LAWRENCE LIVERMORE LABORATORY
University of California/Livermore, California/94550

UCRL-51265

**FINITE BETA AND FINITE GEOMETRY EFFECTS ON
MICROINSTABILITIES IN MIRROR-CONFINED PLASMAS**

William M. Tang
(Ph.D. Thesis)

MS. date: August 18, 1972

NOTICE

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

MASTER

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

fy

Contents

Abstract	1
I. Introduction	2
1.1 General Comments on Plasma Stability	2
1.2 Motivation	3
1.3 Survey of Linear Microinstability Theory	3
1.3.1 Fundamental Considerations	3
1.3.2 Infinite Medium Analysis	5
1.3.3 Bounded Plasmas	8
1.3.4 Spatial Inhomogeneity in Finite Systems	10
1.3.5 Finite Beta Effects on Plasma Stability	12
1.4 Scope and Organization of Thesis	13
II. Flute Mode Formalism	15
2.1 General Approach	15
2.1.1 Basic Equations and Approximations	15
2.1.2 Particle Orbits	17
2.1.3 Integral Wave Equation	19
2.1.4 Density Response of Electrons and Ions	20
2.1.5 Flute Modes	22
2.2 Electromagnetic Modifications	24
2.2.1 Infinite Medium Analysis	24
2.2.2 High Frequency Modes	25
2.2.3 Low Frequency Modes	27
2.3 Further Justification of Finite Beta Flute Formalism	35
III. Dispersion Relation Derivation	40
3.1 Equilibrium Properties and Parametric Constraints	40
3.2 Finite Beta Modifications of Particle Orbits	41
3.3 Electromagnetic Flute Dispersion Relation	42
IV. Dory-Guest-Harris Instability	50
4.1 Introduction	50
4.2 Equilibrium Distributions	53
4.3 Dispersion Relation	56
4.4 Stability Analysis and Results	59
4.4.1 Instability Thresholds and Growth Rate	59
4.4.2 Energy Distributions from Experimental Data	61
4.4.3 Energy Distributions from Fokker-Planck Calculations	63
4.4.4 Analytic Results and Stability Requirements	64
V. Drift-Cone Instability	72
5.1 Introduction	72
5.2 Dispersion Relation	72

5.3	Stability Analysis and Results	74
5.3.1	Infinite Medium Limit	74
5.3.2	Finite Geometry Effects	78
VI.	Resonant Electromagnetic Instability	82
6.1	Introduction	82
6.2	Dispersion Relation	82
6.3	Stability Analysis and Results	83
6.3.1	Infinite Medium Limit	83
6.3.2	Finite Geometry Effects: Fluid Limit	87
6.3.3	Finite Geometry Effects: Ion Transit Analysis	91
VII.	Conclusions	98
	Acknowledgments	100
	References	101
	Appendix A. Phase Integrals	106
	Appendix B. Details in Dory-Guest-Harris Calculations	112
	Appendix C. Evaluation of Ion Transit Integrals	115

FINITE BETA AND FINITE GEOMETRY EFFECTS ON MICROINSTABILITIES IN MIRROR-CONFINED PLASMAS

Abstract

Linear microinstabilities in collisionless, axisymmetric, mirror-confined plasmas are considered. After a review of earlier work in this area, the appropriate formalism for treating flute type instabilities in a bounded, inhomogeneous, finite beta plasma is determined. Conditions for the applicability of the method to both low and high frequency modes are given, and it is shown that finite beta modifications do not alter the basic procedure, previously derived in the electrostatic limit. The general electromagnetic flute dispersion relation including finite beta corrections to the equilibrium orbits of the particles is then derived and applied to the analysis of the Dory-Guest-Harris, drift-cone, and resonant electromagnetic instabilities.

In addition to finite geometry and finite beta modifications, the stability analysis of the Dory-Guest-Harris mode, which is sensitive to details of the particle distribution, includes effects due to the presence of an ambipolar potential and involves the use of both theoretically-evaluated and experimentally-measured distributions. Stability criteria from numerical as well as analytic calculations are obtained and applied to the 2X experiment. It is found that, within the uncertainty of the data, the existence of this instability in the 2X plasma is a borderline proposition, and that at higher mirror ratios ($R_0 \geq 3$), the mode should be eliminated.

For the case of the drift-cone instability, which had previously been analyzed only in the electrostatic limit, it is demonstrated that finite beta effects can stabilize the mode. An analytic expression for the stability criterion is derived using a mirror ratio dependent loss-cone ion distribution. This result is then used to determine beta limits for stability, which prove to be insensitive to changes in mirror ratio and plasma temperature. Application of the analysis to experiment indicates that the 2X system should be unstable to the drift-cone mode and that a plasma radius on the order of 50 ion Larmor radii is required for stability.

In the analysis of the resonant electromagnetic instability, it is first shown that this mode can be derived analytically in the infinite medium limit. This is verified by the close agreement between the analytic and previously-calculated numerical results. It is then found that the inclusion of detailed finite geometry effects lead to stabilization. Application of the results indicate this highly resonant electromagnetic mode should not appear in existent mirror systems.

Chapter I

Introduction

1.1 GENERAL COMMENTS ON PLASMA STABILITY

A plasma is in a state of thermal equilibrium when it has filled the entire space available to it and has relaxed to a Maxwellian distribution. It is therefore clear that a hot, confined plasma does not exist in such a state. In controlled fusion research attention is accordingly focused on the question of how long the inevitable approach to equilibrium can be delayed. An obvious relaxation mechanism is that of collisions, which occur at rates proportional to $T^{-3/2}$ at the high temperatures (T) of fusion interest.^{1,2} A far more dangerous threat to confinement, however, is the rapidly growing disturbances due to the charged particles in the plasma. Since the characteristic time scales of these phenomena are much shorter than classical collision periods, the particle distribution can be considered as steady state to a good approximation. Growing disturbances, which remove the plasma from such an "equilibrium" state, are called instabilities,³ and are customarily divided into two general categories, macroinstabilities and microinstabilities.

Macroinstabilities^{3,4} are associated with the production of large-scale fields that lead to swift destruction of confinement. They are derived from the fluid equations of a plasma with characteristic frequencies much less than either the ion-cyclotron frequency and the electron-plasma frequency, and wavelengths much larger than the ion Larmor radius. Fortunately, it has been demonstrated⁵ that these gross disturbances can be eliminated in confined plasmas. This is accomplished in mirror systems⁶ by placing the plasma in a magnetic field that increases in strength in every direction away from the plasma. Such "magnetic well" or "minimum-B" configurations provide magnetic pressure that impedes gross motion of the plasma away from the confinement region.

Microinstabilities^{3,7} are smaller-scale disturbances that result from the interaction between particles and waves. Since one must account for detailed particle behavior in the analysis of such modes, the fluid treatment is no longer adequate, and it becomes necessary to consider the particle distribution. The principal sources of free energy⁸ available to drive microinstabilities in a hot, confined plasma are (1) anisotropy in the velocity distribution; (2) inhomogeneity in density, magnetic field, and temperature; and (3) diamagnetic field energy. In the presence of these energy reservoirs, particles tend to surrender a portion of their energy to the waves, thus amplifying the disturbances. This occurs, for example, when the frequency of a wave corresponds to some natural period of particle motion such as the ion-cyclotron frequency. If the free energy available is substantial, this resonant interaction enables the disturbance to grow to large amplitudes. From linear stability analysis it is found that microinstabilities have characteristically short wavelengths and frequencies

typically on the order of the ion-cyclotron frequency. As a result, the kinetic energy of motion, even for small amplitude disturbances, soon exceeds the energy available to drive the instabilities.⁸ At this stage there should be a qualitative deviation from linear behavior. In short, modes are expected to evolve quickly into the nonlinear regime where their growth is limited. So, in contrast to macroinstabilities, microinstabilities are associated with small-scale fluctuating fields which give rise to diffusion of particles out of the magnetic trap. Although not as catastrophic as the gross disturbances, this class of instabilities is nevertheless quite dangerous to plasma confinement and is the subject of much study in fusion research.

1.2 MOTIVATION

As noted in Refs. 3, 7, and 8, linear theory predicts the possible existence of a large variety of microinstabilities. Despite the fact that many of these modes have been experimentally identified and eliminated, it is generally suspected that a confined plasma will never be completely free of all such disturbances. The practical goal is to reduce the free energy reservoirs to the point where they cannot stimulate instabilities that would be strong enough to prevent the production of net thermonuclear power. So, the basic theoretical questions involve (1) the identification of which sources of free energy lead to disturbances that are the most detrimental to confinement; and (2) the determination of what restrictions on the dimensions, distribution, and other physical characteristics of the plasma are necessary to suppress or at least limit the disruptive effect of the instabilities. The present research project is addressed to this general problem with the particular aim of obtaining stability criteria for mirror devices. In the following section a review of previous work in this area is given.

1.3 SURVEY OF LINEAR MICROINSTABILITY THEORY

1.3.1 Fundamental Considerations

The microscopic treatment of a plasma entails the determination of the distribution function $f_j(\mathbf{r}, \mathbf{v}, t)$ for particles of species j in six-dimensional phase space. To do this it is necessary to solve the Boltzman equation

$$\frac{\partial}{\partial t} f_j + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} f_j + \frac{1}{m_j} \mathbf{F}_j \cdot \frac{\partial}{\partial \mathbf{v}} f_j = \left(\frac{\partial f_j}{\partial t} \right)_c, \quad (1.1)$$

where $\mathbf{F}_j$ is the force on a particle, m_j its mass, and $(\partial f_j / \partial t)_c$ is the rate of change of the distribution function due to collisions. In a hot plasma field fluctuations arising from collective plasma oscillations can occur on a time scale short compared to the time required for large-angle Coulomb scattering. In this regime of very infrequent collisions, it is appropriate to drop the collision term in Eq. (1.1), thus giving the collisionless Boltzmann or Vlasov equation.⁹ Use of this equation is justified¹⁰ provided

$$n_0 \lambda_D^3 \gg 1, \quad (1.2)$$

where $\lambda_D = (\kappa T / 4\pi n_0 e^2)^{1/2}$ is the Debye length, and n_0 is the plasma density. Since F_j is due primarily to electromagnetic forces specified by Maxwell's equations, the group of equations to be analyzed take the form

$$\frac{\partial}{\partial t} f_j + \underline{v} \cdot \frac{\partial}{\partial \underline{r}} f_j + (e_j/m_j) \left(\underline{E} + \frac{1}{c} \underline{v} \times \underline{B} \right) \cdot \frac{\partial}{\partial \underline{v}} f_j \quad (1.3)$$

$$\nabla \cdot \underline{E} = \sum_j 4\pi e_j \int d^3v f_j \quad (1.4)$$

$$\nabla \cdot \underline{B} = 0 \quad (1.5)$$

$$\underline{v} \times \underline{E} = -\frac{1}{c} \frac{\partial}{\partial t} \underline{B} \quad (1.6)$$

$$\underline{v} \times \underline{B} = \frac{1}{c} \frac{\partial}{\partial t} \underline{E} + \frac{4\pi}{c} \sum_j \int d^3v \underline{v} f_j. \quad (1.7)$$

In stability studies the concern is with small departures from equilibrium. Hence one can take $f_j = f_{0j} + f_{1j}$, $\underline{B} = \underline{B}_0 + \underline{B}_1$, and $\underline{E} = 0 + \underline{E}_1$, where zero subscripts refer to equilibrium or time averaged values and unity subscripts to very small perturbations. Substitution into Eq. (1.3) gives, respectively, the equilibrium Vlasov equation and the linearized Vlasov equation,

$$\frac{d}{dt} f_{0j}(\underline{r}, \underline{v}) = \left[\underline{v} \cdot \frac{\partial}{\partial \underline{r}} + (e_j/m_j) \left(\frac{1}{c} \underline{v} \times \underline{B}_0 \right) \cdot \frac{\partial}{\partial \underline{v}} \right] f_{0j}(\underline{r}, \underline{v}) = 0 \quad (1.8)$$

and

$$\left[\frac{\partial}{\partial t} + \underline{v} \cdot \frac{\partial}{\partial \underline{r}} + (e_j/m_j) \left(\frac{1}{c} \underline{v} \times \underline{B}_0 \right) \cdot \frac{\partial}{\partial \underline{v}} \right] f_{1j}(\underline{r}, \underline{v}, t) + (e_j/m_j) \left(\underline{E}_1 + \frac{1}{c} \underline{v} \times \underline{B}_1 \right) \cdot \frac{\partial}{\partial \underline{v}} f_{0j}(\underline{r}, \underline{v}) = 0. \quad (1.9)$$

Particle equilibrium distributions must satisfy Eq. (1.8), while Eq. (1.9), together with Maxwell's equations, comprise the basic set of equations to be solved.

The general technique for attacking this class of problems was developed by Landau¹¹ to study electrostatic oscillations in a homogeneous, unmagnetized plasma. Treating the calculation as an initial value problem, he performed a Fourier analysis in space and a Laplace analysis in time on the perturbed quantities. Solutions of the resultant dispersion relation indicated that wave-particle interactions can cause longitudinal waves to be damped. This collisionless phenomenon is called Landau damping and has since been verified experimentally.¹²

The first comprehensive analysis of a homogeneous plasma in the presence of a uniform magnetic field was performed by Bernstein.¹³ He applied Landau's method in specifying the source term, $\underline{J}$, in the vector wave equation,

$$\underline{k} \times (\underline{k} \times \underline{E}) + \frac{\omega^2}{c^2} \underline{E} + \frac{4\pi i \omega}{c^2} \underline{J} = 0, \quad (1.10)$$

where unit subscripts have been dropped and $\underline{J}$ is the total current density. Noting that Eq. (1.10) has a nontrivial solution only when the determinant of the matrix coefficient of $\underline{E}$ vanishes, Bernstein obtained an equation relating frequency and wave number for allowable waves. He then solved this general dispersion relation for the case of a Maxwellian distribution and found the waves to be stable ("Bernstein Modes").

In the study of microinstabilities it is customary to further subdivide the subject into the categories of electrostatic (longitudinal) modes and electromagnetic (transverse) modes. Electrostatic instabilities arise from charge separation phenomena and are derived with the restriction that the curl of the electric field vanish ($\nabla \times \underline{E} = 0$). When this constraint is removed ($\nabla \times \underline{E} \neq 0$), it becomes necessary to consider additional effects associated with bending, compression, and expansion of the magnetic field. However, for sufficiently small values of beta (the ratio of plasma pressure to magnetic pressure) it has been demonstrated^{13,14} that the effect of transverse wave coupling to electrostatic waves can be ignored. In addition, instabilities are produced that, in the strict sense, are coupled to the electrostatic modes but are primarily electromagnetic in character. These "electromagnetic" instabilities³ generally require large values of beta ($\beta > 1$) for their growth rates to be significant.

Since mirror machines are basically low beta devices, the theoretical studies in this area have been focused on electrostatic modes. In such open-ended systems the principal sources of free energy available to drive these instabilities are velocity space anisotropy and spatial inhomogeneity. The latter source is common to all confined plasmas, while the former one is particularly evident in mirror systems because the unavoidable end-losses cause depletion of low energy states. In what follows a review will be given of research directed at finding conditions under which these free energy sources can give rise to unstable waves in a mirror plasma.

1.3.2 Infinite Medium Analysis

The initial efforts in mirror microinstability theory dealt with the determination of stability criteria for an unbounded, uniform plasma. Harris¹⁵ specialized the problem considered by Bernstein to the case of electrostatic waves and focused

attention on the unstable modes associated with anisotropic velocity distributions. He derived the following form for the electrostatic dispersion relation:

$$D(\omega, k) = 1 + \sum_j \left(\frac{\omega_{pj}^2}{k^2} \right) \sum_{n=-\infty}^{\infty} \int d^3v J_n^2(\alpha_j) (\omega - k_{\parallel} v_{\parallel} - n\omega_{cj})^{-1} \times \left(\frac{n}{v_{\parallel}} \omega_{cj} \frac{\partial}{\partial v_{\parallel}} + k_{\parallel} \frac{\partial}{\partial v_{\parallel}} \right) f_{0j}(v_{\perp}^2, v_{\parallel}^2) = 0, \quad (1.11)$$

where the $(\parallel)$ and $(\perp)$ subscripts refer to directions parallel and perpendicular to the applied magnetic field, $\omega_p^2 = 4\pi n_0 e^2/m$ is the plasma frequency, $\omega_c = eB/mc$ is the cyclotron frequency, $\alpha = k_{\perp} v_{\perp} / \omega_c$ is the argument of the Bessel function, and $f_{0j}(v_{\perp}^2, v_{\parallel}^2)$ is the equilibrium velocity distribution.

With the Harris equation as the starting point, many authors have examined the large number of possible modes initiated by anisotropy in velocity space. The unstable waves, which arise from the inequality of mean velocities parallel and perpendicular to the magnetic field, are known as temperature anisotropy instabilities. Dnestrovskiy *et al.*¹⁶ found that these modes occur with characteristic frequencies close to harmonics of the ion-cyclotron frequency, provided $T_{\perp i} > 2T_{\parallel i}$, where $T_{\perp i}$ and $T_{\parallel i}$ refer to the perpendicular and parallel ion temperatures. They also showed that the threshold density of these waves increases with electron temperature. Hall *et al.*¹⁷ included a more rigorous derivation of these results in a paper which treated in addition a number of other velocity space instabilities.

Unstable waves associated with nonmonotonic perpendicular velocity distributions are called loss-cone instabilities. As pointed out by Krasovitskii and Stepanov¹⁸ and, independently, by Rosenbluth and Post,¹⁹ systems with loss-cone distributions can support very high frequency instabilities which propagate along the magnetic field lines ($k_{\parallel} \neq 0$). These waves are characterized by perpendicular wavelengths that are short compared to the ion Larmor radius ($k_{\perp} a_i \gg 1$, $k_{\perp} \gg k_{\parallel}$) and frequencies greatly exceeding the ion cyclotron frequency ($\omega \gg \omega_{ci}$). It was subsequently shown by several authors^{20,21} that these loss-cone modes can also exist at the lower frequencies close to harmonics of the ion-cyclotron frequency.

Instabilities that propagate in a direction perpendicular to the applied magnetic field ($k_{\parallel} = 0$) are known as flute modes. Dory, Guest, and Harris²² discovered that such instabilities can appear in mirror devices at zero frequency and near the ion-cyclotron harmonics. An ion loss-cone distribution of the form,

$$F_0^{(j)}(v_{\perp}^2) = \int_{-\infty}^{\infty} dv_{\parallel} f_0(v_{\perp}^2, v_{\parallel}^2) = (\pi a^2 j!)^{-1} \left(\frac{v_{\perp}}{a} \right)^{2j} \exp(-v_{\perp}^2/a^2), \quad (1.12)$$

where $j = 1, 2, \dots$, was considered, and it was shown that with $j < 3$, the distribution is sufficiently broad to prevent instability. Noting this sensitivity to the shape of the function, Post and Rosenbluth²³ went on to investigate more realistic models based

upon theoretical calculations of collisionally equilibrated distributions for mirror-confined plasmas. In the present work additional modifications due to the effects of finite geometry, finite beta, and ambipolar potential are taken into account. This analysis, which will be presented in Chapter IV, includes the use of distributions obtained from both experimental data and detailed numerical (Fokker-Planck) calculations.

Another flute mode arising primarily from the loss-cone free energy source is the multicomponent instability.^{24,25,17} In this case a hot plasma with a loss-cone distribution sufficiently broad to avoid the Dory-Guest-Harris (DGH) mode can be driven unstable when a low temperature Maxwellian plasma is added. Here the colder plasma stimulates the instability by providing a dissipative medium to which the main hot component can give up energy. Such a mixture can occur, for example, when there is charge exchange between fast ions and cold neutral particles in mirror systems. Moreover, the possibility of stabilizing a number of loss-cone modes by the introduction of a colder isotropic species has been proposed by several authors.^{26,27} In later work²⁸ it was shown that in order to avoid the multicomponent instability, it is necessary to either maintain the ratio of mean energies of the warm to hot species above 0.1 or to keep the fraction of the colder plasma below 1%.

To better understand the mechanisms of many important plasma instabilities, it is useful to adopt the concept of negative energy waves.²⁹ The energy of a wave in a dispersive medium is proportional to $(\partial/\partial\omega)(\omega D)$, where D is dielectric constant.³⁰ Kadomtsev et al.³¹ have pointed out that this energy can be negative in systems that are not in thermodynamic equilibrium with the consequence that "the energy of the medium is lower in the presence of the wave than in the absence of the wave." Since these negative energy waves grow when they dissipate energy, ordinary dissipation due, for example, to collisions and Landau damping, will serve to amplify rather than damp them. In the application^{32,33} of this concept to velocity space instabilities, one can associate the physical mechanism driving the disturbances with the interaction between negative and positive energy waves. As a specific example, the multicomponent flute instability described earlier can be viewed in terms of the coupling between negative energy waves arising from the hot, loss-cone plasma and positive energy upper hybrid oscillations from the cold plasma component.

As noted earlier, a confined plasma is necessarily subject to spatial inhomogeneities which constitute another principal source of free energy available to drive instabilities. In particular, radial density gradients [(A) to B] can give rise to positive energy drift waves from which unstable modes can draw energy.^{34,35} Low frequency ($\omega \ll \omega_{ci}$) drift instabilities are suppressed in mirror machines by the minimum-B configuration of the magnetic trap. However, as pointed out by Mikhailovskii and Timofeev,³⁶ density gradients can also generate high frequency ($\omega \sim \omega_{ci}$), flute type modes. These instabilities, which are known as drift-cyclotron modes, were derived with the assumption that the plasma velocity distribution is Maxwellian. Mikhailovskii³⁷ and Shima and Fowler³⁸ then went on to show that for mirror systems the combination

of the loss-cone and spatial inhomogeneity sources of free energy lead to flute type instabilities with highly restrictive stability criteria. These modes are called "drift-cone" instabilities³⁹ and arise from the coupling between the positive energy drift waves and the negative energy loss-cone waves. They have been analyzed in great detail by Post and Rosenbluth,²³ who derived the following stability criterion:

$$\frac{\langle a_i \rangle}{n_i} \frac{dn_i}{dx} \sim \frac{a_i}{R_p} < 0.38 \left(\frac{m_e}{M_i} + \frac{\omega_{ci}^2}{\omega_{pi}^2} \right)^{2/3}, \quad (1.13)$$

where

$$\langle a_i \rangle = \left[\omega_{ci}^3 \int d^3 v f_{oi} (v_{\perp}^2, v_{\parallel}^2) v_{\perp}^{-3} \right]^{1/3} \approx a_i.$$

$n_i (dn_i/dx)$ is the density gradient, and R_p is the plasma radius. At the high densities necessary for fusion reactors this condition indicates that a very large plasma radius (several hundred ion Larmor radii) is required for stability. However, it has been pointed out in recent work,⁴⁰ which comprises a portion of this thesis, that the electrostatic treatment of these instabilities is no longer valid in such a regime and that electromagnetic effects must be taken into account. This finite beta analysis will be discussed in Section 1.3.5 and presented in detail in Chapter V.

1.3.3 Bounded Plasmas

From the preceding section it is seen that infinite medium theory indicates the possible existence of numerous instabilities in mirror systems. In applying the analysis to actual bounded devices, however, it is necessary to further determine whether the unstable waves are absolute or convective in nature.^{41,42} This is done by studying a localized initial disturbance (a wave packet) instead of a single Fourier mode extending through infinite space. Wave packets, which grow in amplitude while propagating away from their point of origin, are classified as "convective" instabilities. "Absolute" instabilities, on the other hand, are growing disturbances with zero group velocity, i.e., nonpropagating wave packets.

The convective type $k_{\parallel} \neq 0$ modes have a limited disruptive effect on confinement since they can propagate out of the plasma along the magnetic field lines before growing to significant amplitudes. The distance over which such modes grow by a factor of e is called the growth or e -folding length. Their importance is dependent on the magnitude of this value compared to the plasma length and on the degree to which they are reflected at the boundaries.

The problem of wave reflection at the ends of a mirror system in which the electrons are assumed to be Maxwellian has been treated by Berk et al.⁴³ using the WKB approach.⁴⁴ It was found that the reflection coefficient is usually small provided the characteristic plasma inhomogeneity length is large compared to the ion Larmor

radius. Numerical solutions⁴⁵ of the Vlasov-Poisson equations for waves under the influence of a linear confining potential confirmed the conclusions of the WKB analysis. Baldwin⁴⁶ then noted that reflection is enhanced if there are abrupt changes in the confining potential. Consequences of this effect were investigated by Berk et al.⁴⁷ and it was concluded that in most cases wave reflection is still sufficiently limited to prevent serious disruption of confinement.

In the absence of appreciable reflection, the stability criterion for convective instabilities is that the plasma length be less than several growth lengths. Rosenbluth and Post^{19,23} specified the condition that such modes can only grow to large enough amplitudes to disrupt confinement if the plasma length exceeds ten e-folding lengths. Noting that the very high frequency loss-cone instabilities (described earlier) are convective, they applied the above criterion and showed the plasma to be stable against these modes provided,

$$L < L_{\text{crit}} \sim 100 a_i \left[1 + (\omega_{ce}/\omega_{pe})^2 \right]^{1/2}, \quad (1.14)$$

where L is the plasma length and a_i the ion Larmor radius. However, even if the plasma length is less than L_{crit} the presence of such convective waves can lead to enhanced particle scattering. Baldwin and Callen⁴⁸ have recently emphasized this point and have noted that such effects could account for the anomalous loss observed in current mirror experiments such as the 2X device.^{49,50}

Analysis has shown that the loss-cone modes,²⁰ characterized by $k_{\parallel} \neq 0$ and frequencies close to the ion cyclotron harmonics, are convective at low densities ($\omega_{pi}^2/\omega_{ci}^2 < 1$).²¹ Beasley and Cordey⁵¹ treated in great detail the problem of plasmas with distributions having both temperature anisotropy and a loss-cone and found that at low densities the associated instabilities are also convective. These modes, which have been experimentally identified in the Phoenix⁵² and Alice⁵³ mirror devices, have long growth lengths and require strong reflections from the ends of the plasma to sustain the disturbances.⁵⁴ It was pointed out by Taylor that collisional electron distributions, which are sharply cut off at the tail, can lead to such strong reflections. In subsequent work, Berk⁵⁵ treated a collisional mirror distribution for the electrons and showed from dimensional considerations that wave damping without reflection can be achieved experimentally.

At higher densities ($\omega_{pi}^2/\omega_{ci}^2 > 1$) the above modes become absolute in character with very long wavelengths along the magnetic field.⁵¹ McCune and Callen^{56,57} have calculated this "transition density" for a number of instabilities. As noted earlier, absolute modes, in contrast to the convective type, do not propagate out of the plasma. Flute modes ($k_{\parallel} = 0$), for example, fall into this category. The Bers-Briggs criterion⁴² for the existence of absolute instabilities requires first that

$$D(\omega, k) = \frac{\partial D(\theta, k)}{\partial k} = 0, \quad (1.15)$$

where $D(\omega, k)$ is the dielectric function of Eq. (1.11), and second that the "pole pinch" sufficiency condition be satisfied. This latter condition means that if a mode (ω_o, k_o) satisfying Eq. (1.15) is an absolute instability, it must have the property that the $k(\omega)$ solutions of $D(\omega, k) = 0$, which coalesce at (ω_o, k_o) (yielding $\partial D / \partial k = 0$), at least two originate from opposite sides of the Fourier inversion contour (real k -axis) as ω is moved away from the original Laplace inversion contour (in the complex ω -plane) toward ω_o .⁵⁶

In applying the concept of an absolute mode to finite systems, the basic requirement is that there exist a pair of internal "turning points" near which the wave can reflect and thus set up a standing wave.³⁷ This standing wave is comprised of two waves with group velocities of opposite sign between the turning points and zero group velocity at the turning point where they coalesce. If there is only a single pair of turning points where $k_{\parallel} = 0$ in the system, then a standing wave solution occurs if the following phase integral condition is satisfied between these points

$$\oint k_{\parallel} ds = (2n + 1) \pi. \quad (1.16)$$

However, in the general case of axially inhomogeneous systems, the problem is complicated by the existence of multiple turning point structures where the group velocity can vanish without $k_{\parallel} = 0$.⁵⁸ Here it is necessary to determine the reflection and transmission properties of the waves through each structure under the condition that the relationship between the phases and amplitudes of the waves be unique.⁵⁹ In the next section specific absolute modes in spatially nonuniform finite plasmas will be discussed.

1.3.4 Spatial Inhomogeneity in Finite Systems

When spatial inhomogeneity is included in the stability analysis of a bounded plasma, it becomes necessary to determine the spatial structure of the waves. This is a complicated problem that generally involves the solution of an integro-differential dispersion equation. In treating plasmas that are uniform in the axial direction and slightly inhomogeneous in the radial direction, the most common approach is to adopt the "local approximation." The basic requirement here is that the characteristic scale size of the plasma inhomogeneity (R_p) be much larger than the scale size for the spatial variation of the wave ($1/k_{\perp}$).³⁴ It is then assumed that the perpendicular spatial dependence of the perturbations can be expressed in the eikonal form, $\exp(i\xi)$, where $\xi \equiv k_{\perp} \cdot r_{\perp}$ is independent of position along the field line and $r_{\perp}$ is the perpendicular distance to the field line. In addition, since the magnetic field confining the plasma is surrounded by a vacuum, it is also implicitly assumed that waves in the radial direction are perfectly reflected. Many authors³⁴ have demonstrated that for axisymmetric systems one can obtain standing wave solutions justifying the local approximation. However, if the nonaxisymmetric property of minimum-B configurations is included,

the wave equation is not separable in its two radial components. The resultant highly complicated problem of obtaining standing wave solutions in such a system has not been resolved to date.

In applying the local approximation to the radial gradient problem, the perturbation quantities are taken to be independent of the coordinate in the inhomogeneous direction (x), and the equilibrium distribution is expanded in a Taylor's series about a reference position that maximizes the growth rate. This leads to a dispersion relation of the same form as Eq. (1.11) except that the bracketed factor in the integrand now contains the additional drift term, $k_y/R_p \omega_{ci}$. Analysis of this equation yields the drift modes mentioned earlier.

The first extensive stability analysis of a finite length, axially inhomogeneous plasma was carried out by Rosenbluth et al.⁶⁰ using the WKB method.^{44,61} This work was applied to the determination of length criteria for mirror systems.²⁶ The analysis showed that the absolute resonant loss-cone modes are stabilized in plasmas with $L/a_i \leq 100$ where L is the axial scale length and a_i the ion Larmor radius. It was later discovered that the presence of spatial variation along the magnetic field introduces dissipation which can destabilize the negative energy $(\partial/\partial\omega)(\omega D) < 0$ loss-cone waves.⁶² In an axially inhomogeneous plasma, stable negative energy waves in the central region gradually convert to positive energy waves near the ends of the system. Since the outgoing positive energy waves carry off energy from the interior, they serve as a dissipation mechanism to drive the negative energy waves to higher amplitude. Berk et al.⁵⁹ used a modified WKB technique, which includes the effect of wave amplification due to the presence of singularities in the dielectric function, to obtain the following length criterion for stabilizing these modes:

$$L_m/a_i \leq \left[(M_i/m_e) (1/n^2) (F_n k_{\perp}^2 a_i^2)^{-1} \right]^{1/2} \quad (1.17)$$

with

$$F_n \equiv (2\omega_{ci}^2/k_{\perp}^2) \int d^3v \left(\partial f_{oi} / \partial v_{\perp}^2 \right) J_n^2(k_{\perp} v_{\perp} / \omega_{ci}),$$

where L_m is the axial scale length, n is the harmonic number, and typically for a mirror ratio of 2, $F_n(k_{\perp} a_i)^3 \lesssim 1/8$ and $k_{\perp} a_i > n$. It was also noted that the density requirement for the onset of these instabilities corresponds to the threshold condition for the absolute loss-cone modes^{26,51}

$$(\omega_{pi}/\omega_{ci})^3 > (k_{\perp} a_i) (M_e/m_j)^{1/2} / F_n. \quad (1.18)$$

The authors pointed out, however, that the above stability conditions are not applicable to the flute type negative energy modes with eigenfrequencies lying just below the first harmonic. For this case Baldwin et al.⁶³ have found the appropriate stability criterion to be

$$L/a_i < 3(R-1)^{1/2} |1 - \Omega_i|^{-3/2}, \quad (1.19)$$

where $\Omega_i = \omega/\omega_{ci}$ and R is the mirror ratio.

In the class of problems considered above, the instabilities are characterized by frequencies that are typically greater than both the mean and thermal spread in the bounce and drift frequencies. This allows one to employ the straight-line orbit approximation, which ignores the bounce and drift motions of the ions. The validity of the basic approach has been demonstrated by Berk and Pearlstein,⁶⁴ who showed that phase mixing over sufficiently long time scales effectively quenches the regenerative behavior of the waves.⁵⁸ In cases where this approximation breaks down, it is necessary to consider closed orbits. Several authors^{65,66} have noted that in this regime the axial inhomogeneity of the magnetic field can cause the ion cyclotron frequency to be velocity dependent. This in turn can lead to flute type instabilities in plasmas with highly anisotropic velocity distributions. These modes are called modified negative mass instabilities and have been experimentally identified in the DCX-2 (Ref. 67) and PR-5 (Ref. 68) mirror experiments. A rigorous analysis of this and other types of negative mass instabilities has been carried out by Callen and Horton,⁶⁹ who demonstrate that sufficient thermal spread in the distributions can quench the extreme anisotropy required to drive these disturbances. They show that the modified negative mass instabilities are stabilized for $T_{\parallel}/T_{\perp} \geq a_i/L$.⁷⁰ This result is also quoted by Horton et al.⁷¹ in a paper which includes a rigorous derivation of the appropriate electrostatic dispersion relation for inhomogeneous, axisymmetric mirror systems.

1.3.5 Finite Beta Effects on Plasma Stability

Since mirror machines are basically low beta devices, the stability studies in this area have dealt primarily with electrostatic modes. However, as reactor conditions are approached with hotter and denser plasmas, the coupling between the electrostatic and the transverse electromagnetic waves becomes progressively stronger. Although mirror plasmas approaching fusion conditions should still be free of significant disruptions caused by the high beta type electromagnetic instabilities,³ this coupling can nevertheless introduce significant modifications to electrostatic stability criteria. Callen and Guest⁷² have emphasized the point that such effects become important when $\omega_{pe}^2 \geq k^2 c^2$ (Ref. 13). In their analysis of an infinite, inhomogeneous plasma they have shown that the coupling to transverse waves has a destabilizing influence on the high frequency convective loss-cone modes and a stabilizing effect on the Dory-Guest-Harris instability. In the former case it was found that the parallel wavelengths of the convective waves decrease, thus allowing them to fit into a shorter plasma. In the latter case the introduction of the term $(\omega_{pe}/kc)^2$ (ignored in the electrostatic limit) leads to stability at very high densities, i.e., there exists a high- as well as low-density cutoff for the Dory-Guest-Harris modes. It will be demonstrated in Chapter IV that this modification has little influence on plasmas with very peaked distributions but can lead to stability for the broader distributions.

