \right) E_x^2 - \frac{1}{2\mu} B_y^2, \quad (29)$$

$$S_{yy} \approx -\frac{1}{2} \epsilon_0 E_x^2 + \frac{1}{2\mu} B_y^2, \quad (30)$$

$$S_{zz} \approx -\frac{1}{2} \epsilon_0 E_x^2 - \frac{1}{2\mu} B_y^2. \quad (31)$$

However, from Eqs. (14) and (23), we have

$$\frac{B_y^2}{\mu} \approx \epsilon E_x^2 \quad (32)$$

from which we find

$$S_{yy} \approx S_{xx} \approx -\frac{1}{2} (\epsilon_0 - \epsilon) E^2, \quad (33)$$

$$S_{zz} \approx -\frac{1}{2} (\epsilon_0 + \epsilon) E^2, \quad (34)$$

where we have put $E_x \approx E$ in view of our plane-wave approximation [Eq. (26)].

Thus the transverse components of the stress tensor are equal in this approximation, but differ from the longitudinal component. We can also easily verify that the off-diagonal terms are small compared to these diagonal terms.

These results, however, do not yet constitute a solution for Eq. (24). That will be discussed in the following sections.

An Erroneous Derivation of the Electromagnetic Force

The derivation given here is one way that one can obtain a plausible but erroneous result for the components of the equation of motion. Several people, including the writer, were misled by such reasoning; it is hoped that this publication will serve to warn others.

The derivation is done by substituting Eqs. (33) and (34) back into the equation of motion (24), and calculating the x , y , z components of the force. One obtains:

$$\langle \underline{\nabla} \cdot \underline{S} \rangle_{xx} \stackrel{?}{=} -\frac{1}{2} \left\langle \frac{\partial}{\partial x} (\epsilon_0 - \epsilon) E^2 \right\rangle, \quad (35)$$

$$\langle \underline{\nabla} \cdot \underline{S} \rangle_{yy} \stackrel{?}{=} -\frac{1}{2} \left\langle \frac{\partial}{\partial y} (\epsilon_0 - \epsilon) E^2 \right\rangle, \quad (36)$$

$$\langle \underline{\nabla} \cdot \underline{S} \rangle_{zz} \stackrel{?}{=} -\frac{1}{2} \left\langle \frac{\partial}{\partial z} (\epsilon_0 + \epsilon) E^2 \right\rangle, \quad (37)$$

The diagonal components are small, and so are neglected.

However, the correct derivation of the force, given in Eq. (25) of our recent paper,¹ leads to a different and simpler result:

$$\langle \underline{\nabla} \cdot \underline{S} \rangle = -\frac{1}{2} (\epsilon_0 - \epsilon) \langle \underline{\nabla} E^2 \rangle. \quad (38)$$

(Remember, we are neglecting the electromagnetic momentum term.) Here the force has been reduced from a tensor to the gradient of a scalar; furthermore, the density-dependent function ϵ is outside the gradient operator.

What went wrong? Why are Eqs. (35)-(37) erroneous? The answer is found when one reexamines the small terms that were neglected. We should not have

dropped these small terms before taking the derivative because there is no reason to think that the derivatives of E_y and E_z , say, will be small compared to the derivatives of E_x .

In fact, one should expect that the derivatives will be the same order of magnitude for all three components. To understand this, remember that in our plasma the vectors $\underline{D}$ and $\underline{B}$ are solenoidal:

$$\underline{\nabla} \cdot (\epsilon \underline{E}) = 0 \text{ and } \underline{\nabla} \cdot \underline{B} = 0. \quad (39)$$

In component form, for example, we have

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0. \quad (40)$$

This equation implies that if $\partial B_y / \partial y$ is large, at least one of the other two spatial

derivatives must also be large. Similar reasoning can be applied to the vector $\underline{E}$. Thus, no matter how large one component may be, one cannot, in general, neglect any of the derivatives.

In the papers by Hora,² the longitudinal and transverse components of the force are equal to our erroneous longitudinal component:

$$\begin{aligned} \langle \underline{\nabla} \cdot \underline{S} \rangle &\stackrel{?}{=} -\frac{1}{2} \underline{\nabla} \cdot (\epsilon_0 E^2 + B^2 / \mu) \\ &\approx -\frac{1}{2} \underline{\nabla} \cdot (\epsilon_0 + \epsilon) E^2. \end{aligned} \quad (41)$$

We have not been able to determine the reasons why Hora's result differs from both our correct result [Eq. (38)], and our erroneous results [Eqs. (35)-(37)].

A Corrected Derivation in Component Form

We now return to the stress tensor $\underline{S}$, given in Eqs. (24)-(25). We want to find the divergence of this tensor, averaged over time. In this derivation, we do not assume anything about relative magnitudes; all components are retained and no plane parallel wave assumption is needed. Let us calculate the x-component:

$$(\underline{\nabla} \cdot \underline{S})_x = \frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{yx}}{\partial y} + \frac{\partial S_{zx}}{\partial z} \quad (42)$$

$$\begin{aligned} S_{xx} &= \epsilon E_x^2 + \frac{1}{\mu} B_x^2 - \frac{1}{2} \epsilon_0 E^2 \\ &\quad - \frac{1}{2} \frac{1}{\mu} B^2 \end{aligned} \quad (43)$$

$$S_{yx} = \epsilon E_y E_x + \frac{1}{\mu} B_y B_x \quad (44)$$

$$S_{zx} = \epsilon E_z E_x + \frac{1}{\mu} B_z B_x \quad (45)$$

$$E^2 = E_x^2 + E_y^2 + E_z^2 \quad (46)$$

$$B^2 = B_x^2 + B_y^2 + B_z^2. \quad (47)$$

Next we plug in Eqs. (43)-(47), and after some algebra, Eq. (42) becomes

$$\begin{aligned} (\underline{\nabla} \cdot \underline{S})_x &= \epsilon \left(E_x \frac{\partial E_x^*}{\partial x} + E_y \frac{\partial E_x^*}{\partial y} \right. \\ &\quad \left. + E_z \frac{\partial E_x^*}{\partial z} \right) - \epsilon_0 \frac{\partial}{\partial x} (E^2) \\ &\quad + \frac{B_y^*}{\mu} \left(\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} \right) \\ &\quad + \frac{B_z^*}{\mu} \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right). \end{aligned} \quad (48)$$