In earlier work Pearlstein⁷³ noted that finite beta effects stabilize the absolute loss-cone instabilities at the higher harmonics in an infinite, homogeneous system. However, at high densities, the dominant instabilities, when finite geometry effects are considered, are the unstable negative energy waves.^{59,62} Berk et al.⁵⁹ showed that these modes are destabilized by electromagnetic corrections because the parallel wavelength is reduced by a factor $[1 + (\omega_{pe}/k_c)^2]^{-1/2}$, which in turn leads to a corresponding reduction in the axial length criterion of Eq. (1.17).

A significant demonstration of the stabilizing effect of finite beta was found to be its influence on the drift-cone mode.⁴⁰ As noted earlier, the stability criterion of Eq. (1.13) is not valid at the high densities of fusion interest. With the inclusion of electromagnetic effects in the calculation, the appropriate radial length requirement is

$$R_p \frac{c}{\omega_{ci}} < \frac{2}{1 + (\beta_i/2)} \left[1 + \left(\omega_{pe}^2 / \omega_{ce}^2 \right) \right]^{1/2} \quad (1.20)$$

for

$$\beta_i > \beta_{crit} \sim 4 \left[(\omega_{ci} / \omega_{pi})^2 + (m_e / M_i) \right]^{1/3}.$$

The analysis of the electromagnetic dispersion relation leading to this result will be presented in Chapter V together with stability criteria for lower values of β_i and modifications of the results due to finite geometry effects.

At the beginning of this section it was pointed out that mirror systems should not be affected by most instabilities, which are primarily electromagnetic in character, because the growth rate of these modes generally becomes significant only for large values of beta ($\beta > 1$). An exception to this is the highly resonant flute instabilities that can propagate in infinite, homogeneous, low beta plasmas with loss-cone distributions.⁷⁴ However, in Chapter VI it will be demonstrated, through a detailed analysis of finite geometry effects and ion motion along the magnetic field, that such modes should not appear in existent mirror systems.

1.4 SCOPE AND ORGANIZATION OF THESIS

The development of higher quality laboratory plasmas in recent years has necessitated the inclusion of more detailed effects in stability theory. Research performed in this thesis was directed to this end, and, as indicated at several points in the preceding survey, supplements earlier work in this area. With regard to basic formalism, the current project involved the determination of the appropriate procedure for treating flute modes in a bounded, inhomogeneous, finite beta plasma and the derivation of the general electromagnetic flute dispersion relation including finite beta corrections to the particle orbits. These results were applied to the analysis of the Dory-Guest-Harris, drift-cone, and resonant electromagnetic instabilities. The dominant stabilizing influence in the case of the drift-cone mode was found to come

from the finite beta modification, while the inclusion of detailed finite geometry effects led to stabilization of the resonant electromagnetic mode. For the Dory-Guest-Harris instability, the sensitivity of the mode to the details of equilibrium distributions dictated the use of functions closely approximating actual collisionally equilibrated distributions. So, in addition to the finite beta and finite geometry modifications, the calculation of stability criteria for the DGH mode included effects due to the presence of an ambipolar potential and involved the use of both theoretically-evaluated and experimentally-measured distributions. In what follows the organization of the work in the thesis is presented.

The appropriate formalism for analyzing flute modes is the subject of Chapter II. After reviewing the derivation of the integral wave equation and the method for treating electrostatic flute instabilities in finite mirror systems, electromagnetic effects are considered. It is shown in detail that the finite beta modifications do not alter the basic procedure for dealing with either the low frequency instabilities, such as the "zero frequency" Dory-Guest-Harris mode, or the high frequency ($\omega \geq \omega_{ci}$) instabilities.

With the formalism established, Chapter III proceeds with the derivation of the electromagnetic flute dispersion relation for an axisymmetric, inhomogeneous plasma. After a discussion of equilibrium properties and parametric constraints, the finite beta modifications to the equilibrium orbits of the particles are calculated. Finally, in Section 3.3 the appropriate dispersion relation is presented.

The next three chapters deal with the application of the theory developed in Chapters II and III to particular instabilities. In Chapter IV the zero frequency Dory-Guest-Harris mode is studied. As noted in the survey, the instability is very sensitive to the shape of the plasma distribution function. This aspect is explored in great detail with the analysis applied to both experimentally-measured distributions as well as theoretically-calculated ones.

In Chapter V the drift-cone instability is analyzed with attention focused on the finite beta stabilization of the mode. Since the unstable waves have characteristically high frequencies and short wavelengths, it is shown that the complicated dispersion relation of Chapter III can be reduced to a relatively simple equation. In the infinite medium limit the stability criterion of Eq. (1.20) is derived together with a general analytic expression giving the critical radial density gradient for moderate beta ($\beta \lesssim 1$) plasmas. Finite geometry modifications of these results are also presented, together with a discussion of length restrictions imposed on confined plasmas by this instability and the negative energy flute mode.

Highly resonant electromagnetic instabilities, driven by the loss-cone source of free energy, are treated in Chapter VI. The appropriate infinite medium dispersion equation is solved analytically and the answers shown to agree with previously calculated numerical results. It is then demonstrated that the effects of axial inhomogeneity and ion motion along the magnetic field lead to stabilization.

In the final chapter a summary of the results of the work presented in this thesis is given.

Chapter II

Flute Mode Formalism

2.1 GENERAL APPROACH

2.1.1 Basic Equations and Approximations

In this chapter the appropriate formalism for analyzing flute type instabilities in a bounded, inhomogeneous, finite beta plasma is presented. The present section deals with a detailed review of the derivation of the basic integral wave equation and the form of the ion and electron density operators in the low and high frequency regimes. A description is then given of flute formalism in the electrostatic limit. In Section 2.2 finite beta modifications are introduced, and it is shown that the basic procedure for dealing with both low and high frequency flute modes remains unchanged. Finally, in Section 2.3 a calculation bridging the low and high frequency regimes is presented. Results indicate the finite beta flute formalism to be justified at both limits.

The general form of the integro-differential dispersion equation for electrostatic modes is

$$-\nabla^2 \phi = \sum_j 4\pi n_j e_j \int d^3 v f_{1j} \quad (2.1)$$

with

$$f_{1j} = (e_j/m_j) \int_{-\infty}^t dt' \nabla' \phi \cdot \frac{\partial}{\partial \underline{v}} f_{0j}. \quad (2.2)$$

The perturbed distributions f_{1j} come from solving the linearized Vlasov equation by the method of characteristics.⁷⁵ In this case the characteristics are the orbits of the particles in the unperturbed fields. An effective approach to determining these orbits is to use the small Larmor radius expansion technique and to work with the magnetic flux coordinate system illustrated in Fig. 1.⁷¹ These coordinates are related to the local coordinates r, s_θ, s by the following differential definitions:

$$\begin{aligned} d\psi &= BRdr \\ Rd\theta &= ds_\theta \\ d\chi &= Bds, \end{aligned} \quad (2.3)$$

Fig. 1. Contours of the magnetic flux coordinates for an axisymmetric mirror system in a plane containing the axis of symmetry. The coordinate χ is the magnetic scalar potential, and the coordinate ψ labels surfaces containing the magnetic flux ($2\pi\psi$) which is defined as zero at the axis of symmetry.

where s specifies position along the axis of magnetic field line, r the radial positions from this line, and s_θ the azimuthal position. The expansion parameter ϵ is assumed small with the following ordering for mirror systems:

$$\epsilon \sim a_i/L_B, (a_i/R_c)^{1/2}, \quad (2.4)$$

where

$$L_B = \{B(\partial/\partial X)\ln(B)\}^{-1} \text{ and } R_C = \{BR(\partial/\partial\psi)\ln(B)\}^{-1}.$$

This ordering corresponds to an approximately cylindrical magnetic field configuration with a magnetic flux that is spatially constant to lowest order, i.e.,

$$\psi = (BR^2/2) [1 + O(R/R_C)] = (B_0 R_0^2/2) [1 + O(\epsilon)], \quad (2.5)$$

where zero subscripts denote values at the center of the system. The condition $R/R_C \sim \epsilon$ implies that the particles considered are far off the axis of symmetry and do not encircle it. It is also assumed that the potential $\Phi(\psi, X)$ determining the electric field is weakly inhomogeneous, i.e.,

$$a_1 [B(\partial/\partial X)\ln(\psi)] \sim \epsilon \text{ and } a_1 [BR(\partial/\partial\psi)\ln(\psi)] \sim \epsilon^2. \quad (2.6)$$

With the above specifications, the particle orbits can be obtained from the Euler-Lagrange equations of motion by an expansion procedure that has been carried out in great detail in Ref. 71. The main features of this calculation are summarized in the following section.

2.1.2 Particle Orbits

In the magnetic flux coordinate system the magnetic field, magnetic vector potential, and particle velocity are, respectively,

$$\begin{aligned} \underline{B} &= \underline{\nabla} \times \underline{A} = \underline{\nabla}\psi \times \underline{\nabla}\theta, \\ \underline{A} &= (\psi/R)\hat{\theta}, \text{ and} \\ \underline{v} &= (\dot{\psi}/RB)\hat{\psi} + (R\dot{\theta})\hat{\theta} + (\dot{X}/B)\hat{X}. \end{aligned}$$

Hence the appropriate Lagrangian⁷⁶ is

$$\begin{aligned} L &= (mv^2/2) + (e/c)(\underline{A} \cdot \underline{v}) - e\Phi \\ &= (m/2)[(\dot{\psi}/RB)^2 + (R\dot{\theta})^2 + (\dot{X}/B)^2] + (e\dot{\theta}\psi/c) - e\Phi \end{aligned} \quad (2.7)$$

with the dot denoting a total time derivative. The corresponding constants of motion are the energy, $E = (mv^2/2) + e\Phi$, and the canonical angular momentum $p_\theta = mR^2\dot{\theta} + (e\psi/c)$. In terms of the coordinate ψ , the latter condition can be expressed as

$$\psi_g = \psi + (mR^2\dot{\theta}/e). \quad (2.8)$$

This shows that the guiding center of a particle always remains on a given surface specified by ψ_g . Substitution of Eq. (2.8) into the Euler-Lagrange equation for ψ leads to

$$\ddot{\psi} - \psi [\dot{\chi} (\partial/\partial \chi) \ln(R^2 B^2) + (\psi - \psi_0) \omega_c^2 = (\dot{\psi}^2/2) (\partial/\partial \psi) \ln(R^2 B^2) + (\omega_c^2/2) (\psi - \psi_0)^2 (\partial/\partial \psi) \ln(R^2) - (\dot{\chi}^2/2) R^2 (\partial/\partial \psi) \ln(B^2) - (e/m) (RB)^2 (\partial/\partial \psi) \phi, \quad (2.9)$$

where ω_c is the local cyclotron frequency. From the conservation of energy the equation for χ is

$$\dot{\chi}^2 = \frac{2B^2}{m} (E - E_1 - e\phi), \quad (2.10)$$

where

$$E_1 = \frac{m}{2} v_1^2 = \frac{m}{2} \left[(\dot{\psi}/RB)^2 + (R\dot{\theta})^2 \right].$$

Equations (2.8) and (2.10) completely determine the particle orbits.

To solve Eq. (2.9) an expansion is made about the guiding center that leads to the form of a harmonic oscillator equation with small anharmonic terms. From the asymptotic solutions to this type of equation, one can construct an appropriate trial solution, substitute back into the equation, and equate like powers of ϵ to arrive at the answer. The $\psi(t)$ orbit thus obtained is then combined with Eq. (2.8) and the resultant expression integrated over time to give to lowest order the following orbit for $\theta(t)$:

$$\theta(t^1) = \theta(t) - \frac{(2\mu B/m)^{1/2}}{R\omega_c} \bigg|_{\chi(t^1)} \sin \left(- \int_t^{t^1} dt'' \omega_c(t'') + \phi_1 \right) + \frac{(2\mu B/m)^{1/2}}{R\omega_c} \bigg|_{\chi(t)} \sin(\phi_1), \quad (2.11)$$

where ϕ_1 is the perpendicular velocity phase angle and $\mu \equiv mv^2/2B$ is the magnetic moment, which is also a constant of motion to lowest order.⁷¹ The initial conditions here are that $(\psi^1, \theta^1, \chi^1) \rightarrow (\psi, \theta, \chi)$ at $t = t^1$. Orbit modifications due to cross field drifts caused, for example, by radial field curvature and $\underline{E} \times \underline{B}$ particle motion, are ignored since they are of order ϵ^2 and higher. Finally, substitution of the above results into Eq. (2.10) determines the $\chi(t)$ orbit, which can be expressed as

$$t^1 - t = \int_{\chi}^{\chi^1} \frac{d\chi''}{B(\chi'')} \left[\frac{2}{m} [E - \mu B(\chi'') - e\phi(\chi'')] \right]^{-1/2} = \int_{\chi}^{\chi^1} \frac{d\chi''}{B(\chi'') v_{\parallel}(\chi'')} \quad (2.12)$$

or equivalently

$$s(t^1) = s(t) + \int_t^{t^1} dt'' v_{\parallel}(t''). \quad (2.13)$$

2.1.3 Integral Wave Equation

In terms of the magnetic flux coordinates, the perturbed distribution given by Eq. (2.2) can be written as

$$f_1 = e \int_{-\infty}^t dt' \left(\frac{i\ell}{R} \frac{\partial}{\partial \theta} + \underline{\nabla} \phi \frac{\partial}{\partial \psi} + \underline{\nabla} \times \frac{\partial}{\partial \chi} \right) \phi \cdot \left(\underline{v} \frac{\partial}{\partial E} + \underline{v}_\perp \frac{1}{B} \frac{\partial}{\partial \mu} + \frac{\partial}{\partial \theta} R \frac{\partial}{\partial \rho_\theta} \right) \times f_0 \exp \{ i\ell [\theta(t') - \theta(t)] - i\omega(t' - t) \}, \quad (2.14)$$

where a perturbed potential of the following form has been considered:

$$\phi(\psi, \theta, \chi) = \phi(\psi, \chi) \exp [i\ell \theta - i\omega t], \quad (2.15)$$

with $k_\perp \approx \ell/R$. From Eq. (2.4), $\psi \approx BR^2/2$ is a constant to lowest order. This implies that R and thus $1/k_\perp$ vary spatially as $B^{-1/2}$.

To determine the charge density, the equilibrium particle orbits calculated in the preceding section were substituted into Eq. (2.14). In accordance with the equilibrium Vlasov equation [Eq. (1.8)], the distribution f_0 is a function of the constants of motion E , μ , and ψ . Its normalization is such that $f_0 d^3v$ is the density of particles with guiding centers at ψ_g and velocities in the range $d^3v = \sum \pm d\phi_\perp dE duB/|v_\parallel|$, where $\sum \pm$ denotes summation over positive and negative parallel velocities for particles with given E , μ . After the term $\underline{v} \cdot \underline{\nabla} \phi$ is integrated by parts, with $RB \partial \phi / \partial \psi$ ignored in comparison with ℓ/R , the θ -orbit given by Eq. (2.11) is substituted into the result, and the expression for the phase factor $\exp \{ i\ell [\theta(t)] \}$ is expanded in terms of the Bessel function identity,⁷⁷

$$\exp(iA \sin \alpha) = \sum_{n=-\infty}^{\infty} J_n(A) \exp(in\alpha). \quad (2.16)$$

Averaging this form of f_1 over the phase angle $\phi_\perp$ and substituting into Poisson's equation, Eq. (2.1), gives the basic integral wave equation

$$(\ell/R)^2 \phi = 4\pi p(\phi) = \sum_j \omega_{pj}^2 \sum_{\pm} \iint dE d\mu \frac{B}{|v_\parallel|} \left[\frac{\partial f_0}{\partial E} \phi + \sum_{n=-\infty}^{\infty} J_n^2(\lambda) \right] \times \left(\omega \frac{\partial}{\partial E} + \frac{n\omega_c}{B} \frac{\partial}{\partial \mu} + \frac{mc\ell}{e} \frac{\partial}{\partial \psi} \right) f_0 i \int_{-\infty}^t dt' \exp \left[-i\omega(t' - t) + in \int_t^{t'} dt'' \omega_1(t'') \right] \phi [s(t')], \quad (2.17)$$

where $\lambda \equiv 1(2\mu B)^{1/2}/R\omega_c = k_\perp v_\perp/\omega_c$ is the argument of the Bessel function and is spatially constant to lowest order. In the calculations that follow, spatial variations perpendicular to the magnetic field lines are treated in the local approximation, while detailed axial variations [along $B(s)$] are taken into account by adopting a "slab model".

2.1.4 Density Response of Electrons and Ions

The general form of the density operator given in Eq. (2.17) can be considerably simplified for the parameter ranges associated with most mirror instabilities. Since the electrons are generally far more mobile than the ions along the field lines, the electron density response dominates the integral wave equation and determines the character of the eigenfunction solutions. To obtain the form of this integral operator, one first notes that for the regimes of interest, ω/ω_{ce} and ka_e are much less than unity. If, in addition, the frequency is much larger than the electron bounce frequency, $\omega_{be} \sim v_{the}/L$, then it is valid to use the fluid description, which simply entails combining the continuity and force equations for the electrons. Carrying out this calculation and allowing for a radial gradient in the density then gives

$$4\pi\rho_e(s) = -\frac{B(s)}{\omega^2} \frac{\partial}{\partial s} \left(\frac{\omega_{pe}^2(s)}{B(s)} \frac{\partial}{\partial s} \right) \phi(s) - k_{\perp 0}^2 \frac{\omega_{pe}^2(s)\phi(s)}{\omega_{ce}(s)\omega_{ceo}} \\ + (k_{y0}/n)(\partial n/\partial x_g) \frac{\omega_{pi}^2(s)\phi(s)}{\omega\omega_{cio}}, \quad (2.18)$$

where $\mathbf{x}_g \equiv [\mathbf{x} + (v_y/\omega_c)]_0$ is the guiding center coordinate, x and y are orthogonal coordinates perpendicular to the local axial coordinate s , zero subscripts refer to quantities evaluated at the midplane ($s = 0$), and the density gradient is in the direction $\hat{\mathbf{x}}$ with $\hat{\mathbf{y}} = -\hat{\mathbf{x}} \times \mathbf{B}/|\mathbf{B}|$. In Ref. 71 it is demonstrated in detail that the general integral operator of Eq. (2.17) reduces to the fluid result of Eq. (2.18) for electrons with $\omega \gg \omega_{be}$.

Since the ion cyclotron frequency is usually much greater than ω_{be} in most mirror machines, the high frequency instabilities ($\omega \gtrsim n\omega_{ci}$) can be effectively treated by the above analysis. However, the low frequency analysis, such as that associated with threshold calculations for the Dory-Guest-Harris instability, involves a different prescription. In the regime where $\omega \ll \omega_{be}$, the electron density response is highly nonlocal, and it becomes necessary to time average the perturbed potential over the particle orbit. This calculation is presented in detail in Section 2.2.3, where the appropriate form of the operator is derived, taking into account finite beta corrections.

In treating the ion density response, multiple ion bounces are ignored and the straight line orbit approximation for the unperturbed guiding center trajectories is used. Proof is given in Ref. 63 that this method is valid provided the thermal spread $\Delta\phi_B$ of the ion bounce period is large, i. e.

$$\Delta\phi_B \equiv \left| \omega\tau_B - \int_0^{\tau_B} dt' \omega_{ci}(t') \right| \gg 1, \quad (2.19)$$

where τ_B is the bounce period. The above requirement is clearly satisfied for the loss-cone modes since the system considered is a slowly varying medium with $\Delta\phi_B$ typically of order $L_m/a_1 \sim 1/\epsilon \gg 1$. In the analysis that follows terms of order ϵ are

dropped, and the ratio of parallel to perpendicular temperature is taken to be of order unity. Also, ion Landau damping can be neglected provided $k_{\parallel} v_{thi} > \omega$. Since $k_{\parallel} \sim (1/\phi)(\partial\phi/\partial s) \sim \epsilon$, this condition implies that the lowest frequencies considered must be greater than a_i/L_m .

The ion density operator has the form

$$4\pi\rho_i(s) = (2\pi\omega_{pi0}^2/n_0) \sum_{\pm} \iint dE d\mu \frac{B(s)}{|v_{\parallel}(s)|} \left\{ \left[\frac{\partial}{\partial E} + \frac{1}{B(s)} \frac{\partial}{\partial \mu} \right] f_0 \phi(s) \right. \\ \left. + \sum_{n=-\infty}^{\infty} J_n^2(\lambda) \left[\frac{\partial}{\partial E} + \frac{1}{B(s)} \frac{\partial}{\partial \mu} + \frac{k_{y0}}{M_i \omega \omega_{ci0}} \frac{\partial}{\partial x_g} \right] f_0 S_n(s) \right\} \equiv k_I^2(s) H(s) \phi(s), \quad (2.20)$$

where

$$S_n(s) = i\omega \int_{-\infty}^t dt' \phi(s') \exp \left\{ -i \int_t^{t'} dt'' [\omega - n\omega_{ci}(t'')] \right\} \\ = \frac{i\omega}{v_{\parallel}} \int_{-\infty}^{\infty} ds' \phi(s') \exp \left\{ \frac{i}{v_{\parallel}} \psi_n(s, s') \theta(s - s') \right\} \quad (2.21)$$

with

$$\psi_n(s, s') \equiv \int_s^{s'} ds'' [\omega - n\omega_{ci}(s'')]$$

and

$$\theta(x) = \begin{cases} 1 & \text{for } x > 0 \\ -1 & \text{for } x < 0 \end{cases}.$$

As shown in Ref. 59, the effective range of the s' integral is determined from the phase condition $|\psi_n(s, s')/v_{\parallel}| \gtrsim 1$. When the potential $\phi(s')$ is expanded about s , one obtains

$$S_n(s) = \frac{i\omega\phi(s)}{v_{\parallel}} \int_{-\infty}^{\infty} ds' \exp \left\{ \frac{i}{v_{\parallel}} \psi_n(s, s') \theta(s - s') \right\}, \quad (2.22)$$

provided

$$\Delta \equiv \text{Min} \left[\left| \frac{k_{\parallel} v_{thi}}{\omega - n\omega_{ci}} \right|, \left| \frac{k_{\parallel}^2 v_{thi}}{n\omega_{ci}} \right|, \left| \frac{k_{\parallel}^3 v_{thi}}{n\omega_{ci}} \right| \right] < 1, \quad (2.23)$$

where primes denote differentiation with respect to s and $k_{\parallel} \sim \phi'/\phi \sim \epsilon$. In addition, it is found that if

$$|(\omega - n\omega_{ci})/\omega_{ci}|^3 > nv_{thi}^2 \omega_{ci}'' \quad (2.24)$$

then S_n can be accurately approximated by

$$S_n(s) = - \frac{\omega \phi(s)}{\omega - n\omega_{ci}(s)} \quad (2.25)$$

The above results imply that one can essentially ignore ion motion parallel to the magnetic field ($v_{\parallel i} \rightarrow 0$) provided the nonresonance condition of Eq. (2.24) is satisfied. Hence the simple form for S_n given in Eq. (2.25) can be employed in analyzing instabilities such as the Dory-Guest-Harris and drift-cone modes. However, in the case of unstable waves that are resonant near the midplane ($s = 0$), a different prescription is necessary. In dealing with this type of problem, one can expand $\omega_{ci}(s)$ about $s = 0$ to obtain

$$\psi_n(s, s') = \left[\omega - n\omega_{ci} \left(1 + \frac{s^2 + s'^2 + ss'}{3L_m^2} \right) \right] |s' - s|, \quad (2.26)$$

where $L_m \equiv \omega_{ci}/\omega_{ci}'|_{s=0}$. This result is used in the analysis of the highly resonant electromagnetic flute modes treated in Chapter VI.

2.1.5 Flute Modes

With the form of the density operators determined, the derivation of the flute formalism in the electrostatic limit will now be reviewed. The appropriate form of the mode equation in the high frequency regime [$\omega > n\omega_{ci} \approx \omega_{be}$, where $\omega_{be} \equiv$ electron bounce frequency $\approx k_{\parallel} v_{the}$] is

$$\left[\frac{B(s)}{\omega^2} \frac{\partial}{\partial s} \left(\frac{\omega_{pe}^2(s)}{B(s)} \frac{\partial}{\partial s} \right) + \frac{\omega_{pe}^2(s)}{\omega^2} k_{\parallel}^2(\omega, s) \right] \phi(s) = 0 \quad (2.27)$$

or

$$\left[\frac{B(s)}{\omega_{pe}^2(s)} \frac{\partial}{\partial s} \left(\frac{\omega_{pe}^2(s)}{B(s)} \frac{\partial}{\partial s} \right) + k_{\parallel}^2(\omega, s) \right] \phi(s) = 0,$$

where

$$k_{\parallel}^2(\omega, s) \equiv \frac{\omega^2 k_{10}^2}{\omega_{ci}(s)\omega_{cio}} \frac{m_e}{M_i} \left\{ \frac{\omega_{ci}^2(s)}{\omega_{pi}^2(s)} \left[1 + H(\omega, s) \right] + \frac{m_e}{M_i} + \frac{k_{y0}\omega_{ci}(s)}{k_{10}^2 R_p \omega} \right\}$$

and $R_p \equiv -[(1/n)(\partial n/\partial x_g)]^{-1}$ is the scale length of the radial density gradient. In a spatially homogeneous plasma the solution is just $\phi(s) = \exp(ik_{\parallel}s)$. One can recover the infinite medium flute dispersion relation²² by simply substituting this result into

Eq. (2.27) and setting $k_{\parallel} = 0$. It is obvious, however, that for waves in inhomogeneous systems there exists an "effective $k_{\parallel}$ ".⁷⁸ As a consequence, it is appropriate to consider propagation perpendicular to the magnetic field only if the parallel wavelength of the mode exceeds the plasma length, i. e.,

$$k_{\parallel}(\omega, s)L/\pi < 1. \quad (2.28)$$

For sufficiently short plasmas one can obtain flute type solutions to Eq. (2.27) characterized by $\phi(s) = \phi_0 + \phi_1(s)$, where ϕ_0 is spatially constant and ϕ_1/ϕ_0 is small.^{63,71} This is demonstrated in detail in Ref. 71, where it is pointed out that the symmetry in the kernel of the general density response integrals allows one to formulate the solutions of Eq. (2.17) as the stationary values of the following quadratic expression:

$$\Lambda(\phi) = \int \frac{ds}{B} \left[k_1^2 \phi^2 - 4\pi\phi\rho(\phi) \right]. \quad (2.29)$$

The fact that $\Lambda(\phi)$ is stationary at the solutions of Eq. (2.17) is then used to show that a change ϕ_1 in the wave function leads to a correction of order $|\phi_1/\phi_0|^2$ in the frequency of the mode. In dealing with Eq. (2.27) ϕ is expanded in terms of the parameter $\lambda \equiv \omega^2 k_1^2 L^2 / \omega_{pe}^2$, which is assumed to be small, and it is found that to lowest order the appropriate dispersion relation is

$$\int_{-L/2}^{L/2} \frac{ds}{B(s)} \left\{ \frac{B(s)}{B_0} + \frac{\omega_{pi}^2(s)}{\omega_{cio}^2} \frac{B_0}{B(s)} \left[\frac{m_e}{M_1} + \frac{k_{y0} \omega_{ci}(s)}{k_{10}^2 R_p \omega} + \frac{\omega_{ci}^2(s)}{\omega_{pi}^2(s)} H(\omega, s) \right] \right\} = 0. \quad (2.30)$$

For the low frequency case the same basic approach is adopted with $k_1^2 \lambda_{De}^2 \ll 1$ as the expansion parameter. To lowest order this also leads to the result of Eq. (2.30).

Provided the condition $k_{\parallel}(\omega, s)L/\pi < 1$ is satisfied, Eq. (2.27) can also be solved by an iteration method originally introduced by Roberts and Taylor⁷⁹ in connection with the analysis of MHD-type flute modes. First it is assumed that there is a sharp boundary at the edge of the plasma ($s = \pm L/2$) where the parallel current is small. One then takes $\phi(s) = \phi_0 + \phi_1(s)$, where ϕ_0 is the lowest order solution and ϕ_1 is the first order correction. Substituting this into Eq. (2.27) and performing the $\int ds/B$ line average on the resultant expression leads to the following first order relation at the edge of the plasma:

$$\begin{aligned} \frac{B_0}{L} \int_{-L/2}^{L/2} \frac{ds}{B(s)} D(\omega, s) &= - \frac{B_0}{B(s)} \frac{\omega_{pe}^2(s)}{\omega_{k10}^2 L \phi_0} \frac{\partial \phi_1(s)}{\partial s} \bigg|_{\pm L/2} \\ &= - \frac{B_0}{B(s)} \frac{4\pi i J_{\parallel}(s)}{k_{10}^2 L \phi_0 \omega} \bigg|_{\pm L/2}. \end{aligned} \quad (2.31)$$

where $D(\omega, s)$ is the term in braces, $\{ \}$, in Eq. (2.30).

From the boundary condition the term on the right is small and to lowest order is equal to zero. Hence, the dispersion relation given by Eq. (2.30) is recovered. In the next section the effect of electromagnetic modifications on the basic flute formalism will be determined.

2.2 ELECTROMAGNETIC MODIFICATIONS

2.2.1 Infinite Medium Analysis

Electromagnetic effects can significantly alter the stability criteria for electrostatic modes in a sufficiently dense plasma ($\omega_{pe}^2 > k^2 c^2$).⁷² In analyzing these modifications, which result from the current-induced coupling between the electrostatic and transverse waves, one must consider the total electric field $\underline{\epsilon} = \underline{E} + \underline{E}^T$, where $\underline{E} = -\nabla\phi$ is the electrostatic field and $\underline{E}^T$ the electromagnetic electric field. Since the electron velocity is generally much greater than that of the ions in mirror systems, the dominant coupling is due to the electron currents. In the parameter range of interest, ω/ω_{ce} and $k_{\perp} a_e$ are much less than unity, so that it is appropriate to treat perpendicular electron motion in the fluid limit. With regard to the parallel currents, the fluid approximation can be used in the high frequency regime ($\omega/\omega_{be} \gg 1$) but is not applicable at low frequencies ($\omega/\omega_{be} \ll 1$).

The current density in the fluid approximation is just $\underline{J} = en_e \underline{v}$, where $\underline{v}$ is determined from the force equation. With the assumption $\underline{J} \sim \exp(ik_{\perp} y - i\omega t)$, $k_x = 0$, and $\omega/\omega_{ce} \ll 1$, one obtains

$$4\pi \underline{J} = \frac{\omega_{pe}^2}{\omega} \left[\left(\frac{\omega}{\omega_{ce}} \epsilon_y \right) \hat{x} + \left(-\frac{\omega}{\omega_{ce}} \epsilon_x \right) \hat{y} + (i\epsilon_z) \hat{z} \right], \quad (2.32)$$

where $\hat{z} \equiv \underline{B}(s)/|B(s)|$ is the unit vector along the applied magnetic field. Maxwell's equations can be combined to yield

$$\underline{\epsilon} = \underline{E} + \frac{4\pi i \omega}{k_{\perp}^2 c^2} \underline{J}, \quad (2.33)$$

where it is assumed $(\omega/k_{\perp} c)^2 \ll 1$. Substitution of Eq. (2.32) into this equation gives

$$\underline{\epsilon} = \underline{E} + \left(\frac{\omega_{pe}}{k_{\perp} c} \right)^2 \left[\left(i \frac{\omega}{\omega_{ce}} \epsilon_y \right) \hat{x} + \left(-i \frac{\omega}{\omega_{ce}} \epsilon_x \right) \hat{y} + (i \epsilon_z) \hat{z} \right]. \quad (2.34)$$

Noting that $E_x = 0$ and equating the components of the above equation leads to the following result for the total electric field:

$$\underline{\epsilon} = \left[\frac{\omega}{\omega_{ce}} \left(\frac{\omega_{pe}}{k_{\perp} c} \right)^2 k_{\perp} \phi \right] \hat{x} + \left[(-ik_{\perp} \phi) \right] \hat{y} + \left[- \left[1 + \left(\frac{\omega_{pe}}{k_{\perp} c} \right)^2 \right]^{-1} \frac{\partial \phi}{\partial z} \right] \hat{z}. \quad (2.35)$$

with $(\omega/\omega_{ce})(\omega_{pe}/k_{\perp} c)^2$ taken to be less than unity.

Since the parallel current contribution to $\epsilon_{\parallel}$ cannot be determined at low frequencies from a fluid treatment, one must calculate the electric field ($E_{\parallel}^T$) due to such effects by considering the parallel component of the wave equation, i. e.,

$$E_{\parallel}^T = - \frac{i\omega}{k_{\perp}^2 c^2} 4\pi J_{\parallel} = - \frac{i\omega}{k_{\perp}^2 c^2} \omega_{pe}^2 \int_{-\infty}^{\infty} dv_{\parallel} \frac{\partial f_{oe}}{\partial v_{\parallel}} \int_{-\infty}^t dt' v_{\parallel}' \epsilon_{\parallel}', \quad (2.36)$$

where $\epsilon_{\parallel} = - \frac{\partial \phi}{\partial z} + E_{\parallel}^T$ = total parallel electric field. With $E_{\parallel}^T = - (\partial/\partial z)$, Eq. (2.41) can be written as

$$E_{\parallel}^T = - \frac{i\omega}{k_{\perp}^2 c^2} \omega_{pe}^2 \int_{-\infty}^{\infty} dv_{\parallel} \frac{\partial f_{oe}}{\partial v_{\parallel}} \int_{-\infty}^t dt' v_{\parallel}' \frac{\partial}{\partial z'} (\phi + \psi) \quad (2.37)$$

or

$$E_{\parallel}^T = - \frac{i\omega}{k_{\perp}^2 c^2} \omega_{pe}^2 I \left(1 - \frac{i\omega \omega_{pe}^2}{k_{\perp}^2 c^2} I \right)^{-1} \phi, \quad (2.38)$$

where

$$I = \frac{\omega}{k_{\parallel}} \int_{-\infty}^{\infty} dv_{\parallel} \frac{\partial f_{oe}}{\partial v_{\parallel}} \frac{1}{v_{\parallel} - (\omega/k_{\parallel})} = - \frac{\omega}{k_{\parallel}} Z'(\xi),$$

$\xi \equiv \omega/k_{\parallel} v_{the}$, Z' is the derivative of the plasma dispersion function,⁸⁰ and an electrostatic potential of the form, $\phi = \phi_0 \exp(ik_{\parallel}z - i\omega t)$, has been considered. Hence, the additional field due to the parallel current is

$$E_{\parallel}^T = \frac{\partial \phi}{\partial z} \frac{\omega_{pe}^2}{k_{\perp}^2 c^2} \xi^2 Z'(\xi) \left[1 + \frac{\omega_{pe}^2}{k_{\perp}^2 c^2} \xi^2 Z'(\xi) \right]^{-1}. \quad (2.39)$$

In the low frequency regime, $\xi \approx \omega/\omega_{pe} \ll 1$ and $Z'(\xi) \approx -2$, so that $E_{\parallel}^T \ll \partial\phi/\partial z$. The parallel current contribution is therefore negligible at low frequencies. At the other limit, $\xi \approx \omega/\omega_{pe} \gg 1$ and $Z'(\xi) \approx 1/\xi^2$, which leads to $\epsilon_{\parallel} = -(\partial\phi/\partial z) [1 + (\omega_{pe}/k_{\perp}c)^2]^{-1}$. This is just the fluid result given in Eq. (2.35). In Section 2.3 an appropriate potential for a bounded system will be considered, and it will be shown that these estimates of the parallel current modifications remain valid at the high and low frequency limits. The calculations that follow proceed as before except that the electrostatic field is now replaced by the total electric field $\underline{E}$.

2.2.2 High Frequency Modes

As noted earlier, the electron density response in the high frequency regime ($\omega \gg \omega_{ci}$) can be obtained from the continuity equation

$$4\pi \rho_e = \frac{1}{i\omega} \nabla \cdot 4\pi \underline{J}. \quad (2.40)$$

Substituting Eqs. (2.32) and (2.35) into this equation and allowing for a radial density gradient then gives

$$4\pi\rho_e(s) = - \left\{ \frac{B(s)}{\omega^2} \frac{\partial}{\partial s} \left(\frac{\tilde{\omega}_{pe}(s)}{B(s)} \frac{\partial}{\partial s} \right) + \frac{k_{10}^2 \omega_{pe}^2(s)}{\omega_{ce}(s) \omega_{ceo}} \right. \\ \left. \times \left[1 + \left(\frac{\omega_{pe}(s)}{k_1(s)c} \right)^2 \right] + \frac{k_{y0} \omega_{pi}^2(s)}{R_p \omega_{cio}} \right\} \phi(s), \quad (2.41)$$

where

$$\tilde{\omega}_{pe}(s) \equiv \omega_{pe}^2(s) \left[1 + \left(\frac{\omega_{pe}(s)}{k_1(s)c} \right)^2 \right]^{-1}.$$

It is obvious that the above result, when combined with Poisson's equation, leads to a differential equation of the same form as the one encountered in the electrostatic case. Hence, one can apply the same technique described earlier to obtain the dispersion relation

$$\int_{-L/2}^{L/2} \frac{ds}{B(s)} \left\{ \frac{B(s)}{B_0} + \frac{\omega_{pi}^2(s)}{\omega_{cio}^2} \frac{B_0}{B(s)} \left[\frac{m_e}{M_1} \left(1 + \frac{\omega_{pe}^2(s)}{k_1^2(s)c^2} \right) + \frac{k_{y0} \omega_{ci}(s)}{k_{10}^2 R_p \omega} \right. \right. \\ \left. \left. + \frac{\omega_{ci}^2(s)}{\omega_{pi}^2(s)} H(\omega, s) \right] \right\} = 0. \quad (2.42)$$

It should be noted, however, that the condition for the validity of the flute expansion, $k_{||}(\omega, s)L/\pi < 1$, is more difficult to satisfy for finite beta because the parallel wavelength is reduced by the factor $[1 + (\omega_{pe}/k_1 c)^2]$, i.e., the appropriate expression for $k_{||}$, is now given by

$$k_{||}^2(\omega, s) = \left(1 + \frac{\omega_{pe}^2(s)}{k_1^2(s)c^2} \right) \frac{\omega_{ci}^2 k_{10}^2}{\omega_{ci}(s) \omega_{cio}} \frac{m_e}{M_1} \left\{ \frac{\omega_{ci}^2(s)}{\omega_{pi}^2(s)} \left[1 + H(\omega, s) \right] \right. \\ \left. + \frac{m_e}{M_1} \left(1 + \frac{\omega_{pe}^2(s)}{k_1^2(s)c^2} \right) + \frac{k_{y0} \omega_{ci}(s)}{k_{10}^2 R_p \omega} \right\}. \quad (2.43)$$

Since the ion contribution, $(\omega_{ci}/\omega_{pi})^2 H$, is typically of order 0.1, the length criterion for using the flute formalism at high density ($\omega_{pi} > 3\omega_{ci}$) is approximately

$$L < \frac{\pi}{k_1} \left\{ 10 \frac{M_1}{m_e} \left[1 + \frac{\beta_i}{(k_1 a_1)^2} \right]^{-1} \right\}^{1/2}. \quad (2.44)$$

where $\beta_i \equiv [(\omega_{pi}/\omega_{ci})(v_{thi}/c)]^2$.

2.2.3 Low Frequency Modes

In the absence of a radial density gradient, infinite medium theory indicates that low frequency ($\omega < \omega_{ce}$) flute modes can appear for sufficiently peaked velocity distributions.²² To deal with such instabilities in a finite system it is necessary to time average the perturbed potential over the electron orbits because the electron density response is highly nonlocal in the low frequency regime.⁷¹ In the analysis that follows, it will be rigorously demonstrated that the potential ϕ is a constant to order $k_{\perp}^2 \lambda_{De}^2 \ll 1$. With regard to finite beta effects, the total perpendicular electric field obtained in the fluid limit can be used (since ω/ω_{ce} , $k_{\perp} a_e \ll 1$), and, as shown in Section 2.2.1, modifications due to the parallel current can be ignored. In Section 2.3 the parallel current is included in a calculation that is valid over frequencies spanning the high and low frequency regimes, and it is found that the modification is indeed negligible in the low frequency limit.

The density operator for electrons with a Maxwellian distribution can be expressed as

$$4\pi\rho_e = \frac{m_e \omega_{pe}^2}{k T_e} \left\langle \int_{-\infty}^t dt' \underline{v} \cdot \underline{\varepsilon}' \right\rangle_e \quad (2.45)$$

where $\langle A \rangle_e \equiv (1/n_0) \int d^3 v f_{0e} A$, $f_{0e} = (m_e/2\pi k T_e)^{3/2} \exp(-m_e v^2/2k T_e)$.

Combining the perpendicular component of $\underline{\varepsilon}$ [from Eq. (2.35)] with Eq. (2.45) then gives

$$\begin{aligned} 4\pi\rho_e &= \frac{i m_e \omega_{pe}^2}{k T_e} \left\langle \int_{-\infty}^t dt' \left[k_{\perp} (-i \hat{c} v'_x + v'_y) \phi' - i v'_z (\partial/\partial z') \phi' \right] \right\rangle_e \\ &= - \left(i m \omega_{pe}^2 / k T_e \right) \left\langle k_{\perp}^2 - i I_2 \right\rangle_e, \end{aligned} \quad (2.46)$$

where $\hat{c} \equiv (\omega/\omega_{ce})(\omega_{pe}/k_{\perp} c)^2$ and $\phi' = \phi(z') \exp[ik_{\perp}(y' - y) + i\omega(t' - t)]$.

To lowest order the electron orbits are⁸¹

$$\begin{aligned} y' - y &= (v_{\perp}/\omega_{ce})(\sin \phi_1' - \sin \phi_1) \\ \phi_1' - \phi_1 &= \omega_{ce}(t' - t) \\ v_x' &= v_{\perp} \sin[\phi_1 + \omega_{ce}(t' - t)] \\ v_y' &= v_{\perp} \cos[\phi_1 + \omega_{ce}(t' - t)]. \end{aligned} \quad (2.47)$$

Using these orbits and averaging over the phase angle ϕ_1 then gives

$$I_1 = \sum_n \left[\frac{n J_n^2(\lambda)}{\lambda} - \hat{c} J_n(\lambda) J_n'(\lambda) \right] \int_{-\infty}^0 dt \phi[z(t)] \exp [i(\omega + n\omega_{ce}) t]. \quad (2.48)$$

where $\lambda \equiv k_1 v_1 / \omega_{ce} \ll 1$ and the following Bessel function identities⁷⁷ are employed:

$$\begin{aligned} \exp(i\lambda \sin \theta_1) &= \sum_n J_n(\lambda) \exp(in\theta_1) \\ J_{n+1}(\lambda) + J_{n-1}(\lambda) &= (2n/\lambda) J_n(\lambda) \\ J_{n+1}(\lambda) - J_{n-1}(\lambda) &= -2J_n'(\lambda). \end{aligned} \quad (2.49)$$

Since ω/ω_{be} and ω_{be}/ω_{ce} are small, the integral in Eq. (2.48) can be performed by expanding $\phi[z(t)]$ in harmonics of the orbit periodicity, i.e.,

$$\phi[z(t)] = \sum_l \phi^{(l)} \exp(il\omega_{be}t), \quad (2.50)$$

where

$$\phi^{(l)} = \frac{1}{T_{be}} \int_0^{T_{be}} dt' \phi[z(t')] \exp(il\omega_{be}t'), \quad t' \equiv \int^z dz/v_z, \quad \text{and } \omega_{be} \equiv \pi/T_{be}.$$

The integral $\equiv K_n$ thus becomes

$$\begin{aligned} K_n &= \sum_l \phi^{(l)} \int_{-\infty}^0 dt \exp[i(l\omega_{be} + n\omega_{ce} + \omega)t] \\ &= \sum_l \phi^{(l)} \sum_{p=0}^{\infty} \exp[-ip(l\omega_{be} - n\omega_{ce} + \omega)T_{be}] \int_0^{T_{be}} dt \exp[i(l\omega_{be} + n\omega_{ce} + \omega)t] \\ &= \frac{1}{i(\omega + n\omega_{ce})} \sum_l \left(\frac{\phi^{(l)}}{1 + \frac{l\omega_{be}}{\omega + n\omega_{ce}}} \right) \exp \left[i2\pi \left(\frac{\omega + n\omega_{ce}}{\omega_{be}} \right) \right] \\ &= \frac{\phi^{(0)}}{i(\omega + n\omega_{ce})}. \end{aligned} \quad (2.51)$$

Hence, Eq. (2.48) yields

$$I_1 = -i\phi^{(0)} \sum_n \left[\frac{J_n^2(\lambda)}{k_1 v_1} \left(1 - \frac{\omega}{\omega + n\omega_{ce}} \right) - \frac{J_n(\lambda) J_n^*(\lambda)}{\omega + n\omega_{ce}} \right] \\ = \frac{\phi^{(0)} \lambda^2}{2ik_1 v_1} \left(1 + \frac{\omega_{pe}^2}{k_1^2 c^2} \right). \quad (2.52)$$

Applying the same procedure to the I_2 -term in Eq. (2.46) leads to

$$I_2 = \phi - \phi^{(0)}. \quad (2.53)$$

Finally, use of the above results gives the following expression for the electron density response:

$$4\pi\rho_e = -\left(\frac{\omega_{pe}}{\omega_{ce}}\right)_0^2 \left(1 + \frac{\omega_{pe}^2(s)}{k_1^2(s)c^2} \right) \frac{m_e k_{10}^2}{2n_0 \kappa T_e} \left\langle v_{10}^2 \phi^{(0)} \right\rangle_e - \frac{1}{\lambda_{De}^2} \left\langle \phi(s) - \phi^{(0)} \right\rangle_e, \quad (2.54)$$

where

$$\lambda_{De}^2 \equiv \kappa T_e / m_e \omega_{pe}^2.$$

Since the electromagnetic corrections are mainly due to the electrons, the electrostatic ion density operator can be used here. This approximation is known to be valid⁸⁰ for treating nonresonant modes in spatially homogeneous systems. In the next chapter finite beta corrections to the ion orbits are taken into account in the derivation of the general electromagnetic flute dispersion relation for an inhomogeneous plasma. Since the instabilities of interest here are nonresonant ($\omega \ll \omega_{ci}$) and occur in the absence of radial gradients, the equation to be solved has the form

$$F(s)\phi(s) + G(s) + \frac{1}{\epsilon} \left\langle \phi(s) - \phi^{(0)} \right\rangle_e = 0, \quad (2.55)$$

where

$$F(s) \equiv \frac{B(s)}{B_0} + \frac{\omega_{pi0}^2}{k_1^2} \Omega(s) \sum_n \left\langle \frac{J_n^2(\lambda)}{n + \Omega(s)} \frac{1}{v_1} \frac{\partial}{\partial v_1} \right\rangle_1$$

$$\Omega(s) \equiv \frac{\omega}{\omega_{ci}(s)}$$

$$G(s) = \frac{\omega_{peo}^2}{\omega_{ceo}^2} \frac{m_e \beta_o}{2kT_e} \left\langle v_{1o}^2 \phi(o) \right\rangle$$

$$\beta_o = \left[1 + \left(\frac{\omega_{pe}}{k_1 c} \right)^2 \right]$$

$$\frac{\omega_{pe}^2(s)}{k_1^2(s)} \sim \frac{\omega_{peo}^2}{k_{1o}^2}$$

$$\langle A \rangle_{l,e} \equiv \int d^3v f_{oi,e} A = 2\pi \iint dE d\mu \frac{B(s)}{v_{||}(s)} A f_{oi,e}(E, \mu)$$

$$\phi(o) \equiv \frac{1}{\pi_{be}(E, \mu)} \int \frac{ds}{v_{||}(s)} \phi(s)$$

$$\epsilon \equiv k_{1o}^2 \lambda_{De}^2$$

The method that will be employed to solve the above integral equation was originally used by Coppi⁵² to analyze flute-type low frequency drift modes. In this approach one first separates the terms of the equation into spatially averaged and periodic parts, i. e.,

$$A(s) = \bar{A} + \tilde{A}(s)$$

with

$$\bar{A} \equiv \frac{1}{\xi_o} \int \frac{ds}{B(s)} A(s)$$

$$\xi_o \equiv \int \frac{ds}{B(s)}.$$

Substitution into Eq. (2.51) gives

$$\bar{F}\bar{\phi} + \bar{F}\tilde{\phi} + \tilde{F}\bar{\phi} + \tilde{F}\tilde{\phi} + \bar{G} + \tilde{G} + \frac{1}{\epsilon} \left\langle \tilde{\phi} - \tilde{\phi}(o) \right\rangle_e = 0, \quad (2.57)$$

where

$$\phi(o) = \frac{1}{T_{be}} \int \frac{ds}{v_{||}(s)} [\bar{\phi} + \tilde{\phi}(s)] = \bar{\phi} + \tilde{\phi}(o). \quad (2.58)$$

The $\int ds/B$ line average of this result gives

$$\overline{F\phi} + \overline{G} = -\overline{F\tilde{\phi}} = -\overline{F\tilde{\phi}}. \quad (2.59)$$

In what follows it will be demonstrated that the eigenfunctions are of the flute type, i. e., $(\tilde{\phi}/\phi) \sim k_{\perp}^2 \lambda_{De}^2 \ll 1$, and that the appropriate dispersion relation to lowest order is just Eq. (2.54) with the right-hand term set equal to zero.

When Eqs. (2.57) and (2.58) are combined, one obtains the following inhomogeneous integral equation:

$$F\tilde{\phi} + \overline{F\phi} + \tilde{G} + \frac{1}{e} \left\langle \tilde{\phi} - \tilde{\phi}(0) \right\rangle_e - \overline{F\tilde{\phi}} = 0. \quad (2.60)$$

In terms of its Fourier components $\tilde{\phi}$ can be expressed in the general form⁸³

$$\tilde{\phi}(s) = \sum_{l=1}^{\infty} \hat{\phi}_1 \cos [\pi l \xi(s) / \xi_0], \quad (2.61)$$

where

$$\xi(s) = \int_0^s \frac{ds}{B(s)} = \xi(E, \mu) \cos [\omega_{be}(E, \mu)t(s)]$$

$$\xi(E, \mu) = \int_{-L(E, \mu)}^{L(E, \mu)} \frac{ds}{B(s)}$$

$$\xi_0 = \oint \frac{ds}{B(s)}$$

$$\omega_{be}(E, \mu) = \pi / T_{be}(E, \mu)$$

$$T_{be}(E, \mu) = \int_{-L(E, \mu)}^{L(E, \mu)} \frac{ds}{v_{\parallel}(s)}.$$

This in turn implies that

$$\begin{aligned} \left\langle \tilde{\phi} - \tilde{\phi}(0) \right\rangle_e &= \sum_{l=1}^{\infty} \hat{\phi}_1 \left\langle \cos [A_1 \cos (\omega_{be}t)] - \frac{1}{T_{be}} \int_0^{T_{be}} dt \cos [A_1 \cos \omega_{be}t] \right\rangle_e \\ &= \sum_{l=1}^{\infty} \hat{\phi}_1 \left\langle \cos [A_1 \cos (\omega_{be}t)] - J_0(A_1) \right\rangle_e \end{aligned} \quad (2.62)$$

with $A_1 = [\pi l \xi(E, \mu) / \xi_0]$. Substitution into Eq. (2.56) then gives

$$\sum_{l=1}^{\infty} \hat{\phi}_1 \left\langle \cos [A_1 \cos (\omega_{be} t)] - J_0(A_1) \right\rangle_e = -\epsilon (F\tilde{\phi} + \tilde{F}\bar{\phi} + \tilde{G} - \bar{F}\bar{\phi}). \quad (2.63)$$

Operating on this result with $(1/\xi_0) \int ds \tilde{\phi}/B$ yields

$$\begin{aligned} \sum_{l=1}^{\infty} \hat{\phi}_1 \hat{\phi}_1 \frac{1}{\xi_0} \iint dE d\mu f_{oe} T_{be} \left[\frac{1}{\pi} \int_0^{\pi} d\theta \cos(A_1 \cos \theta) \cos(A_1 \cos \theta) - J_0(A_1) J_0(A_1) \right] = \\ -\epsilon \left\{ \bar{F} \sum_1 \hat{\phi}_1^2 + \sum_{l=1}^{\infty} \hat{\phi}_1 \hat{\phi}_1 \frac{1}{\xi_0} \int \frac{ds}{B} \cos(lA_s) \left[G_0 \left\langle \mu B_0 J_0(A_1) \right\rangle_e + \cos(lA_s) \tilde{F}(s) \right] \right. \\ \left. + \bar{\phi} \sum_1 \hat{\phi}_1 \frac{1}{\xi_0} \int \frac{ds}{B} \cos(lA_s) \tilde{F}(s) \right\}, \quad (2.64) \end{aligned}$$

where $G_0 = (\omega_{pe}/\omega_{ce})_0^2 (m_e \beta_0 / 2kT_e)$ and $A_s = \pi \xi(s)/\xi_0$. For the first Fourier component ($l=1$), Eq. (2.60) gives

$$(\hat{\phi}_1/\bar{\phi}) = \frac{-(\epsilon/\xi_0) \int (ds/B) \cos(lA_s) \tilde{F}(s)}{I_1 + O(\epsilon)}, \quad (2.65)$$

where

$$I_1 = \frac{1}{\xi_0} \iint dE d\mu f_{oe} T_{be} \left\{ \frac{1}{2} \left[1 + J_0(2\pi l \xi(E, \mu)/\xi_0) \right] - J_0^2(\pi l \xi(E, \mu)/\xi_0) \right\}.$$

Since $\tilde{F}(s)$ is typically of order 10^{-1} and $\epsilon \ll 10^{-1}$, $\tilde{\phi}_1/\bar{\phi}$ is of order ϵ provided $I_1 \gtrsim 10^{-1}$. I_1 is then calculated for Maxwellian electrons with the assumption that the applied magnetic field has a parabolic spatial dependence given by

$$B(s) = B_0 [1 + R(s/L)^2], \quad (2.66)$$

with $R \equiv R_0 - 1$ and $R_0 =$ mirror ratio. This implies that

$$\xi_0 = \frac{L}{B_0 R^{1/2}} \tan^{-1} (R^{1/2})$$

$$\frac{L(E, \mu)}{L} = \left(\frac{E - \mu B_0}{\mu B_0 R} \right)^{1/2} = \left(\frac{C}{R} \right)^{1/2}$$

$$T_{be} = \frac{\pi}{2} \frac{L}{R^{1/2}} (\mu B_0)^{-1/2}$$

$$\frac{\xi(E, \mu)}{\xi_0} = \frac{\tan^{-1}(C^{1/2})}{\tan^{-1}(R^{1/2})}$$

Substitution of these values into I_1 gives

$$I_1 = \left[\pi^{1/2} / 2 (kT_e)^{3/2} \tan^{-1}(R^{1/2}) \right] \int_0^\infty dE \exp(-E/kT_e) \int_0^E d\mu B_0 (\mu B_0)^{-1/2} \times \left\{ \frac{1}{2} [1 + J_0(2A_1)] - J_0^2(A_1) \right\}, \quad (2.67)$$

where $A_1 = \pi \tan^{-1}(C^{1/2}) / \tan^{-1}(R^{1/2})$. The integrals here were computed numerically using a Gaussian quadrature scheme,⁶⁴ and it was found that for $l = 1$, I_1 is of order 10^{-1} . Hence, $\hat{\phi}_1 / \bar{\phi}$ is of order ϵ .

With $\hat{\phi}_1$ determined, the higher Fourier components ($\hat{\phi}_l$, $l > 1$) can be readily calculated from Eq. (2.64) by an iterative process. For example, with $l = 2$, Eq. (2.64) gives

$$\hat{\phi}_2 / \bar{\phi} = \frac{-(\epsilon / \xi_0) \int (ds/B) \cos(2A_s) \tilde{F}(s)}{I_2 + O(\epsilon)} = 2I_{1,2} \frac{\hat{\phi}_1}{\bar{\phi}}, \quad (2.68)$$

where

$$I_{l,l+1} = I_{l+1,l} = \frac{1}{\xi_0} \iint dE d\mu f_{oe} T_{be} \left[\frac{1}{\pi} \int_0^\pi d\theta \cos(A_1 \cos \theta) \cos(A_{l+1} \cos \theta) - J_0(A_1) J_0(A_{l+1}) \right].$$

After numerically evaluating the integrals as before, it was found $|\hat{\phi}_2| < |\hat{\phi}_1|$. Further calculations then showed the magnitude of the Fourier components decreases with increasing l , so that, to a good approximation, $(\hat{\phi}_l / \bar{\phi})$ is of order ϵ . Finally, use of this result in the dispersion relation [Eq. (2.59)] demonstrates that the correction, $F(\hat{\phi} / \bar{\phi})$, is of order $\epsilon = k_1^2 \lambda_{De}^2 \ll 1$.

Since a uniform square well will be considered in the next section, it is appropriate here to evaluate the correction to the dispersion relation in this limit for the sake of later comparison. For this model, Eq. (2.61) reduces to

$$\tilde{\phi}(s) = \sum_1 \hat{\phi}_1 \exp(ik_1 z), \quad 1 = \pm 1, \pm 2, \dots \quad (2.69)$$

with $\hat{\phi}_1 = \hat{\phi}_{-1}$ and $k_1 = \pi/L$, and Eq. (2.63) becomes

$$\sum_1 \hat{\phi}_1 \exp(ik_1 z) = -\epsilon(F\hat{\phi} + \tilde{F}\hat{\phi} - \tilde{\tilde{F}}\hat{\phi}). \quad (2.70)$$

Operating with $\int_{-L}^L dz$ then gives

$$\sum_1 \hat{\phi}_1 \sum_1 \hat{\phi}_1 \left[\overline{\cos(k_1 z) \cos(k_1 z)} + \epsilon \overline{F(z) \cos(k_1 z) \cos(k_1 z)} \right] + \frac{\epsilon}{2} \tilde{\tilde{F}}(z) \overline{\cos(k_1 z)} = 0. \quad (2.71)$$

Since $\epsilon F(z)$ is typically much less than unity, the second term in the coefficient of $\hat{\phi}_1$ can be ignored, so that the magnitude of the Fourier components is given approximately by

$$\hat{\phi}_1 = -\frac{\epsilon \tilde{\tilde{F}}}{2} \overline{F(z) \cos(k_1 z)}. \quad (2.72)$$

Substitution of this result in Eq. (2.59) shows that the correction to the dispersion relation is

$$\Delta = \tilde{F}(\tilde{\phi}/\phi) = -\epsilon \tilde{F}(z) \sum_{1=1}^{\infty} \cos(k_1 z) \frac{1}{2L} \int_{-L}^L dz' \tilde{F}(z') \cos(k_1 z') = -\epsilon [\tilde{F}(z)]^2, \quad (2.73)$$

where use has been made of the Fourier expansion⁴⁴ for the even function $\tilde{F}(z)$. On the basis of the results in this section, it can be concluded that the eigenfunctions ϕ are flute-type and that the appropriate low frequency dispersion relation is

$$\int_{-L/2}^{L/2} \frac{ds}{B(s)} \left[\frac{B(s)}{B_0} + \frac{\omega_{peo}^2}{k_{10}^2} \Omega(s) \sum_n \left\langle \frac{J_n^2(\lambda)}{n + \Omega(s)} \frac{1}{v_1} \frac{\partial}{\partial v_1} \right\rangle \right. \\ \left. + \frac{\omega_{peo}^2}{k_{10}^2} \left(1 + \frac{\omega_{peo}^2}{k_{10}^2 c^2} \right) \frac{m_e}{2\kappa T_e} \left\langle v_{10}^2 \right\rangle \right] = 0, \quad (2.74)$$

provided

$$|k_{10}^2 \lambda_{De}^2 F(s)| \ll 1. \quad (2.75)$$

2.3 FURTHER JUSTIFICATION OF FINITE BETA FLUTE FORMALISM

As noted earlier, the motion of the electrons along the applied magnetic field determines the character of the eigenfunction solutions to the wave equation. In the preceding analysis the electromagnetic contribution to the total parallel electric field was estimated in the fluid limit in the high frequency regime and ignored in the low frequency regime. The present section examines the question of how the flute formalism for a finite system is affected when these approximations are not made. In what follows the electrons are considered to be contained in a square well, i.e., a finite length uniform slab surrounded by a vacuum.

The parallel component of the wave equation gives

$$k_{\perp}^2 \left(\epsilon_{\parallel} + \frac{\partial \phi}{\partial z} \right) = -\frac{i\omega}{c^2} 4\pi J_{\parallel}, \quad (2.76)$$

where $J_{\parallel} = e \int d^3v v_{\parallel} f_1$ and $\epsilon_{\parallel} = -\frac{\partial \phi}{\partial z} + E_{\parallel}^T$.

Hence, the total parallel electric field can be expressed in the following form:

$$\epsilon_{\parallel} = -\frac{\partial}{\partial z} (\phi + \psi), \quad (2.77)$$

where

$$E_{\parallel}^T = \frac{i\omega}{k_{\perp}^2 c^2} 4\pi J_{\parallel} = -\frac{\partial \psi}{\partial z} \quad (2.78)$$

and

$$\frac{d}{dz} \psi(z) = \frac{i\omega}{k_{\perp}^2 c^2} \frac{4\pi e^2}{m} \int_{-\infty}^{\infty} dv \frac{df_{oe}}{dv} \int_{-\infty}^{\infty} d\tau \exp(-i\omega\tau) v' \frac{d}{dz'} [\phi(z') + \psi(z')]. \quad (2.79)$$

With the assumption that $d^2\phi(z)/dz^2 \ll k_{\perp}^2$, Poisson's equation yields

$$k_{\perp}^2 \mathcal{F}(z) \phi(z) = -\frac{i\omega_{pe}^2}{\omega} \frac{d}{dz} \int_{-\infty}^{\infty} dv \frac{df_{oe}}{dv} \int_{-\infty}^{\infty} d\tau \exp(-i\omega\tau) v' \frac{d}{dz'} [\phi(z') + \psi(z')], \quad (2.80)$$

where

$$\mathcal{F}(z) = 1 + \left(\frac{\omega_{pe}}{\omega_{ce}} \right)^2 \left[1 + \left(\frac{\omega_{pe}}{k_{\perp} c} \right)^2 \right] + \frac{\omega_{pi0}^2}{k_{\perp}^2} \Omega(z) \sum_n \left\langle \frac{J_n^2(\lambda)}{n + \Omega(z)} \frac{1}{v_1} \frac{\partial}{\partial v_1} \right\rangle.$$

The problem is to solve the coupled integral equations, Eqs. (2.79) and (2.80), for $\phi(z)$ on a uniform slab of length L . Since there can be no surface charge at the microscopic level, it is required that the field at the plasma-vacuum interface be continuous. Hence, the appropriate boundary condition at $z = \pm L/2$ is

$$\frac{1}{\phi} \frac{d\phi}{dz} \Big|_{z=\pm L/2} = \mp k_1. \quad (2.81)$$

Equations (2.79) and (2.80) can be combined to give

$$k_1^2 \phi(z) = - \left[I_1 + \left(\frac{\omega_{pe}^2}{k_1^2 c^2} \right) I_2 + \left(\frac{\omega_{pe}^2}{k_1^2 c^2} \right)^2 I_3 + \dots \right], \quad (2.82)$$

with

$$I_1 = \frac{\omega_{pe}^2}{\omega} \frac{d}{dz} \int_{-\infty}^{\infty} dv \frac{df_{oe}}{dv} \int_{-\infty}^0 d\tau \exp(-i\omega\tau) v^1 \frac{d}{dz} \phi(z'),$$

$$I_2 = \omega_{pe}^2 \frac{d}{dz} \int_{-\infty}^{\infty} dv \frac{df_{oe}}{dv} \int_{-\infty}^0 d\tau \exp(-i\omega\tau) v^1 \int_{-\infty}^{\infty} dv'' \frac{df_{oe}}{dv''} \int_{-\infty}^0 d\tau'' \exp(-i\omega\tau'') v'' \frac{d}{dz''} \phi(z''),$$

and $I_3, I_4, \dots$ taking the same general form. To solve this equation it is convenient to adopt a technique first introduced by Landau¹¹ and later applied by Baldwin⁸⁵ to the analysis of electrostatic oscillations in a bounded electron plasma. The basic approach is to convert the integral equation defined within the slab to one defined in infinite space with a difference kernel. This is done by continuing the definition of $\phi(z)$ outside the slab by the following prescription:

$$\phi(z) = \phi(L - z) \text{ for } L/2 < z < 3L/2$$

and

$$\phi(z + 2nL) = \phi(z) \text{ for all } n = 0, \pm 1, \pm 2, \dots \quad (2.83)$$

The appropriate difference kernel is

$$K(|z - z'|) = \int_{-\infty}^{\infty} dv \frac{df_{oe}}{dv} \exp \left[i \frac{\omega}{v} |z - z'| \right]. \quad (2.84)$$

With the potential now periodic in $2L$ and defined in infinite space, one can Fourier analyze it as follows:

$$\phi(z) = \sum_n \phi_n \exp(-ik_n z) \text{ with } k_n = 2\pi n/L, \quad (2.85)$$

where $\phi_n = \phi_{-n}$. Substitution into Eq. (2.82) then gives for I_1

$$I_1 = \omega_{pe}^2 \frac{1}{\omega} \frac{d}{dz} \int_{-\infty}^{\infty} dv \frac{df_{oe}}{dv} \int_{-\infty}^0 d\tau \exp(-i\omega\tau) v^1 \frac{d}{dz'} \phi(z')$$

$$\begin{aligned}
& + \frac{2}{L} \sum_n \left[\delta \left(z + \frac{L}{2} - 2nL \right) \frac{d\phi}{dz} \Big|_{-\frac{L}{2}} - \delta \left(z - \frac{L}{2} + 2nL \right) \frac{d\phi}{dz} \Big|_{\frac{L}{2}} \right] \\
& = \omega_{pe}^2 \frac{1}{\omega} \frac{d}{dz} \int_{-\infty}^{\infty} dz' K(|z - z'|) \frac{d}{dz'} \phi(z') + C(z) \\
& = \omega_{pe}^2 \sum_n \phi_n \exp(-ik_n z) \int_{-\infty}^{\infty} dv \frac{df_{oe}}{dv} \frac{1}{\left(v - \frac{\omega}{k_n}\right)} + C(z)
\end{aligned}$$

with

$$C(z) = \frac{2k_1}{L} \phi \left(\pm \frac{L}{2} \right) \sum_n \left[\delta \left(z + \frac{L}{2} - 2nL \right) + \delta \left(z - \frac{L}{2} + 2nL \right) \right]. \quad (2.86)$$

For I_2 one obtains

$$\begin{aligned}
I_2 & = \omega_{pe}^2 \frac{d}{dz} \int_{-\infty}^{\infty} dz' K(|z - z'|) \sum_n \phi_n \exp(-ik_n z') \frac{\omega}{k_n} \int_{-\infty}^{\infty} dv \frac{df_{oe}}{dv} \frac{1}{\left(v - \frac{\omega}{k_n}\right)} \\
& = \omega_{pe}^2 \sum_n \phi_n \exp(-ik_n z) \left(\frac{\omega}{k_n} \right)^2 \left[\int_{-\infty}^{\infty} dv \frac{df_{oe}}{dv} \frac{1}{\left(v - \frac{\omega}{k_n}\right)} \right]^2.
\end{aligned} \quad (2.87)$$

Similarly, I_3 is given by

$$I_3 = \omega_{pe}^2 \sum_n \phi_n \exp(-ik_n z) \left(\frac{\omega}{k_n} \right)^4 \left[\int_{-\infty}^{\infty} dv \frac{df_{oe}}{dv} \frac{1}{\left(v - \frac{\omega}{k_n}\right)} \right]^3. \quad (2.88)$$

Equation (2.82) can now be written in the form

$$\begin{aligned}
\mathcal{F}(z)\phi(z) & = - \frac{\omega_{pe}^2}{k_1^2} \sum_n \phi_n \exp(-ik_n z) I_n \left[1 + \frac{\omega_{pe}^2}{k_1^2 c^2} \frac{\omega^2}{k_n^2} I_n + \left(\frac{\omega_{pe}^2}{k_1^2 c^2} \frac{\omega^2}{k_n^2} I_n \right)^2 + \dots \right] + C(z) \\
& = - \frac{\omega_{pe}^2}{k_1^2} \sum_n \phi_n \exp(-ik_n z) \frac{I_n}{\left(1 - \frac{\omega_{pe}^2}{k_1^2 c^2} \frac{\omega^2}{k_n^2} I_n \right)} + C(z),
\end{aligned} \quad (2.89)$$

where

$$I_n = \int_{-\infty}^{\infty} dv \frac{df_{oe}}{dv} \frac{1}{(v - \frac{\omega}{k_n})} = -\alpha Z'(\xi_n),$$

$$\xi_n = \frac{\omega}{k_n \alpha^{1/2}},$$

$\alpha = 1/v_{the}^2$, f_{oe} is Maxwellian, and Z' is the derivative of the plasma dispersion function. To lowest order the dispersion relation is just $\overline{\phi_0 \mathcal{F}(z) \phi_0} = 0$, where the bar denotes integration inside the slab over the length of the plasma ($\int dz$). The correction is given by

$$\Delta = \overline{\phi(z) \mathcal{F}(z) \phi(z)} / \phi_0^2 \approx \overline{2 \mathcal{F}(z) \phi(z)} / \phi_0, \quad (2.80)$$

with $\overline{\mathcal{F}(z)} = 0$. Operating on Eq. (2.80) with $\int dz \phi(z)$ yields

$$\sum_n \phi_n \left[\sum_{n'} \phi_{n'}^* \left[\overline{A_n \cos(k_n z) \cos(k_{n'} z)} - \epsilon \overline{\mathcal{F}(z) \cos(k_n z) \cos(k_{n'} z)} \right] - \frac{1}{2} \epsilon \phi_0 \overline{\mathcal{F}(z) \cos(k_n z)} \right] = 0, \quad (2.91)$$

where $A_n = Z'(\xi_n) / [1 + (\omega_{pe}/k_1 c)^2 \xi_n^2 Z'(\xi_n)]$, $\epsilon = k_1^2 \lambda_{De}^2$, and $n, n' = \pm 1, \pm 2, \dots$. The second term in the coefficient of $\phi_{n'}$ is negligible compared to the first since $\epsilon \mathcal{F}(z)$ is typically much less than A_n . Hence, to a good approximation, the magnitude of the Fourier components is

$$\phi_n = \frac{\epsilon \phi_0}{2 A_n} \overline{\mathcal{F}(z) \cos(k_n z)}. \quad (2.92)$$

In the low frequency limit, ξ_n is much less than unity, so that Z' , $A_n \approx -2$, which implies that

$$\phi_n \approx -\frac{\epsilon \phi_0}{4} \overline{\mathcal{F}(z) \cos(k_n z)}. \quad (2.93)$$

Hence, the correction to the dispersion relation is

$$\Delta = -\epsilon \mathcal{F}(z) \sum_{n=1}^{\infty} \cos(k_n z) \frac{1}{L} \int_{-L/2}^{L/2} dz' \mathcal{F}(z') \cos(k_n z')$$

$$= -\epsilon [\mathcal{F}(z)]^2, \quad (2.94)$$

which is just the result of the preceding section [Eq. (2.73)].