A number of terms were dropped because of the following Maxwell equations:

$$\mathbf{E}_x^* \left(\frac{\partial}{\partial x} (\epsilon \mathbf{E}_x) + \frac{\partial}{\partial y} (\epsilon \mathbf{E}_y) + \frac{\partial}{\partial z} (\epsilon \mathbf{E}_z) \right) = \mathbf{E}_x^* (\nabla \cdot \underline{\mathbf{D}}) = 0, \quad (49)$$

$$\frac{1}{\mu} \mathbf{B}_x^* \left(\frac{\partial \mathbf{B}_x}{\partial x} + \frac{\partial \mathbf{B}_y}{\partial y} + \frac{\partial \mathbf{B}_z}{\partial z} \right) = \frac{1}{\mu} \mathbf{B}_x^* (\nabla \cdot \underline{\mathbf{B}}) = 0. \quad (50)$$

Equations (49) and (50) are based on Eq. (39).

The next steps require that we use the remaining Maxwell equations and the time-averaged values of $\mathbf{E}$ and $\mathbf{B}$. We have

$$\frac{\partial}{\partial t} (\epsilon \underline{\mathbf{E}}) = \frac{1}{\mu} \nabla \times \underline{\mathbf{B}}, \quad (51)$$

where we have not treated the current $\mathbf{J}$ explicitly, as in the first section, but have used the plasma dielectric constant ϵ . Using only the time-varying part of Eq. (1), we obtain

$$i\omega \epsilon \mathbf{E}_z = \frac{1}{\mu} \left(\frac{\partial \mathbf{B}_x}{\partial z} - \frac{\partial \mathbf{B}_z}{\partial x} \right), \quad (52)$$

$$i\omega \epsilon \mathbf{E}_y = \frac{1}{\mu} \left(\frac{\partial \mathbf{B}_y}{\partial x} - \frac{\partial \mathbf{B}_x}{\partial y} \right), \quad (53)$$

where we have assumed that ϵ is not a rapidly varying function of time compared to the oscillations of the electromagnetic wave.

Similarly, we have the equation

$$-\frac{\partial \underline{\mathbf{B}}}{\partial t} = \nabla \times \underline{\mathbf{E}}. \quad (54)$$

From this, we obtain the relations

$$i\omega \mathbf{B}_y^* = \frac{\partial \mathbf{E}_x^*}{\partial z} - \frac{\partial \mathbf{E}_z^*}{\partial x}, \quad (55)$$

$$i\omega \mathbf{B}_z^* = \frac{\partial \mathbf{E}_y^*}{\partial x} - \frac{\partial \mathbf{E}_x^*}{\partial y}. \quad (56)$$

Using Eqs. (51)-(56), we can substitute back into the last two terms of Eq. (48) to express the right-hand side solely in terms of the vector $\mathbf{E}$. When one does this, two pairs of terms cancel out, and the final result becomes

$$(\nabla \cdot \underline{\mathbf{S}})_x = \frac{\epsilon - \epsilon_0}{2} \frac{\partial}{\partial x} (\mathbf{E}^2), \quad (57)$$

A similar result is obtained for the y and z components:

$$(\nabla \cdot \underline{\mathbf{S}})_y = \frac{\epsilon - \epsilon_0}{2} \frac{\partial}{\partial y} (\mathbf{E}^2), \quad (58)$$

$$(\nabla \cdot \underline{\mathbf{S}})_z = \frac{\epsilon - \epsilon_0}{2} \frac{\partial}{\partial z} (\mathbf{E}^2). \quad (59)$$

We will not show this here in detail, but one would expect it on general grounds since our original choice of the x -axis was arbitrary, independent of the particular values of the components $\mathbf{E}_x$, $\mathbf{E}_y$, $\mathbf{E}_z$, $\mathbf{B}_x$, $\mathbf{B}_y$, and $\mathbf{B}_z$. Thus by combining Eqs. (57)-(59) we obtain the correct result:

$$\langle \nabla \cdot \underline{\mathbf{S}} \rangle = \frac{1}{2} (\epsilon - \epsilon_0) \langle \nabla (\mathbf{E}^2) \rangle, \quad (60)$$

where the brackets indicate time-averaging which was introduced in Eqs. (40)-(54). This result agrees with Eq. (38) of this report, as it should. The term in the electromagnetic momentum that appears in Eq. (38) of our recent paper¹ does not appear here because we neglected the slow time variation of the average field when calculating Eqs. (51)-(56). The component method, as usual, is more arduous than the vector method used in Ref. 1.

Summary

We have considered at some length the tensor formulation of the forces exerted by an electromagnetic wave on a fully ionized plasma. The result given in a recent paper¹ is supported by a component derivation, and an erroneous derivation has been analyzed.

As a final remark, we should point out that for applications of this work it is unnecessary to use the tensor formulation in the plasma medium because of the simplicity of the result derived from it [Eqs. (38) and (60)].

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