For ϵ_n much larger than unity, $Z' = 1/\epsilon_n^2$, so that in the high frequency regime A_n becomes

$$A_n = \frac{1}{\epsilon_n^2} \left(1 + \frac{\omega_{pe}^2}{k_1^2 c^2} \right)^{-1} \quad (2.95)$$

and Eq. (2.92) reduces to

$$\phi_n \approx \frac{\epsilon_n^2}{2} \epsilon \phi_0 \left(1 + \frac{\omega_{pe}^2}{k_1^2 c^2} \right) \overline{\mathcal{F}(z) \cos(k_n z)}. \quad (2.96)$$

Substitution into Eq. (2.80) then gives the following correction to the dispersion relation:

$$\Delta \approx k_1^2 \overline{[\mathcal{F}(z)]^2} \frac{\omega^2}{\omega_{pe}^2} \frac{L^2}{r^2} \left(1 + \frac{\omega_{pe}^2}{k_1^2 c^2} \right). \quad (2.97)$$

Since $\mathcal{F}(z)$ is typically of order 0.1, the requirement that this correction be small just corresponds to the flute criterion given in Eq. (2.44). It can therefore be concluded that the finite beta flute formalism remains valid in both the high and low frequency regimes.

Chapter III

Dispersion Relation Derivation

3.1 EQUILIBRIUM PROPERTIES AND PARAMETRIC CONSTRAINTS

It has been demonstrated in earlier work²⁶ that in the absence of radial gradients, the electromagnetic contribution of the ions is negligible for nonresonant instabilities. Hence, the appropriate dispersion relation for analyzing the low frequency Dory-Guest-Harris modes is just that given by Eq. (2.74). In this chapter radial gradients will be taken into account in the derivation of the electromagnetic dispersion relation for analyzing high frequency ($\omega \gtrsim n\omega_{ci}$) flute instabilities.

With regard to the properties of the system considered, the basic assumptions and results of the preceding chapter are used and may be briefly summarized as follows:

- (1) The spatial inhomogeneities in the plasmas are of order ϵ with a_i/L_m , $a_i/R_0 \sim \epsilon \ll 1$.
- (2) Higher order effects such as curvature and crossfield ($\underline{E} \times \underline{B}$) drifts are neglected.
- (3) Spatial variations perpendicular to the applied magnetic field are treated in the local approximation.
- (4) Multiple ion bounces are ignored and the straight-line orbit approximation is used.
- (5) The current along the applied magnetic field is due to the electrons, i.e., the ion contribution to the parallel current is taken to be negligible.
- (6) Temperature anisotropy, external forces (e.g., gravitational forces), and relativistic effects are not considered.
- (7) Landau damping is neglected.

The assumption regarding Landau damping is valid provided $k_{\parallel} v_{the} < \omega$ for the electrons and $k_{\parallel} v_{thi} < |\omega - n\omega_{ci}|$ for the ions. These requirements are generally satisfied in the parameter range of interest. Also, since $k_{\perp} a_e$, $\omega/\omega_{ce} \ll 1$, the condition $k_{\parallel} v_{the} < \omega$ allows one to use the fluid approximation for the electrons.

To satisfy the equilibrium Vlasov equation, the distribution function must be a function of the constants of motion of the particles. As noted in the previous chapter, this takes the general form $f_0(E, \mu, \psi)$ or $f_0(v_1^2, v_{\parallel}^2, x_g)$. The conventional approach²³ is to assume the function to be separable, i.e., $f_0(v_1^2, v_{\parallel}^2, \psi) = n(x_g) f(v_1^2) g(v_{\parallel}^2)$. This approximation greatly facilitates calculations and will be adopted here. It should be noted, however, that in the strict sense the perpendicular and parallel velocities are decoupled only in the large mirror ratio limit.²⁹

The parametric constraints on the system are imposed by the requirements for a self-consistent equilibrium. This means that in addition to satisfying the equilibrium Vlasov equation, the distribution must also satisfy Maxwell's equations,

$$\nabla \cdot \underline{E}_0 = 4\pi \sum_j e_j \int d^3v F_{0j} = 0 \quad (3.1)$$

and

$$\nabla \times \underline{B}_0 = \frac{4\pi}{c} \sum_j e_j \int d^3v \underline{v} F_{0j} \quad (3.2)$$

where $\underline{E}_0 = 0$, $\underline{B}_0 = B_0(z)(1 + \epsilon x)\underline{e}_z$, and $\epsilon = (1/B_0)(d/dx)B_0$. The first condition is just the requirement for charge neutrality in the equilibrium state. Following the usual approach,³⁵ one expands F_0 about the point of maximum variation dn/dx , which is chosen for convenience to be $x = 0$. To lowest order in the inhomogeneity parameter ϵ , this procedure leads to

$$F_{0j} = f_{0j}(v_{\perp}^2, v_{\parallel}^2) \{1 - \epsilon_j [x + (v_y/\omega_c)]\} \quad (3.3)$$

where

$$\epsilon_j = |(1/n)(dn/dx)|_{x=0}.$$

When this form is used in Eq. (3.1), the charge neutrality condition reduces to

$$\epsilon_i = \epsilon_e \quad (3.4)$$

The constraint of Eq. (3.2) requires that the diamagnetic current caused by the density gradient be balanced by the curl of the applied magnetic field. Using Eq. (3.3) this can be expressed in the form

$$\epsilon = -\frac{1}{2} \sum_j \beta_j \epsilon_j \quad (3.5)$$

where $\beta_j = 8\pi n_j e_j T_j / B_0^2$ is the ratio of particle pressure to magnetic pressure for species j .

3.2 FINITE BETA MODIFICATIONS OF PARTICLE ORBITS

Since the radial inhomogeneity parameter is a function of beta, it is appropriate to determine how the equilibrium particle orbits are affected when one deals with finite beta plasmas. To do this it is necessary to solve the following coupled equations:

$$\begin{aligned} (d/dt)v_x^i &= \pm \omega_{ci}(1 + \epsilon x^i)v_y^i \\ (d/dt)v_y^i &= \mp \omega_{ci}(1 + \epsilon x^i)v_x^i \end{aligned} \quad (3.6)$$

with the upper sign for electrons, the lower for the ions, and $\omega_{cj} \equiv eB_0(z)/m_j c$. These equations can be combined to yield to first order in ϵ

$$(d^2/dt^2)v_y' \mp \epsilon v_x'(d/dt')v_y' \mp \epsilon v_y'(d/dt')v_x' \pm \hat{\omega}_{cj}(d/dt')v_x' = 0, \quad (3.7)$$

where

$$\hat{\omega}_{cj} = \omega_{cj} \{1 + \epsilon_j [x \pm (v_y/\omega_{cj})]\}.$$

The lowest order solutions to Eq. (3.6) are just $v_x' = v_1 \sin(\phi_1 + \omega_{cj}\tau)$ and $v_y' = v_1 \cos(\phi_1 + \omega_{cj}\tau)$ with $\tau = t' - t$ and ϕ_1 the perpendicular velocity phase angle.⁸¹ Substitution of these results into Eq. (3.7) yields to first order in ϵ

$$[(d^2/d\tau^2) + \hat{\omega}_{cj}^2]v_y' = [1 + 3\cos^2(\phi_1 + \omega_{cj}\tau)](\epsilon/2)\omega_{cj}v_1^2. \quad (3.8)$$

This differential equation can be readily solved by standard techniques⁸⁷ to give

$$v_y' = v_1 \cos(\phi_1 + \hat{\omega}_{cj}\tau) + (\epsilon v_1^2/2\omega_{cj})[1 - \cos 2(\phi_1 + \omega_{cj}\tau) - \cos(\omega_{cj}\tau) + \cos(2\phi_1 + \omega_{cj}\tau)]. \quad (3.9)$$

Integrating over τ one then obtains

$$y' - y = (v_1/\hat{\omega}_{cj})[\sin(\phi_1 + \hat{\omega}_{cj}\tau) - \sin\phi_1] + (\epsilon v_1^2/2\omega_{cj})\tau - (\epsilon v_1^2/4\omega_{cj}^2)[\sin 2(\phi_1 + \omega_{cj}\tau) + \sin 2\phi_1] + (\epsilon v_1^2/2\omega_{cj}^2)[\sin(2\phi_1 + \omega_{cj}\tau) - \sin(\omega_{cj}\tau)]. \quad (3.10)$$

For the x-component the same procedure yields the equation

$$[(d^2/d\tau^2) + \hat{\omega}_{cj}^2]v_x' = (3/2)\epsilon\omega_{cj}v_1^2 \sin 2(\phi_1 + \omega_{cj}\tau). \quad (3.11)$$

This has the solution

$$v_x' = v_1 \sin(\phi_1 + \hat{\omega}_{cj}\tau) + (\epsilon v_1^2/2\omega_{cj})[\sin 2(\phi_1 + \omega_{cj}\tau) - \sin(2\phi_1 + \omega_{cj}\tau)]. \quad (3.12)$$

Equations (3.9), (3.10), and (3.12) determine the unperturbed particle orbits for a finite beta plasma to first order in ϵ .

3.3 ELECTROMAGNETIC FLUTE DISPERSION RELATION

The Vlasov-Maxwell equations can be combined to yield the following wave equation:

$$\nabla(\nabla \cdot \underline{E}) - (\nabla^2 + (\omega/c)^2)\underline{E} = (4\pi i\omega/c^2)\underline{J}, \quad (3.13)$$

where $\underline{J} = \sum_j e n_j \int d^3v \underline{v} f_{1j}$. Using the general distribution of Eq. (3.3), the current density becomes

$$\underline{J} = \sum_j \frac{\omega_{pj}^2}{2\pi n_j} \int d^3v \underline{v} \int_{-\infty}^t dt'' \left\{ \underline{E}_x'' \underline{v}_x \left[\frac{\partial F_{0j}}{\partial v_1^2} + \frac{k_1 \epsilon_j' f_{0j}}{2\omega_{cj}} \right] + \underline{E}_y'' \left[\underline{v}_y \frac{\partial F_{0j}}{\partial v_1^2} - \frac{\epsilon_j' f_{0j}}{2\omega_{cj}} \right] + \underline{E}_z'' \underline{v}_z \left[\frac{\partial F_{0j}}{\partial v_1^2} + \frac{k_1 \epsilon_j' f_{0j}}{2\omega_{cj}} \right] \right\}. \quad (3.14)$$

Since Landau damping is assumed negligible, the perpendicular components of $\underline{J}$ can be expressed as

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \begin{pmatrix} \hat{b}_{11} & \hat{b}_{12} \\ \hat{b}_{21} & \hat{b}_{22} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}, \quad (3.15)$$

with

$$\begin{aligned} \hat{b}_{11} &= \sum_j \frac{\omega_{pj}^2}{2\pi n_j} \int d^3v v_1 \left[\left(\frac{\partial}{\partial v_1^2} + \frac{k_1 \epsilon_j'}{2\omega_{cj}} \right) f_{0j} \langle \sin \phi_1 I_3 \rangle_{\phi_1} - \frac{\partial f_{0j}}{\partial v_1^2} \frac{\epsilon_j' v_1}{\omega_{cj}} \langle \sin \phi_1 \cos \phi_1 I_3 \rangle_{\phi_1} \right], \\ \hat{b}_{12} &= \sum_j \frac{\omega_{pj}^2}{2\pi n_j} \int d^3v v_1 \left[\frac{\partial f_{0j}}{\partial v_1^2} \langle \sin \phi_1 I_1 \rangle_{\phi_1} - \frac{\epsilon_j' f_{0j}}{2\omega_{cj}} \langle \sin \phi_1 I_2 \rangle_{\phi_1} - \frac{\partial f_{0j}}{\partial v_1^2} \frac{\epsilon_j'}{\omega_{cj}} \langle \sin \phi_1 \cos \phi_1 I_1 \rangle_{\phi_1} \right], \\ \hat{b}_{21} &= \sum_j \frac{\omega_{pj}^2}{2\pi n_j} \int d^3v v_1 \left[\left(\frac{\partial}{\partial v_1^2} + \frac{k_1 \epsilon_j'}{2\omega_{cj}} \right) f_{0j} \langle \cos \phi_1 I_3 \rangle_{\phi_1} - \frac{\partial f_{0j}}{\partial v_1^2} \frac{\epsilon_j' v_1}{\omega_{cj}} \langle \cos^2 \phi_1 I_3 \rangle_{\phi_1} \right], \\ \hat{b}_{22} &= \sum_j \frac{\omega_{pj}^2}{2\pi n_j} \int d^3v v_1 \left[\frac{\partial f_{0j}}{\partial v_1^2} \langle \cos \phi_1 I_1 \rangle_{\phi_1} - \frac{\epsilon_j' f_{0j}}{2\omega_{cj}} \langle \cos \phi_1 I_2 \rangle_{\phi_1} - \frac{\partial f_{0j}}{\partial v_1^2} \frac{\epsilon_j'}{\omega_{cj}} \langle \cos^2 \phi_1 I_1 \rangle_{\phi_1} \right], \\ \text{and } \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} &= \int_{-\infty}^t dt' \exp[ik_1(y' - y) + i\omega(t' - t)] \begin{pmatrix} v_y' \\ 1 \\ v_x' \end{pmatrix} [E(s')/E(s)], \end{aligned} \quad (3.16)$$

where $\underline{E} = \underline{E}(s) \exp(ik_1 y + i\omega t)$, $k_y = k_1$, and $\langle A \rangle_{\phi_1} = (1/2\pi) \int_0^{2\pi} d\phi_1 A(\phi_1)$.

In general, ion-cyclotron instabilities such as the drift modes are characterized by frequencies that satisfy the nonresonance requirement, $|\omega - n\omega_{ci}|^3 > n v_{th}^2 \omega_{ci}^{59}$. This implies that the fields are slowly varying compared to the phase functions and allows one to take $E(s') = E(s)$ in performing the time integrations in Eq. (3.16). To calculate these integrals the orbits specified by Eqs. (3.9), (3.10), and (3.12) are used. The additional terms, introduced by the finite beta corrections, lead to Bessel functions

with arguments, $k\epsilon v_1^2/2\omega_{c1}$, which are usually small. Hence one can expand these terms in a power series and obtain to order ϵ

$$\exp\left[i(k\epsilon v_1^2/2\omega_c^2)(\sin(2\phi + \omega_c\tau) - \sin(\omega_c\tau) - (1/2)\sin 2(\phi + \omega_c\tau) - (1/2)\sin 2\phi)\right] \\ = 1 + (k\epsilon v_1^2/4\omega_c)\Pi, \quad (3.17)$$

where

$$\Pi = \exp[i(2\phi + \omega_c\tau)] - \exp[-i(2\phi + \omega_c\tau)] - \exp(i\omega_c\tau) + \exp(-i\omega_c\tau) - (1/2)\{\exp[i2(\phi + \omega_c\tau)] \\ - \exp[-i2(\phi + \omega_c\tau)] + \exp(i2\phi) - \exp(-i2\phi)\}$$

and the subscripts on ω_c and ϕ have been dropped. Using the Bessel function expansion of Eq. (2.49) together with Eqs. (3.17) and (3.10) leads to the following expression for the phase factor:

$$\exp(ik_1(y' - y) + i\omega(t' - t)) = \sum_{n,m} J_n(\hat{\lambda})J_m(\hat{\lambda})\exp[i(n-m)\phi]\exp[i(m\hat{\omega}_c + \chi)]\left[1 + (k\epsilon v_1^2/4\omega_c^2)\Pi\right], \quad (3.18)$$

where

$$\omega_c = \omega_c \left\{ 1 + \epsilon \left[x + \left(\frac{v_1 \cos \phi}{\omega_c} \right) \right] \right\}, \quad \hat{\lambda} = k_1 v_1 / \hat{\omega}_c, \\ \chi = \omega + (k\epsilon v_1^2/2\omega_c).$$

Equations (3.18), (3.9), and (3.12) are then substituted into Eq. (3.16), and the time integrals readily evaluated to give

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \sum_{n,m} J_n(\lambda')J_m(\lambda')e^{i(n-m)\phi} \begin{pmatrix} \frac{1}{T} \left[\frac{v_1}{2} \left[\frac{e^{i\phi}}{(n+1)\hat{\omega}_c + \chi} + \frac{e^{-i\phi}}{(n-1)\hat{\omega}_c - \chi} + \frac{k\epsilon v_1^2}{4\omega_c^2} \Pi_1 \right] + \frac{\epsilon v_1^2}{4\omega_c} \Pi_1 \right] \\ \frac{1}{T} \left[\frac{1}{n\hat{\omega}_c + \chi} + \frac{k\epsilon v_1^2}{4\omega_c^2} \Pi_2 \right] \\ \frac{1}{T} \left[\frac{v_1}{2} \left[\frac{e^{i\phi}}{(n+1)\hat{\omega}_c + \chi} - \frac{e^{-i\phi}}{(n-1)\hat{\omega}_c - \chi} + \frac{k\epsilon v_1^2}{4\omega_c^2} \Pi_3 \right] - \frac{\epsilon v_1^2}{4\omega_c} \Pi_3 \right] \end{pmatrix}, \quad (3.19)$$

where

$$\Pi_1 = \frac{e^{i3\phi} - e^{i\phi}}{(n+2)\omega_c + \chi} - \frac{e^{-i3\phi} - e^{-i\phi}}{(n-2)\omega_c + \chi} + \frac{2(e^{i\phi} - e^{-i\phi})}{n\omega_c + \chi} - \frac{1}{2} \left[\frac{e^{i3\phi}}{(n+3)\omega_c + \chi} - \frac{e^{-i3\phi}}{(n-3)\omega_c + \chi} + \frac{e^{i3\phi} + e^{i\phi} - e^{-i\phi}}{(n+1)\omega_c + \chi} - \frac{e^{-i3\phi} + e^{-i\phi} - e^{i\phi}}{(n-1)\omega_c + \chi} \right],$$

$$\Pi_2 = \frac{e^{i2\phi} - 1}{(n+1)\omega_c + \chi} - \frac{e^{-i2\phi} - 1}{(n-1)\omega_c + \chi} - \frac{1}{2} \left[\frac{e^{i2\phi}}{(n+2)\omega_c + \chi} - \frac{e^{-i2\phi}}{(n-2)\omega_c + \chi} + \frac{e^{i2\phi} - e^{-i2\phi}}{n\omega_c + \chi} \right],$$

$$\Pi_3 = \frac{e^{i3\phi} - e^{i\phi}}{(n+2)\omega_c + \chi} + \frac{e^{-i3\phi} - e^{-i\phi}}{(n-2)\omega_c + \chi} - \frac{1}{2} \left[\frac{e^{i3\phi}}{(n+3)\omega_c + \chi} + \frac{e^{-i3\phi}}{(n-3)\omega_c + \chi} + \frac{e^{i3\phi} - e^{i\phi} - e^{-i\phi}}{(n+1)\omega_c + \chi} + \frac{e^{-i3\phi} - e^{-i\phi} - e^{i\phi}}{(n-1)\omega_c + \chi} \right],$$

$$III_1 = \frac{2}{n\omega_c + \chi} - \left[\frac{e^{i2\phi}}{(n+2)\omega_c + \chi} + \frac{e^{-i2\phi}}{(n-2)\omega_c + \chi} - \frac{e^{i2\phi} - 1}{(n+1)\omega_c + \chi} - \frac{e^{-i2\phi} - 1}{(n-1)\omega_c + \chi} \right],$$

$$III_3 = \left[\frac{e^{i2\phi}}{(n+2)\omega_c + \chi} - \frac{e^{-i2\phi}}{(n-2)\omega_c + \chi} - \frac{e^{i2\phi}}{(n+1)\omega_c + \chi} + \frac{e^{-i2\phi}}{(n-1)\omega_c + \chi} \right].$$

Before performing the phase integrations, terms with $\hat{\lambda}$ and $\hat{\omega}_c$ are expanded around λ and ω_c , respectively, to first order in ϵ , i.e.,

$$J_n(\hat{\lambda})J_m(\hat{\lambda}) = J_n(\lambda)J_m(\lambda) - \epsilon \left[x + \left(\frac{v_1 \cos \phi}{\omega_c} \right) \right] \lambda \frac{\lambda}{d\lambda} [J_n(\lambda)J_m(\lambda)]$$

and

$$(n\hat{\omega}_c + \chi)^{-1} = (n\omega_c + \chi)^{-1} - \epsilon \left[x + \left(\frac{v_1 \cos \phi}{\omega_c} \right) \right] n\omega_c (n\omega_c + \chi)^{-2}. \quad (3.20)$$

Substitution into Eq. (3.18) then gives

$$\begin{pmatrix} \hat{I}_1 \\ I_2 \\ \hat{I}_3 \end{pmatrix} = \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} + \epsilon \left[x + \left(\frac{v_1 \cos \theta}{\omega_c} \right) \right] \sum_{n,m} e^{i(n-m)\theta} \begin{pmatrix} i \frac{v_1}{2} \left[J_n J_m \left(\frac{(n+1)\omega_c e^{i\theta}}{[(n+1)\omega_c + \chi]^2} + \frac{(n-1)\omega_c e^{-i\theta}}{[(n-1)\omega_c + \chi]^2} \right) \right. \\ \left. + \left[\frac{e^{i\theta}}{(n+1)\omega_c + \chi} + \frac{e^{-i\theta}}{(n-1)\omega_c + \chi} \right] \lambda \frac{\lambda}{d\lambda} (J_n J_m) \right] \\ i \left[J_n J_m \frac{\omega_c}{(\omega_c + \chi)^2} + \frac{\lambda}{(\omega_c + \chi)} \frac{\lambda}{d\lambda} (J_n J_m) \right] \\ i \frac{v_1}{2} \left[J_n J_m \left(\frac{(n+1)\omega_c e^{i\theta}}{[(n+1)\omega_c + \chi]^2} + \frac{(n-1)\omega_c e^{-i\theta}}{[(n-1)\omega_c + \chi]^2} \right) \right. \\ \left. + \left[\frac{e^{i\theta}}{(n+1)\omega_c + \chi} - \frac{e^{-i\theta}}{(n-1)\omega_c + \chi} \right] \lambda \frac{\lambda}{d\lambda} (J_n J_m) \right] \end{pmatrix} \quad (3.21)$$

where $\hat{I}_1$, $\hat{I}_2$, and $\hat{I}_3$ are given by I_1 , I_2 , and I_3 in Eq. (3.19) with λ and ω_c substituted for $\hat{\lambda}$ and $\hat{\omega}_c$ respectively.

After Eqs. (3.15) and (3.21) are combined, the phase integrals in the resultant expression are evaluated. The matrix elements, $b_{1m} = [-4\pi i \omega / k_1 c]^2 \hat{b}_{1m}$, to first order in ϵ' , ϵ , thus become

$$\begin{aligned} b_{11} &= -\frac{2i\omega}{k_1^2 c^2} \sum_j \frac{\omega_j^2}{n_j} \int d^3v \left[\frac{\partial f_{0j}}{\partial v_1} \left(P_{31} - \frac{v_1}{\omega_{cj}} P_{31\epsilon'} \right) + \frac{k_1 \epsilon_j' f_{0j}}{2\omega_{cj}} P_{31_0} \right] \\ b_{12} &= -\frac{2i\omega}{k_1^2 c^2} \sum_j \frac{\omega_j^2}{n_j} \int d^3v \left[\frac{\partial f_{0j}}{\partial v_1} \left(P_{11} - \frac{\epsilon_j'}{\omega_{cj}} P_{11\epsilon'} \right) - \frac{\epsilon_j' f_{0j}}{2\omega_{cj}} P_{21_0} \right] \\ b_{21} &= -\frac{2i\omega}{k_1^2 c^2} \sum_j \frac{\omega_j^2}{n_j} \int d^3v \left[\frac{\partial f_{0j}}{\partial v_1} \left(P_{32} - \frac{\epsilon_j' v_1}{\omega_{cj}} P_{32\epsilon'} \right) + \frac{k_1 \epsilon_j' f_{0j}}{2\omega_{cj}} P_{32_0} \right] \\ b_{22} &= -\frac{2i\omega}{k_1^2 c^2} \sum_j \frac{\omega_j^2}{n_j} \int d^3v \left[\frac{\partial f_{0j}}{\partial v_1} \left(P_{12} - \frac{\epsilon_j'}{\omega_{cj}} P_{12\epsilon'} \right) - \frac{\epsilon_j' f_{0j}}{2\omega_{cj}} P_{22_0} \right] \quad (3.22) \end{aligned}$$

with

$$\hat{J} = \frac{1}{2} - J_1^2 + \frac{d}{d\lambda} J_0^2 \quad (3.22a)$$

$$P_{210} = \frac{1}{2} \sum^n \frac{J_n J_n'}{J_n^2 + \chi} \quad (3.22h)$$

$$- \frac{\chi}{3\Omega^3} J_n^2 - \Omega^2 \frac{d}{d\chi} \left(\frac{\chi}{\Omega^2 J_n^2} \right) - \sum^n \frac{1}{z} \frac{\omega_c}{\Omega^2} \left(\frac{\chi}{\Omega^2 J_n^2} - J_n J_n' \right) \quad (3.22g)$$

$$P_{12} = \frac{1}{2} \left(\sum^n \frac{\Omega^2 J_n^2}{\lambda^2} - \frac{\omega_c}{\chi} + \frac{2K_1}{\epsilon} \right) \left(\frac{1}{z} + \frac{\chi}{3\Omega^2} \right) - \sum^n \frac{1}{z} \frac{\omega_c}{\Omega^2} \left(1 + 2\Omega - \Omega^2 \right) \frac{\chi}{J_n J_n'} \quad (3.22f)$$

$$P_{320} = \frac{1}{2} \sum^n \frac{\Omega J_n J_n'}{\Omega^2 J_n^2 + \chi} \quad (3.22e)$$

$$\times \Omega^2 J_n^2 - 2\Omega \frac{d}{d\chi} \left[\left(\frac{\chi}{J_n^2} - \Omega^2 J_n J_n' \right) \right] - \sum^n \frac{1}{z} \frac{\omega_c}{\Omega^2} \left[\left(\frac{\chi}{J_n^2} - \Omega^2 J_n J_n' \right) \right] \quad (3.22d)$$

$$P_{32} = P_{320} - \frac{2K_1}{\epsilon} \left(\frac{1}{\Omega^2} + \frac{1}{z} - \frac{1}{\chi} + \sum^n \frac{1}{z} \frac{\omega_c}{\Omega^2} \left(\frac{\chi}{4\Omega^2 J_n J_n'} + (3 - \Omega^2) J_n^2 + (1 - \Omega^2 + 1) \frac{\chi}{\Omega^2} \right) \right)$$

$$\times J_n^2 - 2\Omega \frac{d}{d\chi} \left(\frac{\chi}{J_n^2} - \Omega^2 J_n J_n' \right) - \sum^n \frac{1}{z} \frac{\omega_c}{\Omega^2} \left(\frac{\chi}{J_n^2} - J_n J_n' \right) \quad (3.22c)$$

$$P_{11} = \frac{1}{2} \left(\sum^n \frac{\Omega J_n J_n'}{\Omega^2 J_n^2 + \chi} + \frac{2K_1}{\epsilon} \left(\frac{\chi}{\Omega^2} - \frac{1}{z} + \sum^n \frac{1}{z} \frac{\omega_c}{\Omega^2} \left(\frac{\chi}{\Omega^2} + \frac{\chi}{6\Omega J_n J_n'} - \Omega \left(1 - \Omega^2 - \frac{\chi}{\Omega^2} \right) \right) \right) \right)$$

$$P_{310} = \frac{1}{2} \sum^n \frac{J_n^2}{J_n^2 + \chi} \quad (3.22b)$$

$$\left(\sum^n \frac{1}{z} \frac{\omega_c}{\Omega^2} \left(\frac{\chi}{2\Omega} \left(\frac{\omega_c}{\Omega^2} + \chi \right) \frac{1}{z} + \frac{\chi}{J_n J_n'} \right) \right) + (1 + 2\Omega - \Omega^2) \left(\frac{\chi}{\Omega^2} - 1 \right) J_n^2 - 2\Omega \frac{d}{d\chi} \left[\left(\frac{\chi}{J_n^2} - \Omega^2 J_n J_n' \right) \right] \quad (3.22a)$$

$$P_{22_0} = iv_1 \left[\sum_n \frac{\Omega_n^2/\lambda}{n\omega_c + \chi} - \frac{1}{\omega_c} \right] \quad (3.22i)$$

$$P_{31_e} = -iv_1^2 \left[\frac{\hat{J}}{\omega_c \lambda} + \sum_n \frac{1}{n\omega_c + \chi} \left(\frac{\Omega}{\lambda} \right) \left[\frac{J_n J_n'}{\lambda} - (J_n')^2 \right] \right] \quad (3.22j)$$

$$P_{11_e} = -v_1^2 \left[\frac{\Omega}{\lambda^3 \omega_c} + \sum_n \frac{1}{n\omega_c + \chi} \left(\frac{\Omega^2}{\lambda^2} \right) \left[J_n J_n - \frac{J_n^2}{\lambda} \right] \right] \quad (3.22k)$$

$$P_{32_e} = -v_1^2 \sum_n \frac{1}{n\omega_c + \chi} \left(\frac{1}{\lambda} \right) \left[\frac{\Omega^2 J_n J_n'}{\lambda} - (J_n')^2 \right] \quad (3.22l)$$

$$P_{12_e} = iv_1^2 \left[\frac{1}{\lambda \omega_c} \left(\frac{1}{2} + \frac{\Omega^2}{\lambda^2} \right) + \sum_n \frac{1}{n\omega_c + \chi} \left(\frac{\Omega}{\lambda^2} \right) \left(\Omega^2 J_n J_n' - \frac{J_n^2}{\lambda} \right) \right] \quad (3.22m)$$

where $\Omega \equiv \omega/\omega_c$ and $\lambda \equiv k_1 v_1/\omega_c =$ argument of Bessel functions. The calculations leading to these results are straightforward but tedious and are presented in Appendix A. If the terms introduced by the finite beta orbit corrections are ignored, one recovers the form of the matrix elements derived by Mikhailovskii.³⁴

With $(\omega/k_1 c)^2$ and $(1/k_1^2 E) \nabla_s^2 E(s)$ taken to be much less than unity, Eqs. (3.13) and (3.15) can be combined to give for the x-component

$$E_x = - \left(\frac{b_{12}}{1+b_{11}} \right) E_y. \quad (3.23)$$

Using this result in the y-component leads to

$$\nabla_s E_{\parallel}(s) = -ik_1 \left(\frac{1}{k_1^2} \nabla_s^2 + \frac{\omega^2}{k_1^2 c^2} - b_{22} + \frac{b_{12} b_{21}}{1+b_{11}} \right) E_y. \quad (3.24)$$

Since the fluid approximation for electrons is valid in the regime of interest, one can use the result of the preceding chapter and take $E_y = -ik_1 \phi(s)$, $E_{\parallel}(s) = - \left\{ 1 + \left[\omega_{pe}(s)/k_1(s)c \right]^2 \right\}^{-1} (s/\partial s) \phi(s)$, $4\pi J_{\parallel}(s) = i\omega_{pe}^2(s) E_{\parallel}(s)/\omega$, and $k_1^2(s) = k_{10}^2 B_0/B(s)$. Operating on the parallel component of the wave equation with ∇_s then yields

$$\nabla_s^2 \phi(s) + \nabla_s E_{\parallel}(s) = \nabla_s \left[\frac{\omega_{pe}^2(s) B_0}{k_{\perp 0}^2 c^2 B(s)} \right] E_{\parallel}(s). \quad (3.25)$$

When Eqs. (3.24) and (3.25) are combined, the wave equation becomes

$$\frac{B(s)}{\omega^2} \frac{\partial}{\partial s} \left(\frac{\omega_{pe}^2(s)}{B(s)} \right) \frac{\partial}{\partial s} \phi(s) = -k_{\perp 0}^2 \left[\frac{B(s)}{B_0} - \left(\frac{k_{\perp 0} c}{\omega} \right)^2 \right] b_{22}(s) + \frac{(k_{\perp 0} c / \omega)^2 b_{12}(s) b_{21}(s)}{1 + b_{11}(s)} \phi(s), \quad (3.26)$$

where $\tilde{\omega}_{pe}^2(s) \equiv \omega_{pe}^2(s) \left\{ 1 + [\omega_{pe}(s)/k_{\perp}(s)c]^2 \right\}^{-1}$. Finally, application of the flute mode formalism, derived in Chapter II, leads to the following equation for the electromagnetic flute dispersion relation:

$$\int_{-L/2}^{L/2} \frac{ds}{B(s)} \left[\frac{B(s)}{B_0} - \left(\frac{k_{\perp 0} c}{\omega} \right)^2 \right] b_{22}(s) + \frac{(k_{\perp 0} c / \omega)^2 b_{12}(s) b_{21}(s)}{1 + b_{11}(s)} \phi(s) = 0, \quad (3.27)$$

where L is the plasma length and the matrix elements, $b_{lm}(s)$, are specified in Eq. (3.22). It is valid to use this result provided the condition $k_{\parallel}(s)L/\pi < 1$ is satisfied, i.e.,

$$L < k_{\perp 0}^{-1} \left(\frac{M_1}{m_e} \right)^{1/2} \pi \left[\left(1 + \frac{\omega_{pe}^2(s)}{k_{\perp 1}^2(s) c^2} \right) F_{\perp}(s) \right]^{-1/2}, \quad (3.28)$$

where $F_{\perp}(s)$ is the bracketed term in the integrand of Eq. (3.27).

Chapter IV

Dory-Guest-Harris Instability

4.1 INTRODUCTION

The Dory-Guest-Harris modes are flute type instabilities driven by the loss-cone source of free energy in a mirror plasma. In the original work²² on this subject the authors considered an infinite, homogeneous plasma with an ion equilibrium distribution of the form

$$F_o^{(j)}(v_1^2) = \int_{-\infty}^{\infty} dv_{\parallel} f_o(v_1^2, v_{\parallel}^2) = (\pi a^2 j!)^{-1} (v_1/a)^{2j} \exp(-v_1^2/a^2), \quad (4.1)$$

where $j = 1, 2, \dots$ and $\delta/\langle v_1 \rangle = j^{-1/2}$ is the relative full width at half maximum of the function. They carried out the analysis in the electrostatic limit and discovered that for sufficiently narrow distributions, instabilities with $k_{\parallel} = 0$ can occur at zero frequency and near harmonics of the ion-cyclotron frequency. In particular it was found that the high frequency ($\omega \gtrsim \omega_{ci}$) modes are stabilized when $j < 6$, while the zero frequency modes require $j < 3$ for stability. Since the latter requirement is more stringent, it is appropriate to focus attention on the zero frequency instabilities. As noted earlier, ion Landau damping is generally considered negligible in the analysis of these modes. This implies that for finite length systems the term "zero frequency" actually refers to modes characterized by low but finite frequencies, i.e., $\omega_{bi} < \omega < \omega_{be}$ where ω_{bi}, e are the ion and electron bounce frequencies.

The density range of instability for the zero frequency Dory-Guest-Harris mode is given by^{22,57}

$$\frac{\omega_{pi}^2}{\omega_{ci}^2} > \left[\frac{m_e}{M_i} + \frac{\omega_{ti}^2}{k_1^2} \int d^3 v J_o^2(\lambda_1) \frac{1}{v_1} \frac{\partial f_{oi}}{\partial v_1} \right]^{-1} \quad (4.2)$$

where $\lambda_1 = k_1 v_1 / \omega_{ci}$ and the right-hand term is required to be positive. From the form of the integrand it is clear that stability is strongly dependent on the shape of the distribution function, especially in the positive slope region. The destabilizing effect described above is just due to the fact that an increase in j corresponds to an increase in the positive slope of the function as well as a decrease in its width.

The minimum density thresholds, $(\omega_{pi}^2/\omega_{ci}^2)_{\min}$, correspond to the lowest densities required for instability and are determined from Eq. (4.2) with an equals sign replacing the inequality. Since the integral is a function of the parameter, $\lambda_1 = k_1 v_1 / \omega_{ci}$, it is also necessary to find the worst wave number k_1 . To illustrate this process it is convenient to deal with the case of $j \rightarrow \infty$ which reduces the distribution of Eq. (4.1) to

a delta function. In this limit the integral in Eq. (4.2) can be performed analytically and the threshold becomes

$$\left(\frac{\omega_{pi}^2}{\omega_{ci}^2} \right) = \left[\frac{m}{M} + \frac{1}{b} \frac{d}{db} J_0^2(b) \right]^{-1}, \quad (4.3)$$

where $b = k_1 a_1$. As shown in Fig. 2, the values of k_1 corresponding to instability fall in bands between successive pairs of zeros of J_0 and J_1 with the worst k_1 at $b = 3$. In Fig. 3 the dependence of the density thresholds on j are plotted as curve a.

Fig. 2. Unstable wave number bands for the Dory-Guest-Harris instability corresponding to a delta function velocity distribution. The ranges of instability lie between successive zeros of J_0 and J_1 where the function plotted is negative.

The general trend of the above results is of course expected since the narrower distributions correspond to larger deviations from the stable Maxwellian (monotonically decreasing) function and therefore represent a greater source of free energy. In addition, it is known from quasilinear theory⁸⁸ that waves with phase velocities in a

Fig. 3. Density thresholds for the DGH instability corresponding to the distributions of Eq. (4.1).

positive slope region will cause particles there to diffuse to lower velocity regions. For conservative systems this behavior just corresponds to the basic wave-particle instability mechanism whereby the particles surrender energy to the waves.

When electromagnetic effects are included in the above analysis, it is found that there exists a high as well as low density threshold for instability.⁷² The extra term, $\omega_{pe}^2/k_{\perp}^2 c^2$, introduced into the dispersion relation changes the equation for the threshold density into a quadratic of the form

$$\left(\frac{v_{thi}}{c}\right)^2 \frac{1}{(k_{\perp} a_i)^2} \left(\frac{\omega_{pi}^2}{\omega_{ci}^2}\right)^2 + \left[\frac{m_e}{M_i} + D_o(k_{\perp})\right] \left(\frac{\omega_{pi}^2}{\omega_{ci}^2}\right) + 1 = 0. \quad (4.4)$$

where

$$D_o(k_{\perp}) = \frac{\omega_{ci}^2}{k_{\perp}^2} \int d^3 v j_o^2(\lambda_i) \frac{1}{v_{\perp}} \frac{\partial f_{oi}}{\partial v_{\perp}}.$$

Using the distribution of Eq. (4.1) and noting that $(v_{thi}/c)^2 \approx 10^{-5}$ for a 10 keV deuterium plasma, one can readily calculate the stability thresholds from Eq. (4.4). These results, which are plotted as curve b on Fig. 3, show that the transverse wave coupling has a stabilizing effect on these modes.

As shown in Chapter II, when finite geometry effects are included in the analysis, the appropriate dispersion relation is obtained from the $\oint (ds/B)$ line average of the infinite medium flute dispersion equation. In previous work⁷¹ it was found that this modification has a very small stabilizing influence on the Dory-Guest-Harris modes. However, only the extreme case of the delta function distribution was considered. In what follows it will be shown that for the broader distributions typical of mirror devices such as 2X, both the finite length and finite beta modifications can be significant.

4.2 EQUILIBRIUM DISTRIBUTIONS

Since the Dory-Guest-Harris instabilities are sensitive to the details of the distributions, it is important to use in the stability analysis functions that reasonably approximate collisionally equilibrated distributions. The exact determination of such distributions is a complicated problem that involves numerically solving the Fokker-Planck equation, which is the Boltzmann equation, Eq. (1.1), with the collision term specified by the Fokker-Planck operator.⁸⁸ Extensive computations have been performed in this area with the most comprehensive work being that of Killeen and Marx.⁹⁰

In the derivation of analytic forms to approximate the numerical solutions, the basic assumption generally made is that the distributions are separable in the v and θ components of velocity, i. e., $f_o(v, \theta) = f(v)M(u)$, where $v^2 = v_{\perp}^2 + v_{\parallel}^2$ and $u = \cos \theta$

$= v_{\parallel}/v$. This procedure leads to an equation for $f(v)$ that still must be solved numerically in the case of ions. The separated equation for $M(u)$ however, is just Legendre's equation. Hence one can obtain closed expressions approximating the angular dependence.

In the absence of a potential, the particle loss boundary in a mirror machine is a cone with angle specified by

$$\sin^2 \theta_{Lc} = \frac{1}{R_0} \quad (4.5)$$

where $R_0 = B_{\max}/B_0$ is the mirror ratio. This implies that the constraint on the solutions for $M(u)$ is

$$M(u) \Big|_{u > u_0} = 0, \quad (4.6)$$

where

$$u_0 = [(R_0 - 1)/R_0]^{1/2} = \cos \theta_{Lc}.$$

Post and Rosenbluth²³ have performed extensive stability calculations using values for $M(u)$ satisfying Eq. (4.6), together with simple analytic forms for $f(v)$, e.g., $f(v) \sim v^2 \exp(-v^2)$. They found that loss-cone instabilities are sensitive to the width of $f(v)$ but are relatively unaffected by changes in $M(u)$. This analysis, however, does not include effects due to finite geometry, finite beta, and the presence of an ambipolar potential.

Because of their greater mobility, the electrons can escape from the ends of a mirror machine faster than the ions. As a result, an ambipolar potential, Φ , is built up that expels the low energy ions and traps low energy electrons.⁹¹ When this behavior is taken into account,⁹⁰ the loss angle for ions becomes

$$\sin^2 \theta_L = [1 + (e\Phi/E)]/R_0. \quad (4.7)$$

where $E = M_i v_i^2/2$. Hence, the minimum ion velocity possible for confinement is given by $E = e\Phi/[R_0 - 1]$, and the effective mirror ratio by

$$R_e = R_0 [1 + (e\Phi/E)]^{-1}. \quad (4.8)$$

As illustrated in Fig. 4, these modifications change the ion loss boundary from a cone to a hyperboloid of one sheet. The constraint on solutions for $M(u)$ is now specified by Eq. (4.6) with R_e replacing R_0 .

For the electrons the loss angle also takes the form specified by Eq. (4.6), except that $e\Phi$ has the opposite sign and $E = m_e v_e^2/2$. This implies that the minimum speed at which an electron can be lost is given by $E = -e\Phi$. In Fig. 4 it is seen that the

Fig. 4. Loss-cone and electron and ion loss boundaries in mirror systems.

modified loss region affects only the tail of the energy distribution. Hence, on the time scales of interest, the electron distribution can be taken to be Maxwellian. As shown by numerical calculations,⁹² this approximation is valid provided the ambipolar potential exists at an energy slightly above the average electron energy. This condition, which insures that the low energy electrons are trapped, is generally considered to be satisfied in mirror machines.^{90,91}

Reasonably accurate closed expressions for $M(u)$ have been derived by Cordey⁸³ with the ambipolar modification taken into account. He found

$$M(u) = 1 - (u^2/u_0^2) \quad \text{for } R_e \leq 1.5$$

$$M(u) = 1 + [\ln(1 - u^2)/\ln R_e] \quad \text{for } R_e > 1.5. \quad (4.9)$$

where R_e is given by Eq. (4.7). In more recent work Holdren⁹⁴ used a successive approximation technique on the Legendre equation and obtained the following result:

$$M(u) = \frac{u_o^2 - u^2 + (3u_o^2 - 1) \ln [(1 - u^2)/(1 - u_o^2)]}{u_o^2 - (3u_o^2 - 1) \ln (1 - u_o^2)} \quad (4.10)$$

Comparison with the exact numerical solutions showed both Eqs. (4.9) and (4.10) to be good approximations with the latter form slightly more accurate at low mirror ratios ($R_e < 1.5$).⁹⁴ In the stability calculations that follow, these expressions will be used together with energy distributions obtained from the Marx-Killeen Fokker-Planck code⁹⁰ as well as with experimentally-measured energy distributions from the 2X device.^{49,95}

4.3 DISPERSION RELATION

The general dispersion relation for analyzing the "zero frequency" ($\omega \ll \omega_{ci}$) Dory-Guest-Harris modes is given by Eq. (2.74) and can be written in the form

$$\left(\frac{\omega_{pio}^2}{\omega_{cio}^2} \right)^2 \left(\frac{\omega_{cio}^2}{k_{lo}^2 c^2} \bar{F}_e \right) + \left(\frac{\omega_{pio}^2}{\omega_{cio}^2} \right) \left(\bar{F}_i + \frac{m_e}{M_i} \bar{F}_i \right) + \frac{\bar{B}(s)}{B_o} = 0, \quad (4.11)$$

where

$$\bar{F}_e = \frac{Me}{2kTe} \overline{\langle v_{lo}^2 \rangle}_e,$$

$$\Omega(s) = \frac{\omega}{\omega_{cio}} \frac{B_o}{B(s)},$$

$$\bar{F}_i = \frac{\omega_{cio}^2}{k_{lo}^2} \Omega(s) \sum_n \left\langle \frac{J_n^2(\Lambda)}{n + \Omega(s)} \frac{1}{v_l} \frac{\partial}{\partial v_l} \right\rangle_i,$$

and the bar denotes the $\oint ds/B$ -line average operation. The spatial dependence of the magnetic field and the ambipolar potential are taken to be

$$\begin{aligned} B(s) &= B_o \{1 + R(s^2/L^2)\} \\ e\Phi(s) &= \Phi_m (s^2/L^2), \end{aligned} \quad (4.12)$$

where $R = R_0 \sim 1$ and $2L$ is the plasma length. This leads to the following expression for the electron integral of Eq. (4.11):

$$\overline{F}_e = \frac{1}{\kappa T_e} \frac{1}{\xi_0} \iint dE d\mu f_{oe} \mu B_0 \oint \frac{ds}{v_{||}(s)}, \quad (4.13)$$

where

$$f_{oe} = \pi^{-1/2} (\kappa T_e)^{-3/2} \exp(-E/\kappa T_e)$$

and

$$\xi_0 = \oint ds/B = \int_{-L}^L ds/B(s) = 2L \tan^{-1}(R^{1/2})/B_0 R^{1/2}.$$

At the turning points, $\pm L(E, \mu)$, $v_{||} = 0$, so that $\mu B = E$. This implies that $L(E, \mu) = L[E - \mu B_0]/(\mu B_0 R)^{1/2}$, which leads to

$$\oint \frac{ds}{v_{||}} = \int_{-L(E, \mu)}^{L(E, \mu)} ds [E - \mu B(s)]^{-1/2} = \frac{\pi L}{(\mu B_0 R)^{1/2}}.$$

Hence, the electron contribution to Eq. (4.11) is specified by

$$\begin{aligned} \overline{F}_e &= \frac{\pi^{1/2}}{2 \tan^{-1}(R^{1/2})} \frac{1}{(\kappa T_e)^{5/2}} \int_0^\infty dE \exp(-E/\kappa T_e) \int_0^E d\mu B_0 (\mu B_0)^{1/2} \\ &= \frac{\pi^{1/2}}{8 \tan^{-1}(R^{1/2})}. \end{aligned} \quad (4.14)$$

Since $\omega \ll \omega_{ci}$, the ion integral of Eq. (4.11) can be expressed as

$$\overline{F}_i = (M_i \omega_{ci}^2 / k_{i0}^2) M_0, \quad (4.15)$$

where

$$M_0 = \frac{1}{\xi_0} \iint dE d\mu J_0^2 [\alpha (\mu B_0)^{1/2}] \left[\left(\frac{\partial f_0}{\partial E} \right) I_1 + \left(\frac{\partial f_0}{\partial \mu B_0} \right) I_2 \right].$$

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \int \frac{ds}{v_{||}(s)} \begin{pmatrix} 1 \\ B_0/B(s) \end{pmatrix} = \int_{-L(E, \mu)}^{L(E, \mu)} ds [E - \mu B(s) + e\Phi(s)]^{-1/2} \begin{pmatrix} 1 \\ B_0/B(s) \end{pmatrix},$$

and $\alpha = (k_{10}/\omega_{c10})(2/M_i)^{1/2}$. Using Eq. (4.12), I_1 and I_2 are integrated, and one obtains $M_0 = [\pi/2 \tan^{-1}(R^{1/2})] D_0$, where

$$D_0 = \iint dE d\mu B_0 J_0^2 [\alpha(\mu B_0)^{1/2}] \frac{(\partial f_0 / \partial E)}{[\mu B_0 - (\phi_m/R)]^{1/2}} + \frac{(\partial f_0 / \partial \mu B_0)}{[E - (\phi_m/R)]^{1/2}}. \quad (4.16)$$

If the angular dependence specified by Eq. (4.9) is employed, the ion equilibrium distribution takes the form

$$f_0(E, \mu) = f(E) \begin{cases} N_1 \left[(\mu B_0 R_0 - E - \phi_m) / (RE - \phi_m) \right] & \text{for } R_e \leq 1.5 \\ N_2 \left\{ 1 + \left[\ln(\mu B_0/E) / \ln(R_e) \right] \right\} & \text{for } R_e > 1.5, \end{cases} \quad (4.17)$$

where $R_e = R_0 E / (E + \phi_m)$, $f(E)$ is the energy distribution, and N_1 and N_2 are constants obtained from normalizing $f_0(E, \mu)$ to unity at the midplane ($s = 0$). As shown in Appendix B, substitution of Eq. (4.17) into Eq. (4.16) yields

$$D_0 = \int_{\frac{\phi_m}{R}}^{\infty} dE \frac{f(E)}{(E - \frac{\phi_m}{R})^{1/2}} \begin{cases} N_1 \left(\frac{R_0}{R} \int_{\frac{1}{R_0}}^1 dy J_0^2 (\alpha \Lambda^{1/2}) - J_0^2 (\alpha E^{1/2}) \right), & R_e \leq 1.5 \\ N_2 \left(\left(\frac{E - \frac{\phi_m}{R}}{\ln(R_0)} \right) \int_{\frac{1}{R_0}}^1 dy \left[\frac{J_0^2 (\alpha \Lambda^{1/2})}{\Lambda} \right] - J_0^2 (\alpha E^{1/2}) \right), & R_e > 1.5, \end{cases} \quad (4.18)$$

where

$$N_1 = \frac{3}{4} \left(\frac{R_0}{R} \right)^{1/2} / \int_{\frac{\phi_m}{R}}^{\infty} dE f(E) \left(E - \frac{\phi_m}{R} \right)^{1/2}$$

$$N_2 = \frac{1}{2} / \int_{\frac{\phi_m}{R}}^{\infty} dE f(E) \frac{E^{1/2}}{\ln(R_e)} \left[\ln \left(\frac{1 + \Lambda_1}{1 - \Lambda_1} \right) - 2\Lambda_1 \right]$$

$$\Lambda = \left(E - \frac{\phi_m}{R} \right) y + \frac{\phi_m}{R}, \quad \Lambda_1 = \left(\frac{RE - \phi_m}{R_0 E} \right)^{1/2}.$$

Alternatively, if Eq. (4.10) is used for the angular dependence, the distribution has the form

$$f_o(E, \mu) = N_3 f(E) \left(\frac{\mu B_o R_o - E - \phi_m + [E(2R - 1) - 3\phi_m] \ln[\mu B_o R_o / (E + \phi_m)]}{RE - \phi_m + [E(2R - 1) - 3\phi_m] \ln(R_e)} \right), \quad (4.19)$$

where N_3 is the normalization constant. In this case one obtains

$$D_o = N_3 \int_{\phi_m/R}^{\infty} dE \frac{f(E)}{(E - \frac{\phi_m}{R})^{1/2}} \left\{ \frac{R_o (RE - \phi_m)}{R \Lambda_2} \left[\int_{\frac{1}{R_o}}^1 dy J_o^2(\alpha \Lambda^{1/2}) \left(1 + \frac{\Lambda_3}{R_o \Lambda} \right) \right] - J_o^2(\alpha E^{1/2}) \right\}, \quad (4.20)$$

where

$$N_3 = \frac{1}{4} / \int_{\phi_m/R}^{\infty} dE f(E) \frac{1}{\Lambda_2} \left\{ \frac{(RE - \phi_m)^{3/2}}{3R_o^{1/2}} + \Lambda_3 E^{1/2} \left[\frac{1 + \Lambda_1}{1 - \Lambda_1} \right] - \Lambda_1 \right\},$$

$$\Lambda_2 = RE - \phi_m + \Lambda_3 \ln R_e,$$

$$\Lambda_3 = E(2R - 1) - 3\phi_m,$$

and Λ , Λ_1 , R_e are defined in Eq. (4.18). The calculations leading to this result are also given in Appendix B. Both Eqs. (4.18) and (4.20) can be readily computed numerically using a Gaussian quadrature scheme to perform the y -integration and a "Simpson's Rule" quadrature formula for unequal intervals⁹⁰ to evaluate the energy integral. Either expression can be used to specify the ion contribution to the dispersion equation.

4.4 STABILITY ANALYSIS AND RESULTS

4.4.1 Instability Thresholds and Growth Rate

It is clear from the quadratic form of the dispersion relation Eq. (4.11), that the general stability criterion for the Dory-Guest-Harris mode is

$$b_2 \geq 0 \text{ and/or } b_2^2 \leq 4b_1b_3. \quad (4.21)$$

where

$$b_1 = \frac{\omega_{cio}^2}{2k_{10}^2 c^2} ,$$

$$b_2 = \frac{2M_1 \omega_{cio}^2}{k_{10}^2} D_0 + \frac{m_e}{2M_1} ,$$

$$b_3 = 4R^{1/2}/\pi ,$$

and D_0 is defined in the preceding section. If Eq. (4.21) is not satisfied, then the lower and upper thresholds, $(\omega_{pio}^2/\omega_{cio}^2)|_{L, U}$ specifying the density range over which the plasma is unstable, is given by

$$\frac{\omega_{pio}^2}{\omega_{cio}^2} \Big|_{L, U} = \frac{-b_2 \mp (b_2^2 - 4b_1 b_3)}{2b_1} . \quad (4.22)$$

Growth rates, $\gamma_0 \equiv \text{Im}(\omega/\omega_{cio})$, associated with the unstable modes, are determined by the ion term, $\overline{F}_i$, of Eq. (4.11). In the threshold calculations only the $n = 0$ term in the summation is important because $\omega/\omega_{cio} \ll 1$ and $\gamma_0 = 0$. However, to calculate growth rates it is necessary to include the $n = \pm 1$ contribution, i. e.,

$$\overline{F}_{i1} = \frac{\omega_{cio}^2}{k_{10}^2} \gamma_0^2 \frac{\overline{B_0^2}}{B(s)^2} \left\langle J_1^2 \frac{1}{v_1} \frac{\partial}{\partial v_1} \right\rangle . \quad (4.23)$$

Following the same procedure as before, it is shown in Appendix B that this expression reduces to

$$\overline{F}_{i1} = \frac{M_1 \omega_{cio}^2}{k_{10}^2} \frac{\pi}{\tan^{-1}(R^{1/2})} \gamma_0^2 D_1 . \quad (4.24)$$

where

$$D_1 = \int_{\frac{\phi_m}{R}}^{\infty} dE \frac{f(E)}{\left(E - \frac{\phi_m}{R}\right)^{1/2}} \left\{ \begin{aligned} & N_1 \left[\frac{R_0}{R} \int_{\frac{1}{R_0}}^1 dy J_1^2(\alpha \Lambda^{1/2}) \left[\frac{9}{8} y^2 + \left(\frac{1}{2} - \frac{3}{4R_0} \right) y + \frac{3}{8} - \frac{1}{4R_0} \right] \right. \right. \\ & \quad \left. \left. - J_1^2(\alpha E^{1/2}) \right] \right\}, \quad R_e \leq 1.5 \\ & N_2 \left[\frac{1}{4 \ln(R_e)} \int_{\frac{1}{R_0}}^1 dy J_1^2(\alpha \Lambda^{1/2}) \left[\ln \left(\frac{R_0 \Lambda}{E + \frac{\phi_m}{R}} \right) (3y + 1) \right. \right. \\ & \quad \left. \left. + \left(\frac{3y^2 + 2y + 3}{2} \right) \left(y + \frac{\phi_m}{RE - \phi_m} \right)^{-1} \right] - J_1^2(\alpha E^{1/2}) \right] \right\}, \quad R_e > 1.5 \end{aligned} \right.$$

and the distribution given by Eq. (4.18) has been employed. Substitution back into Eq. (4.11) then yields the following expression for the growth rate:

$$\gamma_0 = \frac{\omega_{cio}}{2k_{10}} \left(\frac{b_1 x^2 + b_2 x + b_3}{-M_1 D_1} \right)^{1/2}, \quad (4.25)$$

where b_1 , b_2 , b_3 are defined in Eq. (4.21) and $x = (\omega_{pi}/\omega_{cio})^2$ is the density parameter in the unstable regime.

4.4.2 Energy Distributions from Experimental Data

Experimentally-measured energy distributions^{49,95} were used to investigate the stability of the 2X deuterium plasma against the Dory-Guest-Harris instabilities. In the experiment the plasma was injected into the device and magnetically compressed. The time evolutions of representative distributions are displayed in Figs. 5 and 6. These values for $f(E)$ were fed into a computer code implementing the theory presented in the preceding sections. For the functions⁹⁵ in Fig. 5 it was found that the broad 400 μ sec curve is a stable distribution, but that the narrower 200 μ sec curve is unstable in the density range $2.5 \times 10^{12} \leq n_0 \leq 3.5 \times 10^{13}$, where n_0 is the plasma density in units of particles/cm³. The growth rate corresponding to the experimental value of the density at 200 μ sec ($n_0 = 2.3 \times 10^{13}$) was calculated to be $\gamma_0 = 0.2$. When the unstable curve was slightly broadened, the adjusted distribution (dashed curve) was found to be stable. This again illustrates the sensitivity of the Dory-Guest-Harris mode to the shape of the distribution function.

Fig. 5. Experimentally measured energy distribution for the 2X plasma (data associated with Ref. 95).

Fig. 6. Experimentally measured energy distributions for the 2X plasma (data associated with Ref. 49).

In earlier work⁹⁵ the distributions in Fig. 5 were analyzed in the infinite, uniform medium, electrostatic limit by Post. These calculations were carried out using a computer code that did not specifically include effects due to the ambipolar potential, Φ . As noted in Section 4.2, the inclusion of Φ results in a change in the ion loss boundary which in turn leads to a modified angular distribution and to the constraint, $f(E) = 0$ for $E < \phi_m/R$. The latter result can strongly influence stability since it determines the shape of the distribution in the low energy range. In Post's calculations this effect is included in an indirect manner by the fact that the measured energy distributions used already reflect the influence of Φ . However, the omission of the stabilizing effects due to finite beta and finite geometry modifications led to more pessimistic DGH stability estimates. For example, the instability threshold calculated for the 200 μsec curve was a factor of four lower than the present estimate, and the stable 400 μsec curve was found to be only conditionally stable, i. e., the experimental value of the density at 400 μsec (6×10^{12}), was below Post's calculated instability threshold of 2×10^{14} . More significantly, there were several other cases considered in which distributions that were found to be unstable in the infinite medium, electrostatic limit were stabilized by the effects included in the present code. An example of such a case is the 200 μsec curve in Fig. 6.

The energy distributions in Fig. 6 came from the last set of data obtained from the 2X device⁴⁹ before it was rebuilt into 2X-II.⁵⁰ Calculations showed the 200 μsec and 600 μsec curves to be stable, but the 400 μsec curve to be unstable in the density range $4.4 \times 10^{12} \leq n_0 \leq 5.3 \times 10^{13}$. Instability in the latter case was marginal, since the measured density was less than a factor of two above the computed lower threshold. Also, as seen from Fig. 6, there is little difference between the stable 200 μsec and unstable 400 μsec curve. These facts imply that within the uncertainty of the data,⁴⁹ the existence of the Dory-Guest-Harris instability in the 2X plasma is a borderline proposition.

4.4.3 Energy Distributions from Fokker-Planck Calculations

In order to estimate the Dory-Guest-Harris stability requirements in the 2X-II⁵⁰ experiment, it was necessary to determine the effect of injecting neutral beams into a plasma characterized by an ion distribution that is stable against the DGH mode.⁹⁶ Using a variety of neutral beam sources with energies of 5 to 50 keV and currents of 5 to 20 A, equilibrium energy distributions were calculated with the Marx-Killeen Fokker-Planck code^{90,97} for ion densities of 2 to $4 \times 10^{13} \text{ cm}^{-3}$. These distributions were then used as input to the DGH code.

Results of the above calculations showed that for a mirror ratio of 2, source energy ratios ≥ 5 with average energies $> 20 \text{ keV}$ were required for stability. These ratios can be lowered at higher mirror ratios since the contribution of the ambipolar potential to the loss cone (ϕ_m/R) is then reduced.

Instability threshold densities for a number of the cases considered were plotted against δ , which is a function of the width of the distribution. As shown in Fig. 7, this gives an approximate stability boundary for the DGH mode. The width parameter δ is defined as

$$\delta = \Delta E_{fw}/E_p, \quad (4.26)$$

where ΔE_{fw} is the full width at half maximum of $f(E)$, and E_p is the value of the energy at the peak of the distribution. From the figure it is seen that the approximate condition for stability is

$$\delta \geq 2. \quad (4.27)$$

Application of this criterion to the measured distributions in Figs. 5 and 6 gave stability verdicts in complete agreement with the calculations of the preceding section.

4.4.4 Analytic Results and Stability Requirements

Dory-Guest-Harris stability boundaries as a function of ambipolar potential(Φ), average ion energy (E_0), and mirror ratio (R_0) can be estimated analytically by using a distribution of the form

$$f_0(E, \mu) = \hat{N} \exp(-E/E_0) S(\mu E_0 R_0 - E - \phi_m), \quad (4.28)$$

where

$$S(x) = \begin{cases} 1 & \text{for } x > 1 \\ 0 & \text{for } x \leq 1 \end{cases},$$

$$\hat{N} = (E_0)^{-3/2} (R_0/R\pi) \exp(\phi_m/RE_0),$$

and the function has been normalized to unity at the midplane. This distribution also includes the important ambipolar constraint requiring the function to vanish at low energies; i. e., $f_0(E, \mu) = 0$ for $E, \mu E_0 < \phi_m/R$. Substitution of Eq. (4.28) into Eq. (4.16) leads to the following expression for the ion integral:

Fig. 7. Upper and lower DGH thresholds (shown by dots) for 2-source Fokker-Planck distributions. The notations S, M, and U indicate stable, marginal, and unstable cases. Mirror ratio = 2 in all cases displayed.

$$\begin{aligned}
D_0 &= \int_{\frac{\phi_m}{R}}^{\infty} d\mu B_0 \frac{J_0^2 \left[\alpha (\mu B_0)^{1/2} \right]}{(\mu B_0 - \frac{\phi_m}{R})^{1/2}} \left\{ R_0^{1/2} \exp \left[- \left(\frac{\mu B_0 R_0 - \phi_m}{E_0} \right) \right] - \exp \left(- \frac{\mu B_0}{E_0} \right) \right\} \\
&= \frac{2}{E_0} \left(\frac{R_0}{\pi R} \right)^{1/2} \int_0^{\infty} dx J_0^2 \left[\alpha (x^2 + \hat{\phi})^{1/2} \right] \left(e^{-R_0 x^2} - e^{-x^2} \right) \\
&= \frac{2}{E_0} \left(\frac{R_0}{\pi R} \right)^{1/2} \int_0^{\infty} dx e^{-x^2} \left\{ J_0^2 \left[\alpha \left(\frac{x^2}{R_0} + \hat{\phi} \right)^{1/2} \right] - J_0^2 \left[\alpha (x^2 + \hat{\phi})^{1/2} \right] \right\}. \quad (4.29)
\end{aligned}$$

where

$$\alpha = \frac{k_{\perp 0}}{\omega_{c10}} \left(\frac{2E_0}{M_1} \right)^{1/2} \quad \text{and} \quad \hat{\phi} = \frac{\phi_m}{RE_0}.$$

In the low mirror ratio limit ($R_0 \rightarrow 1$, $R \ll 1$), the distribution is highly unstable since the ambipolar cutoff energy is very large, i.e., $\phi_m/RE_0 \gg 1$. The integral in Eq. (4.29) can be readily evaluated in this regime by expanding the first Bessel function in the integrand around the quantity, $s^2 + \hat{\phi}$. This operation gives

$$\begin{aligned}
D_0 &\approx - \frac{\alpha}{E_0} \left(\frac{R}{\pi} \right)^{1/2} \int_0^{\infty} dx e^{-x^2} \left\{ J_0^2 \left[\alpha (x^2 + \hat{\phi})^{1/2} \right] \right\} \frac{x^2}{(x^2 + \hat{\phi})^{1/2}} \\
&\approx - \frac{\alpha^2 R^{1/2}}{E_0 b} \frac{d}{db} J_0^2(b), \quad (4.30)
\end{aligned}$$

where $b \equiv \alpha(\hat{\phi})^{1/2}$. This just corresponds to the delta function distribution result of Eq. (4.3).

For larger mirror ratios, the above expansion breaks down since the parameter $\hat{\phi}$ is less than unity. Assuming the argument of the Bessel functions in Eq. (4.29) to be typically greater than unity, the integral can be written in the approximate form⁹⁸

$$D_0 \approx \frac{2}{E_0 \alpha} \left(\frac{R_0}{\pi R} \right)^{1/2} \frac{1}{\pi} [F_1 + F_2], \quad (4.31)$$

where

$$F_1 = \int_0^{\infty} dx \frac{e^{-x^2}}{\left(\frac{x^2}{R_0} + \hat{\phi}\right)^{1/2}} \left\{ 1 + \sin \left[2b \left(1 + \frac{x^2}{R_0 \hat{\phi}} \right)^{1/2} \right] \right\}$$

$$F_2 \approx \int_0^{\infty} dx \frac{e^{-x^2}}{\left(x^2 + \hat{\phi}\right)^{1/2}} \left\{ 1 + \sin \left[2b \left(1 + \frac{x^2}{\hat{\phi}} \right)^{1/2} \right] \right\},$$

with b defined in Eq. (4.30). The first integrals in F_1 and F_2 are of the form

$$I = \int_0^{\infty} dy \exp(-Ay) [y(1+y)]^{-1/2}. \quad (4.32)$$

From Ref. 99 [p. 319, Eq. (8)], this becomes

$$I = \exp(A/2) K_0(A/2) \quad (4.33)$$

where K_0 is the Kelvin function.

With regard to the second term in F_1 and F_2 , it is noted that the sine function will cut off the integral faster than the exponential, provided its argument is sufficiently large. This condition is satisfied for F_2 since $\hat{\phi}$ is small and $b > 1$. Hence the integral can be approximated by

$$\begin{aligned} \Pi_2 &= \int_0^{\infty} dx (x^2 + \hat{\phi})^{-1/2} \sin \left[2b \left(1 + \frac{x^2}{\hat{\phi}} \right)^{1/2} \right] \\ &= \int_1^{\infty} dy (y^2 - 1)^{-1/2} \sin(2by) \\ &= \frac{\pi}{2} J_0(2b) \end{aligned} \quad (4.34)$$

with the last equality coming from the Bessel function identity given in Ref. 98 [p. 360, Eq. (9.1.24)].

For F_1 the above approximation begins to break down as R_0 increases. This is due to the fact the coefficient of x^2 in the argument of the sine function is reduced by

a factor of $1/R_0$. Hence it becomes necessary to consider the exponential contribution. Expanding this factor leads to

$$\Pi_1 = \int_0^\infty dx \left(\frac{x^2}{R_0} + \hat{\phi} \right)^{-1/2} \sin \left[2b \left(1 + \frac{x^2}{R_0 \hat{\phi}} \right)^{1/2} \right] \left(1 - x^2 + \frac{x^4}{2} - \dots \right). \quad (4.35)$$

The lowest order term is evaluated as in Eq. (4.34) and becomes

$$(\Pi_1)_0 = R_0^{1/2} \frac{\pi}{2} J_0(2b). \quad (4.36)$$

For the first correction term one obtains

$$\begin{aligned} (\Pi_1)_1 &= R_0 \int_0^\infty dx \left(\frac{x^2}{R_0} + \hat{\phi} \right)^{-1/2} \sin \left[2b \left(1 + \frac{x^2}{R_0 \hat{\phi}} \right)^{1/2} \right] \left[\left(\frac{x^2}{R_0} + \hat{\phi} \right) - \hat{\phi} \right] \\ &= R_0 \left[R_0^{1/2} \hat{\phi} \int_1^\infty dy y^2 (y^2 - 1)^{-1/2} \sin(2by) - \hat{\phi} (\Pi_1)_0 \right] \\ &= R_0 \hat{\phi} \left[- R_0^{1/2} \frac{d}{db} \int_1^\infty dy y (y^2 - 1)^{-1/2} \cos(2by) - (\Pi_1)_0 \right]. \end{aligned} \quad (4.37)$$

From Ref. 99 [p. 423, Eq. (13)], the integral is given by

$$\int_1^\infty dy \frac{y \cos(2by)}{(y^2 - 1)^{1/2}} = - \frac{\pi}{2} J_1(2b), \quad (4.38)$$

so that Eq. (4.37) reduces to

$$(\Pi_1)_1 = - \frac{\pi R_0^{1/2}}{2} (R_0 \hat{\phi}) \frac{J_1(2b)}{2b}. \quad (4.39)$$

The second correction term is evaluated in a similar manner and yields

$$(\Pi_1)_2 = - \frac{3\pi R_o^{1/2}}{2} (R_o \hat{\phi})^2 \frac{1}{(2b)^2} \left[J_o(2b) - \frac{J_1(2b)}{b} \right]. \quad (4.40)$$

Collecting the above terms, the expression for Π_1 becomes

$$\Pi_1 = \frac{\pi}{2} R_o^{1/2} \left\{ J_o(2b) - (R_o \hat{\phi}) \frac{J_1(2b)}{2b} - \frac{3(R_o \hat{\phi})^2}{(2b)^2} \left[J_o(2b) - \frac{J_1(2b)}{b} \right] \right\}. \quad (4.41)$$

Finally, use of Eqs. (4.33), (4.34), and (4.41) in Eq. (4.31) gives the following approximate form for the ion integral:

$$D_{i_1} = \frac{1}{E_o b} \left(\frac{R_o}{\pi R \hat{\phi}} \right)^{1/2} \left\{ R_o^{1/2} \exp\left(\frac{R_o}{2\hat{\phi}}\right) K_o\left(\frac{R_o}{2\hat{\phi}}\right) - \exp\left(\frac{1}{2\hat{\phi}}\right) K_o\left(\frac{1}{2\hat{\phi}}\right) \right. \\ \left. + \left(R_o^{1/2} \left[1 - \frac{3}{4b^2} (R_o \hat{\phi})^2 \right] - 1 \right) \pi J_o(2b) - R_o^{3/2} \hat{\phi} \left(1 - \frac{3R_o}{2b^2} \hat{\phi} \right) \frac{\pi J_1(2b)}{2b} \right\}. \quad (4.42)$$

In order to check the accuracy of the analytic form given above, the exact expression for D_o specified by Eq. (4.29) was evaluated numerically over a wide range of parameters. Comparison of the results with those obtained from Eq. (4.42) showed good agreement.

The ion contribution determined above can now be used with Eq. (4.21) to calculate stability requirements for the DGH mode. In Fig. 8 the stability boundary for a 10-keV plasma is plotted as a function of mirror ratio and ambipolar potential. Curve a is the boundary in the electrostatic limit and curve b is the appropriate result with the finite beta modification included. Comparison of the curves illustrates the stabilizing influence of the electromagnetic correction. At higher temperatures the effect is slightly more pronounced. For example, with $R_o = 2$ and $T_i = 30$ keV, the threshold value for the ambipolar parameter (ϕ_m/T_i) is about 0.23, as compared to 0.18 at $T_i = 10$ keV. Some of the results from the two-source Fokker-Planck distribution calculations are also displayed to demonstrate that the analytic stability boundary is a reasonably accurate estimate.

In terms of the ion contribution D_o , the stability criterion of Eq. (4.21) can be approximated by

$$D_o > 0 \text{ and/or } |E_o D_o| \leq 2(2/\pi)^{1/2} R^{1/4} k_{\perp} a_i (v_{thi}/c), \quad (4.43)$$

Fig. 8. Stability boundary for Dory-Guest-Harris instability.

with

$$D_o \approx \frac{1}{E_o b} \left(\frac{R_o}{\pi R} \hat{\phi} \right) \left[R_o^{1/2} \exp\left(\frac{R_o \hat{\phi}}{2}\right) K_o\left(\frac{R_o \hat{\phi}}{2}\right) - \exp\left(\frac{\hat{\phi}}{2}\right) K_o\left(\frac{\hat{\phi}}{2}\right) + R_o^{1/2} \pi J_o(2b) \right], \quad (4.44)$$

where $b \equiv k_{\perp} a_i \hat{\phi}^{1/2}$, $\hat{\phi} \equiv \phi_m / RE_o$, and $R \equiv R_o - 1$. The maximum negative value of D_o occurs near the zero of J_1 ($b \approx 2$) where $J_o \approx -0.4$. Since the arguments of K_o are generally small the function can be approximated by⁹⁹

$$K_o(x) \approx \ln(x/2) - \gamma, \quad (4.45)$$

where $\gamma \approx 0.577$ is Euler's constant. Hence a rough criterion for stability is

$$f(R_o, \hat{\phi}) \geq 0 \text{ and/or } |f(R_o, \hat{\phi})| \leq \frac{2\pi R_o^{3/4}}{R_o} (k_{\perp} a_i)^2 \left(\frac{v_{thi}}{c} \right) (\hat{\phi})^{-1/2}, \quad (4.46)$$

with

$$f(R_o, \hat{\phi}) = -R_o^{1/2} \left\{ \exp\left(\frac{R_o \hat{\phi}}{2}\right) \left[\ln\left(\frac{R_o \hat{\phi}}{4}\right) + \gamma \right] + 1.25 \right\} + \exp\left(\frac{\hat{\phi}}{2}\right) \left[\ln\left(\frac{\hat{\phi}}{4}\right) + \gamma \right]. \quad (4.47)$$

In connection with their investigation of reactor conditions in mirror systems, Futch, et al.¹⁰⁰ have developed a multispecies Fokker-Planck code to calculate important physical parameters in neutral injection mirror experiments. Among their results are estimates of the maximum potential acting on electrons as a function of mirror ratio and injection energy. In Fig. 8 these values are plotted as curves c and d with the approximation that the injection energy is the same as the average ion energy and that the electron potential is equal to the potential acting on the trapped ions. The latter assumption is a pessimistic estimate because the ambipolar effect prevents the trapped ions from occupying the region of the mirror throat where the electron potential is a maximum. Also, neutral injection experiments should, in general, be more susceptible to the DGH mode because the distributions in such systems are usually much more peaked than those common to compression experiments like 2X. From Fig. 8 it is seen that to insure stability the conservative requirements are that $R_o \gtrsim 3$ for D-He³ systems (curve c) and that $R_o \gtrsim 4$ for D-T systems (curve d).

Chapter V

Drift-Cone Instability

5.1 INTRODUCTION

The drift-cone modes³⁷⁻³⁹ are instabilities driven by both the loss-cone and spatial inhomogeneity sources of free energy in a plasma. In previous work they were extensively analyzed in the infinite-medium, electrostatic limit by Post and Rosenbluth,²³ who derived the following stability criterion:

$$\epsilon' \langle a_i \rangle < 0.38 \left(\frac{m_e}{M_i} + \frac{\omega_{ci}^2}{2\omega_{pi}^2} \right)^{2/3}, \quad (5.1)$$

where

$$\langle a_i \rangle \equiv \left[\omega_{ci}^3 \int d^3 v f_{oi} \left(v_{\perp}^2, v_{\parallel}^2 \right) v_{\perp}^{-3} \right]^{-1/3} \approx a_i$$

and ϵ' is the radial density gradient. To arrive at this result the authors used an approximate form for the dispersion relation based on the assumptions that the instabilities are nonresonant, high frequency modes ($\omega \sim \omega_{ci}$) with short wavelengths ($ka_i \gg 1$). Numerical solutions to the exact dispersion equation justified these assumptions and demonstrated that Eq. (5.1) is accurate at sufficiently high densities, ($\omega_{pi}^2/\omega_{ci}^2 > 10$).

In the regime of fusion interest ($\omega_{pi}^2/\omega_{ci}^2 \gtrsim 10^3$), the above criterion indicates that a plasma radius ($R_p \sim 1/\epsilon'$) of several hundred ion Larmor radii is required for stability. However, in such a regime the electromagnetic analysis is no longer valid. In this chapter it will be shown that finite beta effects have a significant stabilizing influence on the drift-cone modes.

5.2 DISPERSION RELATION

The general electromagnetic flute dispersion relation derived in Chapter III can be considerably simplified when applied to the analysis of the drift-cone modes. In the calculations that follow it will be assumed that the parameter range of these instabilities is given by

$$\begin{aligned} ka_e &\ll 1 \ll ka_i \\ \omega_{ce} &\ll \omega \sim \omega_{ci}. \end{aligned} \quad (5.2)$$

It should also be noted again that in the derivation presented in Chapter III, it was assumed that the Bessel functions, introduced by the finite beta orbit corrections, have

small arguments, i. e., $(k v_1^2 / 2 \omega_{ci}^2) < 1$. These assumptions are all known to be justified in the electrostatic limit,²³ and it will be shown a posteriori that they are also valid for moderate values of beta ($\beta \lesssim 1$).

The requirements ω / ω_{ce} , $k a_e \ll 1$ imply that the electrons can be treated in the fluid limit, and the condition $k a_i \gg 1$ allows one to use the following asymptotic forms for the Bessel functions⁹⁹ in the ion terms:

$$J_n^2 \approx \frac{1}{\pi \lambda}, \quad J_n J_n' \approx -\frac{1}{2\pi \lambda^2}, \quad (J_n')^2 \approx \frac{1}{4\pi \lambda^3}, \quad (5.3)$$

where the argument is $\lambda = k_1 v_1 / \omega_{ci} \sim k_1 a_e$. Substitution into Eq. (3.22) reduces the matrix elements in the dispersion relation.

$$b_{22} \approx -\frac{\omega_{pi}^2 \Omega_i^2}{k_1^2 c^2} \left\{ \frac{m_e}{M_i} - \frac{\epsilon'}{k_1 \Omega_i} + \frac{2\omega_{ci}^2}{k_1^2} \Omega_i \sum_n \int d^3 v \frac{[(\partial f_{oi} / \partial v_1^2) + (k_{1o} \epsilon' f_{oi} / 2\omega \omega_{ci})]}{n + \Omega_i + (k_{1o} \epsilon v_1^2 / 2\omega_{ci}^2)} J_n^2(\lambda) \right\}$$

$$b_{12} \approx -b_{21} \approx \frac{i}{\Omega_i} \frac{\omega_{pi}^2}{\omega_{ci}^2} \left(1 + \frac{k_{1o} \epsilon'}{2\Omega_i} \frac{M_i}{m_e} a_e^2 \right)$$

$$b_{11} \approx \frac{\omega_{pi}^2}{k_1^2 c^2} \frac{\epsilon}{k_1} \ll 1, \quad (5.4)$$

where $\Omega_i = \omega / \omega_{ci}$, $\epsilon = -(\beta/2)\epsilon'$, and $\beta = 8\pi k(T_i + T_e)/B^2$. When these results are used in Eq. (3.27), the dispersion relation becomes

$$\int_{-L/2}^{L/2} \frac{ds}{B(s)} \left(\frac{B(s)}{B_o} + \frac{\omega_{pi}^2(s) B_o}{\omega_{cio}^2 B(s)} \left\{ \frac{m_e}{M_i} \left(1 + \frac{\omega_{pe}^2(s)}{k_1^2(s) c^2} \right) + \frac{\epsilon'}{k_{1o} \Omega_i(s)} \left[1 + \frac{\beta_i}{2} \left(1 - \frac{T_e}{T_i} \right) \right] \right. \right. \\ \left. \left. + \frac{\omega_{ci}^2(s)}{\omega_{pi}^2(s)} H(\omega, s) \right\} \right) = 0, \quad (5.5)$$

with

$$H(\omega, s) = \frac{2\omega_{cio}^2}{k_{1o}^2} \Omega_i(s) \sum_n \int d^3 v \frac{[(\partial f_{oi} / \partial v_1^2) + (k_{1o} \epsilon' f_{oi} / 2\omega \omega_{cio})]}{n + \Omega_i + (k_{1o} \epsilon v_1^2 / 2\omega_{cio}^2)} J_n^2(\lambda)$$

This equation must in general be solved numerically. However, for the case of an axially homogeneous plasma, it reduces to a form that can be solved analytically. In Section (5.3.1) this calculation is performed, and the expression giving the critical radial gradient derived. The validity of the result is confirmed by numerical solutions to Eq. (5.5), which are presented in Section (5.3.2).

5.3 STABILITY ANALYSIS AND RESULTS

5.3.1 Infinite Medium Limit

If finite geometry effects are ignored, the dispersion relation becomes

$$\frac{\omega_{ci}^2}{2} + \frac{m_e}{k_i^2} - \frac{\epsilon_i}{k\Omega_i} \left[1 + \frac{\beta_i}{2} \left(1 - \frac{T_e}{T_i} \right) \right] + \frac{\omega_{pi}^2}{k^2 c^2} + \frac{2\omega_{ci}^3}{\pi k^3} \Omega_i$$

$$\times \sum_n \int d^3v \frac{1}{v_{\perp}} \frac{\delta f_{oi}}{\partial v_{\perp}^2} \frac{1}{n + \Omega_i + (k\epsilon v_{\perp}^2 / 2\omega_{ci}^2)}, \quad (5.6)$$

where $k = k_{\perp}$ and the asymptotic form for $J_n^2(\lambda)$ has been employed. Also, the radial gradient term in the ion integral has been dropped since it is down by a factor of $1/ka_i$ from the dominant term, $\epsilon_i/k\Omega_i$.

In the analysis of Eq. (5.6), the loss-cone equilibrium ion distribution considered has the form

$$f_{oi}(v_{\perp}^2) = \int_{-\infty}^{\infty} dv_{\parallel} f_o(v_{\perp}^2, v_{\parallel}^2) = (1/\pi v_{oi}^2) (R_o / (R_o - 1)) \left[\exp(-v_{\perp}^2/v_{oi}^2) \right.$$

$$\left. - \exp(-R_o v_{\perp}^2/v_{oi}^2) \right] \quad (5.7)$$

with $R_o =$ mirror ratio and $v_{oi}^2 = 2kT_i/M_i$. This implies that the mean ion Larmor radius is $a_i = (v_{oi}/\omega_{ci}) [(R_o + 1)/R_o]^{1/2}$. If the drift term in the denominator of the ion integral is less than the resonance term, i. e.,

$$k\epsilon v_{\perp}^2 / 2\omega_{ci}^2 \sim k\epsilon a_i^2 / 2 < |n - \Omega_i|, \quad (5.8)$$

then one can use the expansion

$$\sum_n [n + \Omega_i + (k\epsilon v_{\perp}^2 / 2\omega_{ci}^2)]^{-1} \approx \pi \cot(\pi \Omega_i). \quad (5.9)$$

The dispersion relation thus becomes

$$\frac{\omega_{ci}^2}{\omega_{pi}^2} + \frac{m}{M} + \frac{\beta_i}{k^2 a_i^2} = \frac{\epsilon_i}{k\Omega_i} C2 - \frac{\Omega_i \cot(\pi\Omega_i)}{(ka_i)^3} C1, \quad (5.10)$$

where

$$C1 = a_i^3 \langle v_{\perp}^{-3} \rangle \omega_{ci}^3 = 2\pi^{1/2} (R_o + 1)^{3/2} / (R_o + R_o^{1/2})$$

$$C2 = 1 + (\beta_i/2) [1 - (T_e/T_i)].$$

In analyzing Eq. (5.10), it is convenient to adopt the approach introduced by Post and Rosenbluth.²³ First the equation is written in the form

$$\epsilon' a_i(D1) = W[W \cot W + z + (D2)z^{1/3}]/z^{2/3} = f(z, W), \quad (5.11)$$

where

$$D1 = \pi(C2) (\pi/C1)^{1/3} (X)^{-2/3}$$

$$D2 = \beta_i (\pi/C1)^{1/3} (X)^{-1/3}$$

$$W = \pi\Omega_i, \quad z = (ka_i)^3 (\pi/C1)(X)$$

$$X = [\omega_{ci}/\omega_{pi}]^2 + (m_e/M_i).$$

It is clear that for a given z , the function $f(z, W)$ can have a maximum with respect to the frequency variable W in the range $0 < W < \pi$. This implies that imaginary roots corresponding to instability will appear if $\epsilon' a_i$ exceeds this optimum value of $f(z, W)$. Hence, the critical radial density gradient is given by

$$(\epsilon' a_i)_{crit} = \frac{1}{D1} \min_z \max_W [f(z, W)]. \quad (5.12)$$

Minimizing with respect to z leads to

$$2W \cot W + (D2)z^{1/3} - z = 0, \quad (5.13)$$

and maximizing with respect to W yields

$$W - 2\cos W \sin W - \frac{\sin^2 W}{W} \{z + (D2)z^{1/3}\} = 0. \quad (5.14)$$

The stability criterion for the drift-cone mode thus becomes

$$\epsilon' a_i < (\epsilon' a_i)_{\text{crit}} = W[W \cot W + (D2)z^{1/3}]/(D1)z^{2/3}, \quad (5.15)$$

where

$$W = 2\sin 2W + (D2)4^{1/3} \sin W (\sin W / W)^{1/3} \quad (5.16)$$

$$z = \frac{1}{2} (W / \sin W)^2. \quad (5.17)$$

In the small beta limit the $(D2)$ term in Eq. (5.15) approaches zero, and Eqs. (5.16) and (5.17) give $W = 2\sin 2W$ and $z = 8\cos^2 W$. These are equivalent to the expressions obtained by Post and Rosenbluth²³ and lead to the result given in Eq. (5.1). For the distribution specified in Eq. (5.7) this becomes

$$(\epsilon' a_i)_{\text{crit}} = (0.38)(C1)^{1/3} \left[(\omega_{ci}/\omega_{pi})^2 + (m_e/M_i) \right]^{2/3}. \quad (5.18)$$

As beta approaches unity, the $(D2)$ term dominates in Eq. (5.15). In this limit Eqs. (5.16) and (5.17) give

$$W/\sin W = (D2)^{3/4} (2)^{1/2} = (2z)^{1/2}, \quad (5.19)$$

which in turn yields

$$ka_i = \left\{ \beta_i / \left[(\omega_{ci}/\omega_{pi})^2 + (m_e/M_i) \right] \right\}^{1/2} \quad (5.20)$$

and

$$n - \Omega_i = (C1/2\pi)^{1/2} \left[(\omega_{ci}/\omega_{pi})^2 + (m_e/M_i) \right]^{1/4} \beta_i^{-3/4}. \quad (5.21)$$

With these results the stability criterion of Eq. (5.15) reduces to

$$c/R_p \omega_{ci} < (2/C2) \left[1 + (\omega_{pe}/\omega_{ce})^2 \right]^{1/2}, \quad (5.22)$$

where $R_p \approx 1/\epsilon'$ is the plasma radius. Equation (5.22) is valid provided

$$\beta_i > \beta_{\text{crit}} = 4(C1/\pi)^{2/3} [\omega_{ci}/\omega_{pi}]^2 + (m_e/M_i)]^{1/3}. \quad (5.23)$$

This constraint arises from the requirement that the (D2) term dominate in Eq. (5.15). Combining Eqs. (5.20) and (5.22) gives

$$k\epsilon a_i^2/2 = \beta_i \beta/2(C2). \quad (5.24)$$

Equations (5.21) and (5.24) then show that the resonance constraint of Eq. (5.8), which determines the range of validity of the calculation, can be expressed as

$$\beta_i^{7/4}/(C2) < 2(C1/2\pi)^{1/2} [\omega_{ci}/\omega_{pi}]^2 + (m_e/M_i)]^{1/3}. \quad (5.25)$$

The above results demonstrate that the basic assumptions made in the analysis are justified for the moderate beta ($\beta \lesssim 1$) regime, i.e.

- (1) With (D2) large Eq. (5.19) implies that ω approaches ω_{ci} so that $\omega/\omega_{ce} \ll 1$.
- (2) Eq. (5.24) clearly shows the condition $k\epsilon a_i^2/2 < 1$ is satisfied since $\beta_i \lesssim 1$ in this regime.
- (3) From Eq. (5.20) the unstable wave numbers fall in the range $(\omega_{pi}/\omega_{ci})\beta_i^{1/2} < k a_i < (M_i/m_e)^{1/2}$, thus meeting the requirements $k a_e \ll 1$ and $k a_i \gg 1$.

Critical radial lengths for stability were computed from Eq. (5.15) for various mirror ratios and temperatures. In Figs. 9 and 10 these lengths are plotted as a function of density ($\omega_{ci}^2/\omega_{pi}^2$). Figure 9 shows that the Post-Rosenbluth²³ low beta analysis is valid up to about $\beta_i = 0.01$. Beyond this range the plasma pressure begins to exert a stabilizing influence on the mode. For example, at $\beta_i = 0.5$ the critical radius required for stability is reduced by a factor of five. This figure also shows that the stability criterion becomes quite insensitive to changes in the mirror ratio for $\beta_i > 0.1$.

The effect of plasma temperature on the critical radial length is illustrated in Fig. 10. It is shown here that for $T_i \geq 10$ keV this parameter has a weak influence on the stability criterion. For example, increasing the temperature from 10 keV to 400 keV alters the critical length ($c/R \omega_{ci}$) for the worst density by less than a factor of two. Moreover, at high densities ($\omega_{ci}^2/\omega_{pi}^2 > M/m$) temperature changes have virtually no influence.

Fig. 9. Critical radial length (R_p/a_i) as a function of density ($\omega_{ci}^2/\omega_{pi}^2$) for a 10 keV deuterium plasma at various mirror ratios. Curve a is the Post-Rosenbluth result, Eq. (5.18), for $R_0 = 1$. Curve b is the critical axial scale length $(L/a_i)_c$ from the finite geometry model, Eq. (5.26), for $R_0 = 2$.

5.3.2 Finite Geometry Effects

As noted in the survey in Chapter I, the stability criterion⁶³ for the flute type negative energy modes, which occur below the first ion-cyclotron harmonic, is given by

$$L/\hat{a}_i < (L/\hat{a}_i)_c \approx 3(R_0 - 1)^{1/2} |1 - \Omega_i|^{-3/2}, \quad (5.26)$$

where L is the axial length of the plasma and $\hat{a}_i \equiv v_{oi}/2^{1/2}\omega_{cio} = a_i[R_0/2(R_0 + 1)]^{1/2}$. Since the resonance width for the drift-cone mode decreases as beta increases, the corresponding axial length requirement also becomes less restrictive. For the

Fig. 10. Critical radial length $(c/R_p \omega_{ci})$ as a function of density $(\omega_{ci}^2/\omega_{pi}^2)$ for a deuterium plasma with $R_0 = 2$ at various temperatures.

moderate beta regime Eq. (5.21) can be combined with Eq. (5.26) to yield the following expression for the critical axial length:

$$(L/\hat{a}_i)_c = 3(R_o - 1)^{1/2} \beta_i^{9/8} \left\{ (C1/2\pi)^2 [\omega_{ci}/\omega_{pi}]^2 + (m_e/M_i) \right\}^{-3/8}. \quad (5.27)$$

In order to include the effects of finite geometry in the critical radial length analysis, it is necessary to numerically solve the dispersion relation specified by Eq. (5.5). This calculation was performed by Berk⁴⁰ who considered a separable ion distribution of the form

$$f_{oi}(v_\perp^2) = \int_{-\infty}^{\infty} dv_{\parallel} f_{oi}(v_\perp^2, v_{\parallel}^2) = \frac{n_o}{\pi v_{oi}^2} \frac{B_o}{B(s)} \left[\frac{B_m - B(s)}{B_m - B_o} \right]^{3/2} \left\{ \exp(-v_\perp^2/v_{oi}^2) - \exp\left[-\frac{B(s)}{B_o} v_\perp^2\right] \right\}, \quad (5.28)$$

where

$$B(s) = B_o \left[1 + 2(R_o - 1)(2s/L)^2 - (R_o - 1)(2s/L)^4 \right],$$

R_o is the mirror ratio, B_m the maximum magnetic field, and L the plasma length. With the drift term in the denominator taken to be negligible, the ion velocity integral can be evaluated analytically to give

$$\frac{\omega_{ci}^2(s)}{\omega_{pi}^2(s)} H(\omega, s) = - \sum_n \frac{\Omega_i(s)}{\Omega_i(s) + n} \frac{1}{b_o} \left\{ e^{-b_o} I_n(b_o) - e^{-b(s)} I_n[b(s)] \right\}, \quad (5.29)$$

where $B_o = (k_\perp a_i)_o^2/2$, $b(s) = b_o B_o/B(s)$, and I_n is the modified Bessel function⁹⁹ which, in the large argument limit, becomes⁵⁹

$$I_n(b) \approx (2\pi b)^{-1/2} \exp\left(b - \frac{n^2}{b}\right). \quad (5.30)$$

Since the exponential varies slowly with position compared to the other terms, its spatial dependence was ignored in the numerical evaluation of the spatial integral. Critical radial lengths were then obtained for the worst ω and k by using a Newton-Raphson¹⁰¹ root-finding scheme to solve the dispersion relation, $D(\omega, k, R_p, n_o) = 0$, with $\partial D/\partial \omega = 0$ and $\partial D/\partial k = 0$.

As shown in Fig. 9, the comparison between the values from the numerical analysis and those from the infinite medium calculations demonstrates the stabilizing influence of the finite geometry effects. At the same time it is seen that the results in the two cases are qualitatively similar. In fact, it was found that if one suppressed the spatial variation in the computer code by letting R_o approach unity, the numerical answers were in good agreement with the infinite medium analytic results.

Values for the frequency, obtained from the finite geometry model, were used in Eq. (5.26) to estimate axial restrictions. Critical axial lengths for $R_o = 2$ and $T_i = 10$ keV are shown as curve b in Fig. 9 with L plotted on the same scale as R_p . Here it is seen that a residual negative energy instability persists even when the drift-cone mode is stabilized. The figure also shows that the axial requirement is satisfied near the point where the critical radial length calculations, governed by the resonance constraint of Eq. (5.25), breaks down.

The above analysis, when applied to the 2X mirror device⁴⁹ ($T_i = 10$ keV, $\beta_i = 0.05$, $R_o = 1.33$), indicates that the system can support drift-cone modes with growth rates on the order of a tenth of the ion-cyclotron frequency. For the 2X parameters a plasma radius greater than 115 orbit radii is required for stability. As beta approaches unity, this criterion is reduced by about a factor of two.

Chapter VI

Resonant Electromagnetic Instability

6.1 INTRODUCTION

In general, unstable waves that are purely electromagnetic in character can seriously disrupt confinement only in high beta ($\beta > 1$) plasmas.³ An exception to this is the highly resonant flute modes that have been derived in the infinite medium limit.⁷⁴ These instabilities are driven by the loss-cone source of free energy and are characterized by wavelengths on the order of the ion Larmor radius and frequencies very close to harmonics of the ion-cyclotron frequency. In Ref. 74 the author considered a Maxwellian distribution for electrons and the familiar form, [Eq. (4.1)], used by Dory, Guest, and Harris, for the ion equilibrium loss-cone distribution. He solved the appropriate dispersion relation¹⁵ numerically and found that for values of beta greater than 1%, unstable modes occur with growth rates equal to several percent of the ion-cyclotron frequency.

The fact that these instabilities are highly resonant implies that their existence is strongly dependent on the phase coherence between particles and waves. Hence, finite geometry effects should exert a stabilizing influence on these modes by introducing a time limit over which phase coherence can be maintained and by introducing a variation of the cyclotron frequency over the length of the plasma, which serves to detune the resonances.

6.2 DISPERSION RELATION

The general form of the electromagnetic flute dispersion relation is given by Eq. (3.27)

$$\int \frac{ds}{B(s)} \left[\frac{B(s)}{B_0} \left(\frac{k_{\perp 0} c}{\omega} \right)^2 b_{22}(s) + \frac{(k_{\perp 0} c / \omega)^2 b_{12}(s) b_{21}(s)}{1 + b_{11}(s)} \right] = 0, \quad (6.1)$$

where $b_{lm} \equiv (-4\pi i \omega / k_{\perp 0}^2 c^2) \hat{b}_{lm}$. In the absence of radial gradients the matrix elements, $\hat{b}_{lm}$, are specified by

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \sum_{i,e} \frac{\omega^2 p_{i,e}}{2\pi} \int d^3v \frac{\partial f_{0,i,e}}{\partial v_1^2} \begin{pmatrix} v_x \\ v_y \end{pmatrix} \int_{-\infty}^t dt' (E_x' v_x' + E_y' v_y') = \begin{pmatrix} \hat{b}_{11} & \hat{b}_{12} \\ -\hat{b}_{12} & \hat{b}_{22} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}, \quad (6.2)$$

with $b_{12} = -b_{21}$.

For the case of an infinite, homogeneous plasma with Maxwellian electrons, the above equations reduce to the usual flute dispersion relation^{14,74}

$$1 - b_{22} - \frac{(\omega/kc)^2 (b_{12})^2}{1 + b_{11}} = 0, \quad (6.3)$$

where

$$b_{22} = -(\omega_{pi}/\omega_{ci})^2 \left[\frac{m}{M} + \sum_n \frac{\Omega}{\Omega + n} \left(J_n^2(\lambda) \frac{1}{v_\perp} \frac{\partial f_{oi}}{\partial v_\perp} \right) (v_{th}^2/2b) \right]$$

$$(\omega/kc)^2 (b_{12})^2 = \frac{\beta_i}{4b} (\omega_{pi}/\omega_{ci})^2 \left[1 + \sum_n \frac{\Omega}{\Omega + n} \left(\frac{v_\perp}{2} \frac{d}{d\lambda} J_n^2(\lambda) \frac{1}{v_\perp} \frac{\partial f_{oi}}{\partial v_\perp} \right) \frac{v_{th}}{(2b)^{1/2}} \right]^2$$

$$b_{11} = \frac{\beta_i}{4b} \sum_n \frac{\Omega}{\Omega + n} \left(v_\perp^2 \left[\frac{d}{d\lambda} J_n(\lambda) \right]^2 \frac{1}{v_\perp} \frac{\partial f_{oi}}{\partial v_\perp} \right)$$

$$\lambda = k_\perp v_\perp / \omega_{ci}$$

$$b = (k_\perp a_i)^2 / 2$$

$$a_i = \text{ion Larmor radius} = v_{th} / \omega_{ci}$$

$$\Omega = \omega / \omega_{ci}$$

$$\beta_i = 2(\omega_{pi}/\omega_{ci})^2 (v_{th}/c)^2$$

with the bars indicating averages over perpendicular velocity.

6.3 STABILITY ANALYSIS AND RESULTS

6.3.1 Infinite Medium Limit

In this section it will be demonstrated that one can solve the infinite medium dispersion relation analytically and closely reproduce the numerical results obtained

in earlier calculations.⁷⁴ The form of the ion equilibrium loss-cone distribution considered has the form

$$f_0(v^2) = \int_{-\infty}^{\infty} dv_{\parallel} f_0(v_{\perp}^2, v_{\parallel}^2) = \frac{1}{\pi} \alpha^2 v_{\perp}^2 \exp(-\alpha v_{\perp}^2) \quad (6.4)$$

with $\alpha = m_i / \kappa T_i = 1/v_{th}^2$. In previous numerical work⁷⁴ it was found that more peaked distributions did not appreciably alter the results.

Since these modes are highly resonant, the infinite sums over Bessel functions in Eq. (6.3) can be evaluated analytically. Separation of the $n = -1$ resonant term from the infinite series gives

$$\left(\frac{\omega_{ci}}{\omega_{pi}} \right)^2 + \frac{m}{M} + F + \frac{\Omega}{\Omega-1} F_1 + \frac{(\beta_i/4b) \left[1 + G + \frac{\Omega}{\Omega-1} G_1 \right]^2}{1 + (\beta_i/4b) \left(H + \frac{\Omega}{\Omega-1} H_1 \right)} = 0, \quad (6.5)$$

where

$$F_1 = (v_{th}^2/2b) \left(\frac{1}{v_{\perp}} \frac{\partial f_0}{\partial v_{\perp}} J_1^2 \right)$$

$$G_1 = \frac{v_{th}}{(2b)^{1/2}} \left(\frac{\partial f_0}{\partial v_{\perp}} \frac{d}{d\lambda} J_1^2 \right)$$

$$H_1 = - \left(v_{\perp} \frac{\partial f_0}{\partial v_{\perp}} (J_1')^2 \right)$$

$$F = (v_{th}^2/2b) \left(\frac{1}{v_{\perp}} \frac{\partial f_0}{\partial v_{\perp}} \sum_{n \neq -1} \frac{\Omega}{\Omega+n} J_n^2 \right)$$

$$G = \frac{v_{th}}{(2b)^{1/2}} \left(\frac{\partial f_0}{\partial v_{\perp}} \frac{d}{d\lambda} \sum_{n \neq -1} \frac{\Omega}{\Omega+n} J_n^2 \right)$$

$$H = - \left(v_{\perp} \frac{\partial f_0}{\partial v_{\perp}} \sum_{n \neq -1} \frac{\Omega}{\Omega+n} (J_n')^2 \right)$$

$\lambda = k_{\perp} v_{\perp} / \omega_{ci}$ = argument of Bessel functions $J_n(\lambda)$.

The nonresonant terms can be written as

$$\begin{pmatrix} F \\ G \\ H \end{pmatrix} = \int_0^\infty dv_\perp \frac{1}{v_\perp} \frac{\partial f_{oi}}{\partial v_\perp} \sum_{n \neq -1} \begin{pmatrix} (v_{th}^2/2b) J_n^2 \\ \frac{v_{th} v_\perp}{(2b)^{1/2}} \frac{d}{d\lambda} (J_n^2) \\ -v_\perp^2 (J_n')^2 \end{pmatrix} \left(\frac{1}{1+n} + \frac{n\delta\omega}{1+n^2} - \dots \right), \quad (6.6)$$

where $\delta\omega \equiv \Omega - 1 \ll 1$. Ignoring terms of order $\delta\omega$ and higher, and using standard Bessel function identities (see Appendix A), the sums can be readily performed, i. e.

$$\begin{aligned} (1) \quad \sum_{n \neq -1} \frac{J_n^2}{1+n} &= \sum_{n=1}^\infty \frac{J_{n-1}^2 - J_{n+1}^2}{n} = -\frac{1}{\lambda} \frac{d}{d\lambda} (J_0^2) \\ (2) \quad \frac{d}{d\lambda} \sum_{n \neq -1} \frac{J_n^2}{1+n} - \frac{2}{\lambda} [J_0^2 - J_1^2 + \frac{1}{\lambda} \frac{d}{d\lambda} (J_0^2)] \\ (3) \quad \sum_{n \neq -1} \frac{(J_n')^2}{1+n} &= \sum_{n=1}^\infty \frac{(J_{n-1}')^2 - (J_{n+1}')^2}{n} \\ &= \frac{4}{\lambda} \sum_{n=1}^\infty \left\{ \left(\frac{n^2}{\lambda^2} + \frac{1}{\lambda^2} - 1 \right) J_n J_n' - \frac{(J_n')^2}{\lambda} - \left(\frac{n^2}{\lambda^2} - \frac{1}{\lambda} \right) J_n^2 \right\} \\ &= -\frac{2}{\lambda^2} [J_0^2 - J_1^2 + \frac{1}{2} \left(\frac{1}{\lambda} - \lambda \right) \frac{d}{d\lambda} (J_0^2)] + \frac{1}{\lambda^2}. \end{aligned} \quad (6.7)$$

Hence, one obtains

$$\begin{pmatrix} F \\ G \\ H \end{pmatrix} = \int_0^\infty dv_\perp \frac{\partial f_{oi}}{\partial v_\perp} \begin{pmatrix} -(v_{th}^2/2b) \frac{1}{\lambda} \frac{d}{d\lambda} J_0^2 \\ (v_{th}^2/2b) [J_0^2 - J_1^2 + \frac{1}{\lambda} \frac{d}{d\lambda} J_0^2] \\ (v_{th}^2/b) [J_0^2 - J_1^2 + \frac{1}{2} \left(\frac{1}{\lambda} - \lambda \right) \frac{d}{d\lambda} J_0^2] \end{pmatrix}. \quad (6.8)$$

Using the distribution of Eq. (6.4) then gives

$$\begin{aligned}
 F_1 &= e^{-b} \{ [1 + (1/b)] I_1 - I_0 \} \\
 G_1 &= 2be^{-b} \{ I_0 - [1 + (1/2b) + (1/2b^2)] I_1 \} \\
 H_1 &= be^{-b} \{ [4b + (1/b) - 2] I_0 - [4b + (1/b) + (1/b^2)] I_1 \} \\
 F &= (1/b^2) - (1/b)e^{-b} \{ [1 + (1/b)] I_1 - I_0 \} \\
 G &= -(1/b^2) + e^{-b} \{ [2 + (1/b) + (1/b^2)] I_0 - 2[1 + (1/b)] I_1 \} \\
 H &= -(1/b^2) + e^{-b} \{ [2 + 4b + (1/b) + (1/b^2)] I_0 - [4 + 4b + (3/b)] I_1 \}, \quad (6.9)
 \end{aligned}$$

where $b \equiv (k_1 v_{th}/\omega_{ci})^2/2 = \text{argument of the modified Bessel functions } I_0(b) \text{ and } I_1(b)$.

Substitution of Eq. (6.9) back into Eq. (6.5) reduces the dispersion relation to a quadratic equation of the following form with analytic expressions for its coefficients

$$A(\delta\omega)^2 + B\delta\omega + C = 0, \quad (6.10)$$

where

$$\delta\omega = \Omega - 1$$

$$A = D + F_1 + F + (\beta_1/4b) \{ (D + F_1 + F)(H_1 + H) + (1 + G_1 + G)^2 \}$$

$$B = F_1 + (\beta_1/4b) \{ (D + 2F_1 + F)H_1 + F_1H + 2G_1(1 + G_1 + G) \}$$

$$C = (\beta_1/4b) (G_1^2 + F_1H_1)$$

$$D = (\omega_{ci}/\omega_{pi})^2 + (m/M).$$

For $\beta_1 < 1$ the stability criterion, $4AC \leq B^2$, can be approximated by

$$F_1^2 > (\beta_1/b) [D + F + G_1^2 + F_1(1 + H_1)]. \quad (6.11)$$

So, in the small β_1 regime, the above result indicates that (1) a finite value of β_1 is required to initiate the instability, (2) the b_{11} -matrix element contribution to the dispersion relation is a weak one, (3) unstable modes appear near the zeros of $F_1(b)$, and (4) the growth rates associated with these modes is roughly proportional to $(\beta_1)^{1/2}$.

Solutions to Eq. (6.10) over a wide range of parameters verified these conclusions. As shown in Table 1, the analytic results were found to agree quite well with previous numerical calculations.⁷⁴

Table 1: Comparison between previously calculated numerical results (i) and the analytic results obtained from Eq. (6.10).

β_i		$\text{Im}(\Omega_i)_{\text{max}}$	$\frac{1}{2} (k_{\perp} a_i)^2_{\text{max}}$
10^{-3}	(i)	2.06×10^{-3}	1.54
	(ii)	1.83×10^{-3}	1.54
10^{-2}	(i)	7.26×10^{-3}	1.54
	(ii)	7.16×10^{-3}	1.54
10^{-1}	(i)	2.20×10^{-2}	1.50
	(ii)	2.20×10^{-2}	1.48
0.5	(i)	3.75×10^{-2}	1.30
	(ii)	3.75×10^{-2}	1.30

6.3.2 Finite Geometry Effects: Fluid Limit

If finite geometry effects are included in the stability analysis, the appropriate dispersion relation is specified by Eqs. (6.1) and (6.2). The fact that Ω_i and $k_{\perp} a_i$ are of order unity for these modes justifies the fluid treatment of the electrons. In this section it will be additionally assumed that ion motion along the magnetic field can be neglected ($v_{\parallel i} = 0$). As noted in Chapter II, this fluid approximation for parallel ion motion is strictly valid only if the nonresonance condition of Eq. (2.84) is satisfied. In the next section it will be shown that ion transit effects are indeed important and lead to stabilization of these modes. For the present section, however, the approximation is employed in order to determine how the instabilities are influenced by the detuning effect due to axial variation of the magnetic field.

With parallel ion motion ignored, the time integration in Eq. (6.2) can be readily performed,⁵⁹ so that the dispersion relation becomes

$$\frac{1}{\int_{-L/2}^{L/2} \frac{ds}{B(s)}} \int_{-L/2}^{L/2} \frac{ds}{B(s)} \left\{ \frac{B(s)}{B_0} + \frac{\omega_{pi}^2(s)}{\omega_{cio}^2} \frac{B_0}{B(s)} \left(\frac{m}{M} + \sum_n \frac{\omega F_n}{\omega + n\omega_{ci}(s)} \right) \right. \\ \left. \frac{[\beta_i(s)/4b](B(s)/B_0) \left[1 + \sum_n \frac{\omega G_n}{\omega + n\omega_{ci}(s)} \right]^2}{1 + [\beta_i(s)/4b](B(s)/B_0) \sum_n \frac{\omega H_n}{\omega + n\omega_{ci}(s)}} \right\} = 0, \quad (6.12)$$

where L is the plasma length and F_n , G_n , and H_n are defined as before. Since the resonant terms are dominant, the spatial variation in only these terms will be considered. Eq. (6.12) thus reduces to

$$\left(\frac{\omega_{ci}}{\omega_{pi}} \right)^2 + \frac{m}{M} + F + \frac{1}{L} \int_{-L/2}^{L/2} ds \left(\frac{F_1}{\Omega - \hat{b}(s)} + \frac{(\beta_i/4b)}{1 + b_{11}(s)} \left[1 + G + \frac{G_1}{\Omega - \hat{b}(s)} \right]^2 \right) = 0, \quad (6.13)$$

where

$$b_{11}(s) = (\beta_i/4b) \left[H + \frac{H_1}{\Omega - \hat{b}(s)} \right]$$

$$\hat{b}(s) = B(s)/B_0$$

and F_1 , F , etc., are defined in Eq. (6.9). The magnetic field is taken as

$$B(s) = B_0 \left[\left(\frac{R_0 + 1}{2} \right) - \left(\frac{R_0 - 1}{2} \right) \cos \left(\frac{2\pi s}{L} \right) \right], \quad (6.14)$$

where R_0 is the mirror ratio. Using this in Eq. (6.13) leads to the following form for the line-averaged dispersion relation:

$$A' + \frac{B'}{[(\Omega - 1)(\Omega - R_0)]^{1/2}} + \frac{C'}{[(\Omega - C_2)(\Omega - C_2 R_0)]^{1/2}} = 0, \quad (6.15)$$

where

$$A' = D + F + (C_1/H_1)(1+G)^2$$

$$B' = F_1 + (G_1^2/H_1)$$

$$C' = (-C_2/H_1)[C_1(1 + G) - G_1]^2$$

$$C_1 = (\beta_1/4b)H_1/[1 + (\beta_1/4b)H]$$

$$C_2 = [1 + (\beta_1/4b)H]/[1 + (\beta_1/4b)(H_1 + H)]$$

By setting the mirror ratio to unity, the infinite medium equation, Eq. (6.10), is recovered.

In the small β_1 regime the C_2 parameter approaches unity. Provided $|\delta\omega| < (\beta_1/4b)H_1$, Eq. (6.14) can be expanded around $C_2 = 1$ to give

$$A = \frac{B}{[\delta\omega(1 - R_0)]^{1/2}} + \frac{C}{2[\delta\omega^3(1 - R_0)]^{1/2}} = 0, \quad (6.16)$$

where

$$A = D + F + (\beta_1/4b)[H(D + F) + (1 + G)^2]$$

$$B = F_1 + (\beta_1/4b)[F_1H + H_1(D + F) + 2G_1(1 + G)]$$

$$C = (\beta_1/4b)(G_1^2 + H_1F_1)$$

This result in turn can be written as a cubic equation and readily solved. Solutions to Eq. (6.15) over a wide range of parameters were obtained and compared to the infinite medium results. In Fig. 11 it is shown that the finite geometry modification due to the spatially varying magnetic field reduces the growth rates of the unstable modes but fails to stabilize them. For example, with $\beta_1 = 0.1$ and $R_0 = 2$ the growth rate is reduced by an order of magnitude, and as R_0 approaches infinity, the growth rate goes to zero.

To check the range of validity of Eq. (6.16), solutions to Eq. (6.15) were also obtained. Comparison of the answers indicates that growth rates calculated from the exact equation are uniformly lower than the approximate results. As shown in Fig. 11, there is good correspondence for small values of β_1 and agreement within a factor of three for β_1 of order unity. Hence it can be concluded that the approximate form of the line-averaged dispersion relation is reliable for $\beta_1 < 1$ but will give pessimistic estimates of the growth rate as β_1 approaches unity.

Equation (6.16) can also be obtained from the following analysis. First, Eq. (6.13) is rewritten in the form

Fig. 11. Comparison of growth rates calculated from infinite medium theory, curve a, and the values from the line-averaged results at a mirror ratio of 2 in the fluid limit. Curve b comes from the approximate equation, Eq. (6.16), and curve c from the exact equation, Eq. (6.15).

$$\frac{1}{L} \int_{-L/2}^{L/2} \frac{ds}{1 + b_{11}(s)} \left\{ \left[\left(\frac{\omega_{ci}}{\omega_{pi}} \right)^2 + \frac{m}{M} + F + \frac{F_1}{\Omega - \hat{b}(s)} \right] [1 + b_{11}(s)] \right. \\ \left. + \frac{\beta_i}{4b} \left[1 + G + \frac{G_1}{\Omega - \hat{b}(s)} \right]^2 \right\} = 0. \quad (6.17)$$

As noted in the preceding section, the b_{11} -matrix element has a weak effect on the instability for $\beta_i < 1$. One can accordingly make the approximation that this term can be dropped in the denominator. Performing the line-averaging operation on the resultant expression leads exactly to Eq. (6.16). The main reason for calling attention to this particular approximation and for justifying the use of Eq. (6.16) is that in the next section it will be necessary to adopt this method in evaluating a complicated ion transit integral.

6.3.3 Finite Geometry Effects: Ion Transit Analysis

In the previous section it was pointed out that the fluid approximation for ion motion along the field lines begins to break down as the modes approach resonance. When this occurs, ion transit effects must be taken into account. In performing the time integrations, multiple ion bounces are ignored and the straight-line orbit approximation for the unperturbed guiding center trajectories is used. As noted in Chapter II, this method is valid provided the thermal spread, $\Delta\phi_B$, of the ion bounce period is large, i. e.,

$$\Delta\phi_B = \left| \omega\tau_B - \int_0^{\tau_B} dt' \omega_c(t') \right| \gg 1, \quad (6.18)$$

where τ_B is the bounce period. For the present problem this requirement is clearly satisfied since the frequencies of the mode are close to the ion-cyclotron frequency and $\omega_{ci}\Delta\tau_B$ is the order $(L/a_1) \gg 1$.

Since these instabilities are resonant near the midplane ($s = 0$), one can follow the prescription described in Section 2.1.4 and expand $\omega_{ci}(s)$ about $s = 0$ in performing the time integrations. Using Eq. (2.26), the dispersion relation with ion transit modification included becomes

$$A + BA_1(\Omega, L) + CA_2(\Omega, L) = 0, \quad (6.19)$$

where

$$A_1(\Omega, L) = \frac{i\omega}{2L} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} \frac{dv_{\parallel}}{v_{\parallel}} f_0(v_{\parallel}) \int_{-\infty}^{\infty} ds' \exp\left[\frac{i\omega_{ci0}|s' - s|}{v_{\parallel}}\right] \psi(s, s'),$$

$$A_2(\Omega, L) = \frac{\omega^2}{4L} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} \frac{dv_{\parallel}}{v_{\parallel}} f_0(v_{\parallel}) \int_{-\infty}^{\infty} ds' \exp\left[\frac{i\omega_{ci0}|s' - s|}{v_{\parallel}}\right] \psi(s, s')$$

$$\times \int_{-\infty}^{\infty} \frac{dv_{\parallel}}{v_{\parallel}} f_o(v_{\parallel}') \int_{-\infty}^{\infty} ds'' \exp \left[\frac{i\omega_{cio} |s'' - s|}{v_{\parallel}'} \psi(s, s'') \right],$$

$$\psi(s, s') = \delta\omega - (1/3L_m^2)(s^2 + s'^2 + ss') \text{ with } s'' \text{ for } s' \text{ in } \psi(s, s''),$$

$$L_m^2 = 2\omega_{ci}(s)/\omega_{ci}''(s) \Big|_{s=0} = 0.$$

$$f_o(v_{\parallel}) = (2\pi v_{th})^{-1/2} \exp(-v_{\parallel}^2/2v_{th}^2).$$

and A, B, C are defined in Eq. (6.16). For the magnetic field of Eq. (6.14), $L_m^2 = L^2/[\pi^2(R_o - 1)]$. To arrive at Eq. (6.19), the integrand was written in the form given by Eq. (6.17) with the b_{11} -term taken to be negligible in the denominator. If this method is not used, the term $A_2(\Omega, L)$ becomes a highly intractable multidimensional integral. As shown in the previous section, the approximation is valid for $\beta_1 < 1$ but will lead to pessimistic values for the growth rates as β_1 approaches unity.

To calculate the ion transit integrals, $A_1(\Omega, L)$ and $A_2(\Omega, L)$, it is convenient to make the change of variable, $\xi = \omega_{cio}(s' - s)/v_{\parallel}$. For the first ion transit term this gives

$$\begin{aligned} A_1(\Omega, L) &= \frac{\Omega}{Lv_{th}(2\pi)^{1/2}} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} d\xi \exp \left[i\xi \left(\delta\omega - \frac{s^2}{L_m^2} \right) \right] \\ &\int_{-\infty}^{\infty} dv_{\parallel} \exp \left[- \left(\frac{1}{2v_{th}^2} + \frac{i\xi^3}{3L_m^2\omega_{cio}^2} \right) v_{\parallel}^2 - \frac{i\xi^2 s}{L_m^2\omega_{cio}} v_{\parallel} \right] \\ &= \left[\frac{i\Lambda}{\delta\omega\pi(1 - R_o)} \right]^{1/2} \int_{-\infty}^{\infty} dz \exp(-i\gamma z) [z(1 + iz^3)]^{-1/2}, \end{aligned} \quad (6.20)$$

where $\Lambda \equiv -\delta\omega(6^{1/2}L\omega_{cio}/\pi v_{th})^{2/3}$. For the squared ion transit term one obtains

$$A_2(\Omega, L) = \frac{3}{2} \Omega^2 (L_m/L) (L_m \omega_{cio}/v_{th})^2 \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\xi' \exp[i(\xi + \xi')\delta\omega] \\ \times [i(\xi + \xi')\chi\chi' + (3/4)(\chi'\xi^4 + \chi\xi'^4)]^{-1/2} \quad (6.21)$$

where $\chi = (3/2)(L_m \omega_{cio}/v_{th})^2 + i\xi^3$. With the transformation $\chi = \xi + \xi'$ and $y = (\xi - \xi')/\chi$, this becomes

$$A_2(\Omega, L) = [\pi(1 - R_0)]^{-1/2} (i\Lambda/\delta\omega)^{3/2} \int_{-\infty}^{\infty} dz z^{1/2} \exp(-i\Lambda z) \\ \times \int_0^1 dy [1 + iz^3 f_1(y) - z^6 f_2(y)]^{-1/2}, \quad (6.22)$$

where $f_1(y) = (5 + 6y^2 - 3y^4)/8$ and $f_2(y) = (1 - y^2)^3/16$. Now the dispersion relation with ion transit effects can be written as

$$A(R_0 - 1)^{1/2} + (-\Lambda/\pi\delta\omega)^{1/2} [BI_1(\Lambda) - (\Lambda/\delta\omega)CI_2(\Lambda)] = 0, \quad (6.23)$$

where

$$I_1(\Lambda) = i^{1/2} \int_0^{\infty} dz \exp(-i\Lambda z^2) [1 + iz^6]^{-1/2}$$

$$I_2(\Lambda) = i^{3/2} \int_0^{\infty} dz z^2 \exp(-i\Lambda z^2) \int_0^1 dy [1 + iz^6 f_1(y) - z^{12} f_2(y)]^{-1/2}.$$

For large values of Λ (i. e., very large L), the integrals I_1 and I_2 approach $(\pi/\Lambda)^{1/2}$ and $(\pi/\Lambda^3)^{1/2}/2$ respectively, and the fluid result of Eq. (6.16) is recovered. The numerical evaluation of these integrals for general cases is discussed in Appendix C.

Equation (6.23) was solved over a wide range of parameters and the maximum growth rates (for the worst wave number) plotted as a function of the length parameter, $\tilde{L} = L/a_1$, in Figs. 12 and 13. As illustrated in Fig. 12, one just recovers the fluid limit growth rates of Eq. (6.16) for large values of L , and for sufficiently small values of this parameter, $\text{Im}(\Omega_1)$ goes to zero. In Fig. 13 it is shown that for still shorter lengths, the modes become damped.

Fig. 12. Growth rates from the ion transit analysis as a function of plasma length at various values of β_i and a mirror ratio of 2.

It was found that to a good approximation the length, $\hat{L}_c$, at which the growth rate equals zero, can be determined from

$$\hat{L}_c \approx [2\text{Im}(\Omega_i)_F]^{-3/2}, \quad (6.24)$$

where $\text{Im}(\Omega_i)_F$ is the magnitude of the maximum growth rate in the fluid limit. Use of Eq. (6.24) with values for $\text{Im}(\Omega_i)_F$ obtained from Eq. (6.16) yielded estimates of $\hat{L}_c$ close to those calculated from Eq. (6.23). For reasons noted earlier, these results, which are plotted as curve a in Fig. 14, tend to be pessimistic as β_i approaches unity. To obtain an optimistic estimate of $\hat{L}_c$, values for $\text{Im}(\Omega_i)_F$, calculated from Eq. (6.15), were used in Eq. (6.24). These results are plotted as curve b in Fig. 14. It is not valid, however, to claim that curves a and b give the stability boundary for the resonant electromagnetic mode, because the flute formalism breaks down at these very large values of $\hat{L}_c$.

The range of applicability of the flute formalism is specified by Eq. (2.44),

Fig. 13. Damping of the resonant electromagnetic flute instability at sufficiently short plasma lengths.

Fig. 14. The flute formalism begins to break down to the right of the dashed curve. Points 1 and 2 denote respectively the 2X and Van Allen belt plasma parameters. Curve a gives values for L_c [where $\text{Im}(\Omega_i) = 0$] calculated from Eq. (6.23), and curve b shows the results from Eqs. (6.24) and (6.15).

$$L < L_{\text{crit}} \sim \frac{\pi}{k_{\perp 0}} \left\{ [\mathcal{F}(s)]^{-1} \frac{M_i}{m_e} \left[1 + \frac{\beta_i}{(k_{\perp} a_i)^2} \right]^{-1} \right\}^{1/2} \quad (6.25)$$

with $\mathcal{F}(s)$ corresponding to the integrand of Eq. (6.1). Substitution of typical values for $\mathcal{F}(s)$ and $k_{\perp}$ from the present calculation [$\mathcal{F}(s) \sim 0.01$ and $b \approx (k_{\perp} a_i)^2/2 \sim 2$] leads to values for L_{crit} plotted as the dashed curve in Fig. 14. At these lengths solutions to Eq. (6.23) indicate the modes to be heavily damped, i. e., the system should be stable to the REM modes in the regime where the flute formalism is applicable. Hence, the dashed curve in Fig. 14 represents a sufficient but not necessary criterion for stability. The requirement here is, of course, a very conservative one since the instability is already well damped at such lengths. At the longer lengths approaching threshold, the mode becomes very resonant. In this regime, where the flute formalism breaks down, finite $k_{\parallel}$ effects such as ion Landau damping should account for stabilization.

Application of the preceding results to actual mirror systems indicates that the resonant electromagnetic flute instabilities should not be present in either laboratory devices or the Van Allen belt plasma.¹⁰² For example, the 2X machine,⁵⁰ with β_1 of 5 to 10%, mirror ratio of 2, and $\hat{L}$ about 100, and the Van Allen belt plasma, with $\beta_1 \sim 1$, $R_0 \sim 2$, and $\hat{L} \sim 40$ as representative parameters, are denoted by points 1 and 2, respectively, in Fig. 14. It is seen that both points fall within the stable regime where the flute formalism is valid.

Chapter VII

Conclusions

The research in this thesis has dealt with (1) the development of the appropriate formalism for dealing with flute type instabilities in mirror-confined plasmas, (2) the derivation of the general electromagnetic flute dispersion relation for axisymmetric systems, and (3) the application of the results of (1) and (2) to determine the effects of finite beta and finite geometry on stability criteria for the Dory-Guest-Harris, drift-cone, and resonant electromagnetic instabilities.

The finite beta flute formalism was derived for both the low and high frequency regimes, and the results corroborated by an exact model calculation spanning the frequency range between these limits. Application of the formalism essentially involves taking the $\oint [ds/B(s)]$ -line average of the infinite medium flute dispersion relation after allowing for axial nonuniformity. This procedure is valid in the low frequency regime ($\omega < \omega_{be}$), provided

$$|F(s)k_{\perp 0}^2 \lambda_{De}^2| < 1, \quad (7.1)$$

and in the high frequency regime ($\omega > \omega_{be}$), provided

$$L < L_{crit} \sim \frac{\pi}{k_{\perp 0}} \left\{ [F(s)]^{-1} \frac{M_i}{m_e} \left[1 + \frac{\beta_i}{(k_{\perp 0} a_i)^2} \right]^{-1} \right\}^{1/2}, \quad (7.2)$$

where $F(s)$ is the perpendicular contribution of the ions and electrons to the wave equation [defined as the integrand of Eq. (3.27)].

In the derivation of the general electromagnetic flute dispersion relation, finite beta modifications to the equilibrium orbits of the particles were taken into account. The result is given in Eq. (3.27) with the matrix elements specified by Eq. (3.22). This expression includes terms of order ϵ (the logarithmic derivative of the self-consistent magnetic field) and, to lowest order, just reduces to previously derived results. Application to specific problems leads to considerably less complicated forms for the dispersion relation. Specifically, the DGH and resonant electromagnetic modes do not concern radial gradients, and the drift-cone mode has characteristic parameters that make the contribution of the ϵ -terms small.

The analysis of the Dory-Guest-Harris instability included, in addition to finite geometry and finite beta modifications, effects due to the presence of an ambipolar potential. Extensive numerical and analytic calculations of stability criteria were performed using both theoretically-evaluated and experimentally-measured distributions. The numerical results indicated that an approximate stability criterion related to the specific form of the distribution is given by

$$\delta \approx \Delta E_{fw}/E_p \geq 2, \quad (7.3)$$

where E_{fw} is the full width at half maximum of the energy distribution and E_p is the value of the energy at the peak of the distribution. Stability criteria as a function of mirror ratio and ambipolar potential were determined from an analytic calculation and the relevant boundaries displayed in Fig. 8. In terms of these parameters the stability criterion is approximately

$$f \geq 0 \text{ and/or } |f| \leq \frac{2\pi R_0^{5/4}}{R_0} (ka_i)^2 \left(\frac{v_{thi}}{c}\right) (\hat{\phi})^{-1/2}, \quad (7.4)$$

with

$$f \approx -R_0^{1/2} \left\{ \exp\left(\frac{R_0 \hat{\phi}}{2}\right) \left[\ln\left(\frac{R_0 \hat{\phi}}{4}\right) + \gamma \right] + 1.25 \right\} + \exp\left(\frac{\hat{\phi}}{2}\right) \left[\ln\left(\frac{\hat{\phi}}{4}\right) + \gamma \right],$$

where R_0 = mirror ratio, $R \approx R_0 - 1$, $\hat{\phi} = \phi_m/RE_0$, E_0 is the average ion energy, and $\gamma \approx 0.577$ is Euler's constant. Application of the theoretical results indicate that, within the uncertainty of the experimental data, instability in the 2X plasma is marginal and that at higher mirror ratios ($R_0 \geq 3$) the DGH mode should be eliminated.

The most significant result of the drift-cone mode analysis was the demonstration of the stabilizing effect of finite beta. An analytic expression for the stability criterion [Eq. (5.15)] was derived in the infinite medium limit and used to determine beta limits for stability, which were found to be insensitive to changes in the mirror ratio and plasma temperature. In the moderate beta regime ($\beta \leq 1$) the stability criterion reduces to approximately

$$\frac{c}{R_p \omega_{ci}} < \frac{2}{1 + (\beta_i/2)} \left[1 + (\omega_{pe}^2 / \omega_{ce}^2) \right]^{1/2} \quad (7.5)$$

for

$$\beta_i > \beta_{crit} \sim 4 \left[(\omega_{ci}^2 / \omega_{pi}^2) + (m_e / M_i) \right]^{1/3}.$$

Inclusion of finite geometry effects led to a dispersion relation that had to be solved numerically. As indicated in Fig. 9, comparison of these solutions with the analytic results demonstrates the stabilizing influence of finite geometry, and, at the same time, illustrates that the results in the two cases are qualitatively similar. Suppression of spatial nonuniformity in the numerical model led to answers closely approximating the analytic values. However, a residual negative energy instability can persist in the system even when the drift-cone mode is stabilized (curve b in Fig. 9). Application of the analysis indicated that the 2X device should be unstable to drift-cone mode and that a plasma radius on the order of 50 ion Larmor radii is required for stability.

The resonant electromagnetic instability was derived analytically in the infinite medium limit. Comparison with previous numerical results showed close agreement. With the introduction of axial nonuniformity, which has a detuning effect on the resonances, it was found that the infinite medium growth rates are reduced but not eliminated in the fluid limit ($v_{||1} = 0$). When ion transit effects were also included in the analysis, sufficient but not necessary conditions for stability were obtained (dashed curve in Fig. 14). Exact stability thresholds could not be determined in the present treatment because the criterion for the validity of the flute formalism (Eq. 7.2) is violated at the very long critical lengths obtained here. Application of the analysis indicates that the REM mode should not appear in mirror systems (such as the 2X and Van Allen belt plasmas), since the appropriate parameters for such systems fall within the stable regime where the flute formalism is valid.

Acknowledgments

I am very grateful to my thesis advisor, Dr. L. Donald Pearistein, for his interest and guidance during the course of this research. Working with him was indeed a stimulating educational experience, and I feel quite fortunate to have had the opportunity to do so. A special note of thanks is due to Dr. Herbert L. Berk for many helpful suggestions. The counsel and friendship of both Don and Herb are sincerely appreciated. My thanks also to Prof. James Hurley and Prof. William Knox of the Physics Department at U.C. Davis for their aid and concern during the course of my graduate studies. In addition, I appreciate helpful discussions with Dr. David Baldwin, Dr. William Newcomb, and with Dr. Fred Coensgen and the 2X research group.

I am grateful to the Lawrence Livermore Laboratory for its financial support, to Dr. Wilson Talley and Dr. Richard Post for arranging my appointment here, and to the Applied Science and Physics Departments for their cooperation. Thanks are also extended to Dr. T. K. Fowler and M-Division for providing me with the opportunity to work with the Theoretical Group. Particular thanks go to Sandra Hawley, Lois Barber, and Olga McDonald for typing the draft of this thesis and for their general good cheer during my stay.

Finally, I deeply appreciate the constant support and understanding of my family and friends. I am especially grateful to my father whose encouragement and counsel has proven invaluable.

References

1. R. F. Post, Rev. Mod. Phys. 28, 338 (1956).
2. For derivation see, for example, E. S. Tannenbaum, Plasma Physics (McGraw-Hill, New York, 1970), p. 251.
3. For a survey of plasma instabilities with experimental evidence of their existence see B. Lehnert, Plasma Phys. 9, 301 (1967).
4. For a survey of macroinstabilities see J. D. Jukes, Rep. Prog. Phys. Pt. II, 30, 333 (1967).
5. J. B. Taylor, Phys. Fluids 6, 1529 (1963); Phys. Fluids 7, 161 (1964).
6. Yu B. Gott, M. S. Ioffe, and B. G. Telkovsky, Nucl. Fusion Supp. Pt. III, 1045 (1962), [English transl: Soviet J. At. Energy 14, 459 (1964)].
7. For a survey of microinstabilities see J. D. Jukes, Rep. Prog. Phys. Pt. I, 32, 305 (1968).
8. M. N. Rosenbluth, Plasma Physics, (International Atomic Energy Agency, Trieste, 1964), p. 485.
9. A. A. Vlasov, Zh. Eksp. i. Teor. Fiz. 8, 291 (1938), [English transl.: Theory of Many Particles, (Gordon and Beach, New York, 1960)].
10. D. C. Montgomery and D. A. Tidman, Plasma Kinetic Theory (McGraw-Hill, New York, 1964) p. 41.
11. L. D. Landau, Sov. Phys.-JETP 16, 574 (1946).
12. J. H. Malmberg and C. B. Wharton, Phys. Rev. Letters 13, 184 (1964); Phys. Rev. Letters 17, 175 (1966).
13. I. B. Bernstein, Phys. Rev. 109, 10 (1958).
14. C. N. Lahmore-Davies, Plasma Phys. 11, 271 (1969).
15. E. G. Harris, Phys. Rev. Letters 2, 34 (1959); J. Nucl. Energy Pt. C. 2, 138 (1961).
16. Yu N. Dnestrovsky, D. P. Kostomarov, and V. I. Pistunovich, Nucl. Fusion 3, 30 (1963).
17. L. S. Hall, W. Heckrotte, and T. Kammash, Phys. Rev. 139, 1117 (1965).
18. V. B. Krasovitskii and K. N. Stepanov, Zh. Techn. Fiz. 34, 1013 (1964).
19. M. N. Rosenbluth and R. F. Post, Phys. Fluids 8, 547 (1965).
20. G. E. Guest and R. A. Dory, Phys. Fluids 8, 1853 (1965); Phys. Fluids 11, 1775 (1968).
21. J. G. Cordey, Phys. Letters 23, 228 (1966).
22. R. A. Dory, G. E. Guest, and E. G. Harris, Phys. Rev. Letters 14, 131 (1965).
23. R. F. Post and M. N. Rosenbluth, Phys. Fluids 9, 730 (1966).
24. L. D. Pearlstein, M. N. Rosenbluth, and D. B. Chang, Phys. Fluids 9, 953 (1966).
25. W. M. Farr and R. E. Budwine, Phys. Fluids 11, 883 (1968).

26. H. L. Berk, T. K. Fowler, L. D. Pearlstein, R. F. Post, J. D. Callen, C. W. Horton, and M. N. Rosenbluth, in Plasma Physics and Controlled Nuclear Fusion Research Vol. II, (International Atomic Energy Agency, Vienna, 1969), p. 125.
27. R. A. Dory, W. M. Farr, G. E. Guest, and J. D. Callen, Phys. Fluids **12**, 2117 (1969).
28. G. E. Guest, W. M. Farr, and R. A. Dory, Phys. Fluids **13**, 3038 (1970).
29. P. A. Sturrock, J. Appl. Phys. **31**, 2052 (1960); and W. P. Allis, S. J. Buchsbaum, and A. Bers, Waves in Anisotropic Plasmas, (MIT Press, Cambridge, Mass., 1963).
30. L. D. Landau and E. M. Lifshitz, Electrodynamics of Continuous Media, (Addison-Wesley, Reading, Mass., 1960) p. 253.
31. B. B. Kadomtsev, A. B. Mikhailovskii, and A. V. Timofeev, Zh. Eksp. Teor. Fiz. **47**, (1965) [English transl.: Sov. Phys.-JETP **20**, 1517 (1965)].
32. A. Bers and S. Gruber, Appl. Phys. Letters **6**, 27 (1965).
33. L. S. Hall and W. Heckrotte, Phys. Fluids **9**, 1496 (1966).
34. A. B. Mikhailovskii, Reviews of Plasma Physics, Vol. 3, (Consultants Bureau, New York, 1967), p. 159.
35. N. A. Krall, Advances in Plasma Physics, Vol. 1, (Wiley, New York, 1968), p. 153.
36. A. B. Mikhailovskii and A. V. Timofeev, Zh. Eksp. Teor. Fiz. **44**, 919 (1963) [English transl.: Sov. Phys.-JETP **17**, 626 (1963)].
37. A. B. Mikhailovskii, Nucl. Fusion **5** (1965). [in Russian].
38. Y. Shima and T. K. Fowler, Phys. Fluids **8**, 2245 (1965).
39. A. B. Mikhailovskii, Dokl. Akad. Nauk SSSR **169**, 554 (1966) [English transl.: Sov. Phys.-Dokl. **11**, 603 (1967)].
40. W. M. Tang, L. D. Pearlstein, and H. L. Berk, Phys. Fluids **15**, 1153 (1972).
41. P. A. Sturrock, Phys. Rev. **112**, 1488 (1958).
42. See for example: R. J. Briggs, Electron Stream Interaction with Plasmas, Research Monograph No. 29 (MIT Press, Cambridge, Mass., 1964).
43. H. L. Berk, M. N. Rosenbluth, and R. N. Sudan, Phys. Fluids **9**, 1606 (1966); H. L. Berk, C. W. Horton, M. N. Rosenbluth, and R. N. Sudan, Phys. Fluids **10**, 2003 (1967).
44. P. M. Morse and H. Feshbach, Methods of Theoretical Physics, Vol. 2, (McGraw-Hill, New York, 1953) p. 1092.
45. C. W. Horton, Phys. Fluids **11**, 1154 (1968).
46. D. E. Baldwin, Phys. Rev. Letters **18**, 1119 (1967).
47. H. L. Berk, C. W. Horton, M. N. Rosenbluth, D. E. Baldwin, and R. N. Sudan, Phys. Fluids **11**, 365 (1968).
48. D. E. Baldwin and J. D. Callen, Enhanced Scattering Inherent to "Loss-cone" Particle Distributions, Lawrence Livermore Laboratory Rept. UCRL-73929 (1972).

49. F. H. Coensgen, W. F. Cummins, V. A. Finlayson, W. E. Nexsen, and T. C. Simonen, in Plasma Physics and Controlled Nuclear Fusion Research, Vol. II, (International Atomic Energy Agency, Vienna, 1971), p. 721.
50. F. H. Coensgen, W. F. Cummins, W. E. Nexsen, and T. C. Simonen, Initial Operation of the 2X-II Experiment, Lawrence Livermore Laboratory Rept. UCRL-51208 (1972).
51. C. O. Beasley and J. G. Cordey, Plasma Phys. **10**, 411 (1968).
52. J. G. Cordey, K. Kuo-Petravic, E. G. Murphy, M. Petravic, D. R. Sweetman, and E. Thompson, in Plasma Physics and Controlled Nuclear Fusion Research, Vol. II, (International Atomic Energy Agency, Vienna, 1969), p. 289.
53. C. C. Damm, J. H. Foote, A. H. Futch, A. L. Hunt, K. Moses, R. F. Post, and J. B. Taylor, Phys. Rev. Letters **24**, 489 (1970).
54. J. G. Cordey, Phys. Fluids **12**, 1506 (1969).
55. H. L. Berk, Phys. Fluids **14**, 1036 (1971).
56. J. E. McCune and J. D. Callen, Plasma Phys. **12**, 305 (1970).
57. J. D. Callen, Ph.D. Thesis, Massachusetts Institute of Technology (1968).
58. The phenomenon of plasma wave regeneration has been treated by many authors. See, for example, the following article and the references cited therein:
H. L. Berk and D. L. Book, Phys. Fluids **12**, 649 (1969).
59. H. L. Berk, L. D. Pearlstein, J. G. Cordey, Phys. Fluids **15**, 891 (1971).
60. M. N. Rosenbluth, H. L. Berk, and L. D. Pearlstein, Bull. Am. Phys. Soc. **13**, 297 (1968).
61. J. Heading, An Introduction to Phase Integral Methods, (Wiley, New York, 1962).
62. H. L. Berk, L. D. Pearlstein, J. D. Callen, C. W. Horton, and M. N. Rosenbluth, Phys. Rev. Letters **22**, 876 (1969).
63. D. E. Baldwin, C. O. Beasley, H. L. Berk, W. M. Farr, R. C. Harding, J. E. McCune, L. D. Pearlstein, and A. Sen, in Plasma Physics and Controlled Nuclear Fusion Research, Vol. II, (International Atomic Energy Agency, Vienna, 1971), p. 735.
64. H. L. Berk and L. D. Pearlstein, Phys. Fluids **14**, 1810 (1971).
65. J. F. Clarke and G. G. Kelley, Phys. Rev. Letters **21**, 1401 (1968).
66. B. B. Kadomtsev and O. Pogutse, in Plasma Physics and Controlled Nuclear Fusion Research, Vol. II, (International Atomic Energy Agency, Vienna, 1969), p. 125.
67. J. F. Clarke, G. G. Kelley, J. F. Lyon, and R. F. Stratton, in Plasma Physics and Controlled Nuclear Fusion Research, Vol. II, (International Atomic Energy Agency, Vienna, 1969), p. 291.
68. Yu. T. Baiborodov, Yu. V. Gott, M. S. Ioffe, and R. I. Sobolev, in Plasma Physics and Controlled Nuclear Fusion Research, Vol. II, (International Atomic Energy Agency, Vienna, 1969), p. 213.
69. J. D. Callen and C. W. Horton, Phys. Fluids **13**, 154 (1970).

70. J. D. Callen and C. W. Horton, *Bull. Am. Phys. Soc.* **15**, 489 (1970).
71. C. W. Horton, J. D. Callen, and M. N. Rosenbluth, *Phys. Fluids* **14**, 2019 (1971).
72. J. D. Callen and G. E. Guest, *Phys. Fluids* **14**, 1338 (1971).
73. L. D. Pearlstein, Finite β Stabilization of Absolute Loss-Cone Instabilities, Lawrence Livermore Laboratory Rept. UCRL-50397 (1968).
74. L. C. Himmell, *Phys. Fluids* **14**, 1419 (1971).
75. A. Sommerfeld, Partial Differential Equations in Physics, (Academic Press, New York, 1949), p. 36.
76. H. Goldstein, Classical Mechanics, (Addison-Wesley, Cambridge, Mass., 1953), p. 19.
77. G. N. Watson, Theory of Bessel Functions, (Cambridge Univ. Press, 1962), p. 22.
78. J. B. Taylor and R. J. Hastie, *Plasma Phys.* **10**, 479 (1968).
79. K. V. Roberts and J. B. Taylor, *Phys. Fluids* **8**, 315 (1965).
80. B. D. Fried and S. D. Conte, The Plasma Dispersion Function, (Academic Press, New York, 1961).
81. See, for example, T. H. Stix, The Theory of Plasma Waves, (McGraw-Hill, New York, 1962), p. 171.
82. B. Coppi, *Phys. Rev. Letters* **22**, 59 (1969).
83. J. D. Jukes, Low Frequency Instability in Shallow Magnetic Wells, Gulf-General Atomic, Inc. Rept. GA-8735 (1968).
84. V. I. Krylov, Approximate Calculation of Integrals, (MacMillan, New York, 1962), p. 100.
85. D. E. Baldwin, Low Frequency Electron Oscillations in Bounded Plasmas, Lawrence Livermore Laboratory Rept. UCID-15546 (1969).
86. J. D. Callen and G. E. Guest, submitted (1972) for publication in *Phys. Fluids*.
87. L. R. Ford, Differential Equations, (McGraw-Hill, New York, 1955), p. 70.
88. A. A. Vedenov, E. P. Velikhov, and R. Z. Sagdeev, *Nucl. Fusion* **1**, 82 (1961); 1962 Supplement, Part 2, p. 465; W. E. Drummond and D. Pines, *ibid.*, Part 3, p. 1049.
89. M. N. Rosenbluth, W. M. MacDonald, and D. L. Judd, *Phys. Rev.* **107**, 1 (1957).
90. J. Killeen and K. D. Marx, Methods in Computational Physics, Vol. 9, (Academic Press, New York, 1970), p. 421.
91. R. F. Post, *Phys. Fluids* **4**, 902 (1961).
92. J. Killeen and A. H. Futch, *J. Comput. Phys.* **2**, 236 (1968).
93. J. G. Cordey, submitted (1971) for publication in *Phys. Fluids*.
94. J. P. Holdren, Analytic Approximation to Collisional Distributions in Mirror Plasmas, Lawrence Livermore Laboratory Rept. UCRL-73281 (1971).
95. F. H. Coensgen, W. F. Cummins, R. E. Ellis, and W. E. Nexsen, *Proc. Int. Conf. on Plasma Phys. and Controlled Nucl. Fusion Res.* 8th, Vienna, Vol. II, 225 (1969).

96. W. F. Cummins and W. M. Tang, Bull. Am. Phys. Soc. **16**, 1260 (1971).
97. The original Marx-Killeen code described in Ref. 90 was modified by W. F. Cummins to include a second source.
98. M. Abramowitz and I. A. Stegun, Eds., Handbook of Mathematical Functions, (National Bureau of Standards, Washington, D.C., 1967).
99. I. S. Gradshteyn and I. M. Ryzhik, Tables of Integrals, Series, and Products, (Academic Press, New York, 1965).
100. A. H. Futch, J. P. Hoidren, J. Killeen, and A. A. Mirin, Multi-Species Fokker-Planck Calculations for D-T and D-He³ Mirror Reactors, Lawrence Livermore Laboratory Rept. UCRL-73226 (1971).
101. L. G. Kelly, Handbook of Numerical Methods and Applications, (Addison-Wesley, Massachusetts, 1967), p. 80.
102. For the Van Allen belt plasma the important electromagnetic instabilities are the parallel propagating modes ($k = k_{\parallel}$). See, for example, C. F. Kennell and H. E. Petschek, J. Geophys. Res. **71**, 1 (1966); J. M. Cornwall and M. Schulz, J. Geophys. Res. **76**, 7791 (1971).

Appendix A

Phase Integrals

In this appendix the details of the phase integration leading to Eq. (3.22) are presented. The integrals to be evaluated are specified by Eqs. (3.15) and (3.21) and are labeled as follows:

$$\begin{pmatrix} P_{11} \\ P_{12} \\ P_{11\epsilon'} \\ P_{12\epsilon'} \end{pmatrix} = v_1 \int_0^{2\pi} d\phi \mathbf{i}_1 \begin{pmatrix} \sin\phi \\ \cos\phi \\ \epsilon' \sin\phi \cos\phi \\ \epsilon' \cos^2\phi \end{pmatrix} \quad (\text{A-1})$$

$$\begin{pmatrix} P_{31} \\ P_{32} \\ P_{31\epsilon'} \\ P_{32\epsilon'} \end{pmatrix} = v_1 \int_0^{2\pi} d\phi \mathbf{i}_3 \begin{pmatrix} \sin\phi \\ \cos\phi \\ \epsilon' \sin\phi \cos\phi \\ \epsilon' \cos^2\phi \end{pmatrix} \quad (\text{A-2})$$

$$\begin{pmatrix} P_{21} \\ P_{22} \end{pmatrix} = v_1 \int_0^{2\pi} d\phi \mathbf{i}_2 \begin{pmatrix} \sin\phi \\ \cos\phi \end{pmatrix} \quad (\text{A-3})$$

These algebraically tedious but straightforward calculations are performed with the aid of the Bessel function identities given in Eq. (2.49) together with the following relationships^{77,98,99}:

$$J_{n-2} + J_{n+2} = 4 \left[J_n \left(\frac{n^2}{\lambda^2} - \frac{1}{2} \right) - \frac{J'_n}{\lambda} \right] \quad (\text{A-4})$$

$$J_{n-2} - J_{n+2} = \frac{4n}{\lambda} \left(J'_n - \frac{J_n}{\lambda} \right) \quad (\text{A-5})$$

$$J_{n-3} + J_{n+3} = \frac{8n}{\lambda^2} \left[J_n \left(\frac{n^2}{\lambda^2} + \frac{2}{\lambda} - \frac{3\lambda}{4} \right) - 3J'_n \right] \quad (\text{A-6})$$

$$J_{n-3} - J_{n+3} = -2J'_n + \frac{8}{\lambda^2} \left[J_n \left(\lambda - \frac{3n^2}{\lambda} \right) + J'_n (n^2 + 2) \right] \quad (\text{A-7})$$

$$J_{n-1}^2 + J_{n+1}^2 = 2 \left(J_n^2 + \frac{n^2}{\lambda^2} J_n^2 \right) \quad (\text{A-8})$$

$$J_{n-1}^2 - J_{n+1}^2 = \frac{4n}{\lambda} J_n J'_n \quad (\text{A-9})$$

$$J_n'' = J_n \left(\frac{n^2}{\lambda^2} - 1 \right) - \frac{J_n'}{\lambda} \quad (\text{A-10})$$

$$\sum_n J_n^2 = 1, \quad n = 0, \pm 1, \dots \quad (\text{A-11})$$

$$\sum_{n=1}^{\infty} n^2 J_n^2 = \frac{\lambda^2}{4} \quad (\text{A-12})$$

$$\sum_{n=1}^{\infty} (J_n')^2 = \frac{1}{4} - \frac{J_1^2}{2}, \quad (\text{A-13})$$

where λ = argument of the Bessel functions and the primes denote differentiation with respect to λ . To illustrate the basic method used to evaluate all the integrals in Eqs. (A-1), (A-2), and (A-3), the detailed evaluation of the P_{11} element will be presented.

Starting with Eq. (3.21) and (A-1), the calculation proceeds as follows:

$$P_{11} = \frac{v_1}{2\Gamma} \int_0^{2\pi} d\phi (e^{i\phi} - e^{-i\phi}) I_1 = I_{10} + I_{1A} + I_{1B}, \quad (\text{A-14})$$

where

$$I_{10} = -\frac{v_1}{2} \sum_{n,m} J_n J_m \int_0^{2\pi} d\phi (e^{i\phi} + e^{-i\phi}) \left\{ \frac{v_1}{2} \left[\frac{e^{i\phi}}{(n+1)\omega_c + \chi} + \frac{e^{-i\phi}}{(n-1)\omega_c + \chi} \right. \right. \\ \left. \left. + \frac{k_1 \epsilon v_1^2}{4\omega_c^2} \Pi_1 \right] + \frac{\epsilon v_1^2}{4\omega_c} \text{III}_1 \right\} \quad (\text{A-15})$$

$$I_{1A} = \chi \epsilon \frac{v_1^2}{4} \int_0^{2\pi} d\phi \sum_{n,m} e^{i(n-m)\phi} \left(J_n J_m \left\{ \frac{(n+1)(\omega_c)(e^{i2\phi} - 1)}{[(n+1)\omega_c + \chi]^2} \right. \right. \\ \left. \left. - \frac{(n-1)(\omega_c)(e^{-i2\phi} - 1)}{[(n-1)\omega_c + \chi]^2} \right\} + \left\{ \frac{e^{i2\phi} - 1}{[(n+1)\omega_c + \chi]} - \frac{e^{-i2\phi} - 1}{[(n-1)\omega_c + \chi]} \right\} \right) \\ \times \lambda \frac{d}{d\lambda} (J_n J_m) \quad (\text{A-16})$$

$$I_{1B} = \epsilon \frac{v_{\perp}^3}{8\omega_c} \int_0^{2\pi} d\phi \sum_{n,m} e^{i(n-m)\phi} \left(J_n J_m \left\{ \frac{(n+1)(\omega_c)(e^{i3\phi} - e^{-i\phi})}{[(n+1)\omega_c + \chi]^2} \right. \right. \\ \left. \left. - \frac{(n-1)(\omega_c)(e^{-i3\phi} - e^{i\phi})}{[(n-1)\omega_c + \chi]^2} \right\} + \left\{ \frac{e^{i3\phi} - e^{-i\phi}}{[(n+1)\omega_c + \chi]} - \frac{e^{-i3\phi} - e^{i\phi}}{[(n-1)\omega_c + \chi]} \right\} \right. \\ \left. \times \lambda \frac{d}{d\lambda} (J_n J_m) \right). \quad (A-17)$$

After performing the phase integration, the terms are collected as follows:

$$I_{1A} = \frac{v_{\perp}^2}{4} \sum_m \left\{ \frac{\omega_c(m-1)(J_m J_{m-2} + J_m^2)}{[(m-1)\omega_c + \chi]^2} - \frac{\omega_c(m-1)(J_m J_{m+2} + J_m^2)}{[(m+1)\omega_c + \chi]^2} \right. \\ \left. + \frac{\lambda \frac{d}{d\lambda} (J_m J_{m-2} + J_m^2)}{[(m-1)\omega_c + \chi]} - \frac{\lambda \frac{d}{d\lambda} (J_m J_{m+2} + J_m^2)}{[(m+1)\omega_c + \chi]} \right\} \\ = -\frac{v_{\perp}^2}{4} \sum_n \frac{n\omega_c}{(n\omega_c + \chi)^2} (J_{n-1} + J_{n+1})(J_{n-1} - J_{n+1}) \\ + \frac{\lambda \frac{d}{d\lambda} (J_{n-1} + J_{n+1})(J_{n-1} - J_{n+1})}{(n\omega_c + \chi)} \\ = -\frac{v_{\perp}^2}{4} \sum_n \left[\frac{n\omega_c}{(n\omega_c + \chi)^2} + \frac{\lambda \frac{d}{d\lambda}}{(n\omega_c + \chi)} \right] \left(\frac{4n}{\lambda} J_n J_n' \right) \quad (A-18)$$

$$I_{1B} = \epsilon \frac{v_{\perp}^3}{8\omega_c} \sum_m \left\{ \frac{\omega_c(m-2)J_m(J_{m-3} + J_{m-1})}{[(m-2)\omega_c + \chi]^2} - \frac{\omega_c(m+2)J_m(J_{m+3} + J_{m+1})}{[(m+2)\omega_c + \chi]^2} \right. \\ \left. + \frac{\lambda \frac{d}{d\lambda} [J_m(J_{m-3} + J_{m-1})]}{[(m-2)\omega_c + \chi]} - \frac{\lambda \frac{d}{d\lambda} [J_m(J_{m+3} + J_{m+1})]}{[(m+2)\omega_c + \chi]} \right\}$$

$$= \epsilon \frac{v_1^3}{8\omega_c} \sum_n \left[\frac{n\omega_c}{(n\omega_c + \chi)^2} + \frac{\lambda \frac{d}{d\lambda}}{(n\omega_c + \chi)} \right] \left[\frac{8n^2}{\lambda^2} \left(\frac{J_n^2}{\lambda} - J_n J'_n \right) \right] \quad (\text{A-19})$$

$$I_{10} = -\frac{v_1^2}{4} \sum_m J_m \left\{ \left[\frac{J_{m-2} + J_m}{(m-1)\omega_c + \chi} - \frac{J_{m+2} + J_m}{(m+1)\omega_c + \chi} \right] + \frac{k_1 \epsilon v_1^2}{4\omega_c^2} IV \right\}.$$

where

$$IV = \hat{I}_1 - \frac{1}{2} \hat{I}_2 + \frac{2}{\lambda} \hat{I}_3$$

$$(i) \sum_m J_m \left[\frac{J_{m-2} + J_m}{(m-1)\omega_c + \chi} - \frac{J_{m+2} + J_m}{(m+1)\omega_c + \chi} \right] = -\frac{4}{\lambda} \sum_n \frac{n J_n J'_n}{n\omega_c + \chi}$$

$$(ii) \sum_m J_m \hat{I}_1 = \sum_m J_m \left[\frac{J_{m-4} + J_m + 2J_{m-2}}{(m-2)\omega_c + \chi} + \frac{J_{m+4} + J_m + 2J_{m+2}}{(m+2)\omega_c + \chi} \right. \\ \left. - \frac{2(J_{m+2} + J_{m-2} + 2J_m)}{(m+3)\omega_c + \chi} \right]$$

$$= \sum_n \frac{(J_{n+2} + J_{n-2} + 2J_n)(J_{n+2} + J_{n-2} - 2J_n)}{n\omega_c + \chi}$$

$$= \frac{16}{\lambda^2} \sum_n \left[n^2 J_n^2 \left(\frac{n^2}{\lambda^2} - 1 \right) + (J'_n)^2 - J_n J''_n \left(\frac{2n^2}{\lambda} - \lambda \right) \right] \frac{1}{n\omega_c + \chi}$$

$$(iii) \sum_m J_m \hat{I}_2 = \sum_m J_m \left[\frac{J_{m-4} - J_{m+2} - 2J_m}{(m-1)\omega_c + \chi} + \frac{J_{m+4} - J_{m-2} - 2J_m}{(m+1)\omega_c + \chi} \right. \\ \left. + \frac{J_{m-4} + J_{m-2}}{(m-3)\omega_c + \chi} + \frac{J_{m+4} + J_{m+2}}{(m+3)\omega_c + \chi} \right]$$

$$= - \sum_n \frac{(J_{n-1} - J_{n+1})(J_{n-3} - J_{n+3}) - (J_{n-1} + J_{n+1})(J_{n+3} + J_{n-3}) + 2(J_{n-1}^2 + J_{n+1}^2)}{n\omega_c + \chi}$$

$$= - \frac{16}{\lambda^2} \sum_n \left[\lambda J_n J_n' + (n^2 + 2)(J_n')^2 + n^2 J_n^2 \left(1 - \frac{n^2 + 2}{\lambda^2} \right) \right] \frac{1}{n\omega_c + \chi}$$

$$\begin{aligned} \text{(iv)} \sum_m J_m \hat{I}_3 = \sum_m J_m \left[\frac{2J_{m-1} - J_{m+1} - J_{m-3}}{(m-1)\omega_c + \chi} - \frac{2J_{m+1} - J_{m+3} - J_{m-1}}{(m+1)\omega_c + \chi} \right. \\ \left. + \frac{J_{m-3} - J_{m-1}}{(m-2)\omega_c + \chi} - \frac{J_{m+3} - J_{m+1}}{(m+2)\omega_c + \chi} + \frac{2(J_{m+1} - J_{m-1})}{m\omega_c + \chi} \right] \end{aligned}$$

$$= \sum_n \frac{4J_n(J_{n+1} - J_{n-1}) + 2(J_{n-2} + J_{n+2})(J_{n-1} - J_{n+1})}{n\omega_c + \chi}$$

$$= \frac{16}{\lambda} \sum_n \left\{ J_n J_n' \left(\frac{n^2}{\lambda} - \lambda \right) - (J_n')^2 \right\} \frac{1}{n\omega_c + \chi}$$

Collecting (i) through (iv) gives

$$I_{10} = \frac{v_1^2}{\lambda} \sum_n \frac{n J_n J_n'}{n\omega_c + \chi} + \frac{ev_1^2}{2k_1} \sum_n \frac{\lambda J_n J_n' - n^2 \left[J_n'^2 + J_n^2 \left(\frac{n^2 - 2}{\lambda^2} - 1 \right) \right]}{n\omega_c + \chi} \quad (\text{A-20})$$

The n -factors in the numerators are then eliminated by rewriting the sums as follows:

$$\text{(i)} \sum_n \frac{n J_n J_n'}{n\omega_c + \chi} = \frac{1}{\omega_c} \sum_n J_n J_n' \left[1 - \frac{\chi}{n\omega_c + \chi} \right] = \frac{\chi}{\omega_c} \sum_n \frac{J_n J_n'}{n\omega_c + \chi}$$

$$\sum_n \frac{n^2 (J_n')^2}{n\omega_c + \chi} = \sum_n \frac{n (J_n')^2}{\omega_c} \left[1 - \frac{\chi}{n\omega_c + \chi} \right] = - \frac{\chi}{\omega_c^2} \sum_n (J_n')^2 \left[\frac{\chi}{n\omega_c + \chi} \right]$$

or, using Eq. (A-13),

$$(iii) \sum_n \frac{n^2 (J_n^1)^2}{n\omega_c + \chi} = -\frac{\chi}{\omega_c^2} \left[\frac{1}{2} - J_1(2J_0 + J_1) \right] + \left(\frac{\chi}{\omega_c} \right)^2 \sum_n \frac{(J_n^1)^2}{n\omega_c + \chi}$$

$$(iii) \sum_n \frac{n^2 J_n^2}{n\omega_c + \chi} = -\frac{\chi}{\omega_c^2} + \left(\frac{\chi}{\omega_c} \right)^2 \sum_n \frac{J_n^2}{n\omega_c + \chi}$$

$$(iv) \sum_n \frac{n^4 J_n^2}{n\omega_c + \chi} = -\frac{\chi}{\omega_c^2} \sum_n n^2 J_n^2 \left[1 - \frac{\chi}{n\omega_c + \chi} \right]$$

$$= -\frac{\chi}{\omega_c^2} \left\{ \frac{\lambda^2}{2} + \left(\frac{\chi}{\omega_c} \right)^2 \left[1 - \chi \sum_n \frac{J_n^2}{n\omega_c + \chi} \right] \right\}.$$

where use has been made of Eq. (A-12) in obtaining (iv). Substitution of results (i) through (iv) back into Eq. (A-20) yields to order ϵ ,

$$I_{10} = v_1^2 \left(-\sum_n \frac{n J_n J_n^1 / \lambda}{n\omega_c + \chi} + \frac{\epsilon}{2k_1} \left\{ \frac{n}{\omega_c} \left[\frac{n^2 - 2}{\lambda^2} - J_1(2J_0 + J_1) \right] \right. \right. \\ \left. \left. + \sum \frac{n^2 \left(1 - \frac{n^2 - 2}{\lambda^2} \right) J_n^2 - n(J_n^1)^2}{n\omega_c + \chi} \right\} \right). \quad (A-21)$$

Finally, after repeating the above procedure in eliminating the n -factors in the numerators of Eqs. (A-18) and (A-19), the results are combined with Eq. (A-21) to yield P_{11} . With the local radial factor (χ) set equal to zero for convenience, the answer is given in Chapter III as Eq. (3.22d). The method presented in this appendix is applied in the same manner to arrive at the other results in Eq. (3.22).

Appendix B

Details in Dory-Guest-Harris Calculations

This appendix gives the details of the calculations that yield Eqs. (4.19), (4.20), and (4.24) in connection with the analysis of the Dory-Guest-Harris instability. In particular, the appropriate normalization of the distributions introduced by Cordey [Eq. (4.9)] and Holdren [Eq. (4.10)] is shown. Substituting the respective distributions into Eq. (4.16) and making the change of variable

$$y = \left(\mu B_0 - \frac{\phi_m}{R} \right) / \left(E - \frac{\phi_m}{R} \right) \quad (\text{B-1})$$

leads to the forms of the ion contribution given in Eqs. (4.19) and (4.20). The calculation of the D_1 term, which is needed to estimate the growth rate of the mode, is also presented here.

Cordey's distribution is normalized at the midplane ($s = 0$) as follows:

$$\begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = \frac{1}{\int_{\phi_m/R}^{\infty} dE f(E)} \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \quad \begin{matrix} \text{for } Re \leq 1.5 \\ \text{for } Re > 1.5, \end{matrix} \quad (\text{B-2})$$

where

$$\begin{pmatrix} L_1 \\ L_2 \end{pmatrix} = \int_{\frac{E + \phi_m}{R_0}}^E (E - \mu B_0)^{-1/2} \begin{pmatrix} (\mu B_0 R_0 - E - \phi_m)/(RE - \phi_m) \\ 1 + [\ln(\mu B_0/E)/\ln(R_e)] \end{pmatrix} dy.$$

L_1 and L_2 are then integrated to give

$$\begin{aligned} L_1 &= \frac{1}{RE - \phi_m} \int_0^{(RE - \phi_m)/R_0} dy y^{-1/2} [(RE - \phi_m) - R_0 y] \\ &= \frac{4}{3} \left(\frac{R}{R_0} \right)^{1/2} \left(E - \frac{\phi_m}{R} \right)^{1/2} \end{aligned} \quad (\text{B-3})$$

$$L_2 = 2[(RE - \phi_m)/R_0]^{1/2} + \frac{E^{1/2}}{\ln(R_e)} \int_{R_e}^1 dy \ln(y)(1-y)^{-1/2} \quad (B-4)$$

From Ref. 99 [p. 204, Eq. (5)], one obtains

$$L_2 = \frac{2E^{1/2}}{\ln(R_e)} \left[\ln \left(\frac{1 + \Lambda_1}{1 - \Lambda_1} \right) - 2\Lambda_1 \right], \quad (B-5)$$

where

$$\Lambda_1 = [(RE - \phi_m)/R_0 E]^{1/2}$$

$$R_e = R_0 E / (E + \phi_m).$$

For Holdren's distribution, normalization at the midplane leads to

$$N_3 = 1 / \int_{\frac{\phi_m}{R}}^{\infty} dE f(E) [RE - \phi_m + \Lambda_3 \ln(R_e)]^{-1} L_3, \quad (B-6)$$

where

$$L_3 = \frac{\int_{\frac{E + \phi_m}{R_0}}^E d\mu B_0 (E - \mu B_0)^{-1/2} [\mu B_0 R_0 - E - \phi_m + \Lambda_3 \ln(R_e \mu B_0 / E)]}{R_0}$$

with

$$\Lambda_3 = E(2R - 1) - 3\phi_m.$$

The integration here is performed just as in Eqs. (B-3) and (B-5) above, and the result given in Eq. (4.20).

As shown in Eq. (4.23), the growth rate calculations require that the following integral be evaluated:

$$M_1 = \frac{R^{1/2}}{2L \tan^{-1}(R^{1/2})} \iint dE d\mu B_o J_1^2 [\alpha(\mu B_o)^{1/2}] \left[\left(\frac{\partial f_o}{\partial E} \right) V_1 + \left(\frac{\partial f_o}{\partial \mu B_o} \right) V_2 \right], \quad (B-7)$$

where

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \frac{2L}{(\mu B_o R - \phi_m)^{1/2}} \int_0^{\pi/2} \frac{dx}{(1 + b \sin^2 x)^2} \begin{pmatrix} 1 \\ (1 + b \sin^2 x)^{-1} \end{pmatrix},$$

$$b = \frac{E - \mu B_o}{\mu B_o - \frac{\phi_m}{R}},$$

and the spatial dependence specified by Eq. (4.12) is used. From Ref. 99 (pages 152-3), the V_1 and V_2 integrals become

$$\int_0^{\pi/2} \frac{dx}{(1 + b \sin^2 x)^2} = \frac{\pi}{2} \frac{(1 + b/2)}{(1 + b)^{3/2}} \quad (B-8)$$

$$\int_0^{\pi/2} \frac{dx}{(1 + b \sin^2 x)^3} = \frac{\pi}{16} \left[\frac{3}{(1 + b)^{1/2}} + \frac{2}{(1 + b)^{3/2}} + \frac{3}{(1 + b)^{5/2}} \right] \quad (B-9)$$

After substituting Eqs. (B-8) and (B-9) back into Eq. (B-7), the change of variable specified by Eq. (E-1) is made, and the result given in Eq. (4.24) is obtained.

Appendix C

Evaluation of Ion Transit Integrals

This appendix deals with the evaluation of the ion transit integrals, $I_1(\Lambda)$ and $I_2(\Lambda)$, from Eq. (6.23). As noted in Section 6.3.3, they can be calculated analytically in the large Λ limit to give $(\pi/\Lambda)^{1/2}$ and $\frac{1}{2}(\pi/\Lambda^3)^{1/2}$ respectively. However, for general values of this parameter, these integrals must be evaluated numerically. For unstable and marginally stable modes, $\text{Im}(\Lambda) \leq 0$. The choice of an appropriate contour of integration is thus determined by the real part of Λ and the location of the branch points of the integral in the complex z -plane. To facilitate numerical convergence, one deforms the contour into the upper half-plane for $\text{Re}(\Lambda) < 0$ and into the lower half-plane for $\text{Re}(\Lambda) \geq 0$.

The branch points in the complex z -plane for $I_1(\Lambda)$ come from the zeros of the equation

$$1 + iz^6 = 0. \quad (\text{C-1})$$

It is clear that they are located at magnitude unity and at angles given by

$$\theta_E \approx \exp(i[n + (1/4)]\pi/3), \quad (\text{C-2})$$

where $n = 1, 2, \dots, 6$.

In the case of $I_2(\Lambda)$ the branch points are determined by the zeros of the following equation:

$$f_2(y)z^{12} - if_1(y)z^6 - 1 = 0, \quad (\text{C-3})$$

where

$$f_1(y) = (5 + 6y^2 - 3y^4)/8$$

$$f_2(y) = (1 - y^2)^3/16.$$

Since $0 \leq y \leq 1$, the limits of f_1 and f_2 are

$$5/8 \leq f_1 \leq 1 \text{ and } 0 \leq f_2 \leq 1/16. \quad (\text{C-4})$$

The solution to Eq. (B-3) is just

$$z^6 = (1/2f_2) \left[f_1 \pm (f_1^2 - 4f_2)^{1/2} \right]. \quad (\text{C-5})$$

From Eq. (C-4), the right-hand side of Eq. (C-5) is always a positive, imaginary value. Thus, the branch points in the complex z -plane will be located at angles given by Eq. (C-2) with magnitudes specified by

$$(z_0)_\pm = \left\{ \left[f_1 \pm (f_1^2 - 4f_2)^{1/2} \right] / [2f_2] \right\}^{1/6}. \quad (C-6)$$

In other words, there will be two branch points with magnitude $(z_0)_-$ and $(z_0)_+$ along each of six branch lines at angles θ_B . Since $(z_0)_\pm$ is a function of y , the determination of this value hinders the efficient numerical calculation of the integrals. To avoid this difficulty, one simply takes the contours of integration to lie within the region

$$-\pi/4 < \theta < \pi/12 \text{ and } 0 < |z| < \infty. \quad (C-7)$$

With this choice of contours, no branch lines are crossed, and the ion transit integrals can be readily computed using standard numerical methods.⁹⁹