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A FREE-BOUNDARY TOKAMAK

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On Stability of the Elliptical Cross Section
for a Free-Boundary Tokamak

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ABSTRACT

Analysing stability of the free-boundary cylindrical tokamak of elliptical cross-section with semi-axes a , b to modes with $k_z = 0$, we show instability to vertical displacement ($m=1$) if $b > a$, and to triangulation ($m=3$) if $b/a > 4.5$, but stability to $m = 2, 4$ for any b/a .

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MASTER

Motivated by the desire for stronger Ohmic currents at fixed total rotational transform, there has been recent interest in tokamaks with noncircular plasma cross-section,¹ especially vertically elongated ellipses. When the plasma has a free surface, and the equilibrium is formed by suitable externally-produced shaping fields, an interesting question arises as to the magnetohydrodynamic stability of the elliptical shape against changes of shape, e.g., more or less ellipticity ($m = 2$), triangularity ($m = 3$), etc. This is the topic of the present note. The modes of concern are axisymmetric in the torus, and have $k_z = 0$ in the cylindrical approximation which we employ. They have been considered previously² within the much wider context of kink modes, but certain of the perturbations considered in this earlier treatment violated the incompressibility constraint.

Our equilibrium is an infinitely-long cylindrical plasma of elliptical cross-section: $x^2/a^2 + y^2/b^2 = 1$, with $b > a$. It is surrounded by a vacuum which extends to infinity. Current-carrying conductors in the vacuum maintain the elliptical plasma equilibrium. Provided the currents in these conductors are unaffected by the perturbations, they do not enter our stability analysis. The longitudinal field B_z is assumed to be uniform and much larger than the perpendicular field $\underline{B}_\perp$. The field $\underline{B}_\perp$ may be described by a flux function $\psi(x,y)$, such that $\underline{B}_\perp = \underline{e}_z \times \nabla \psi$ with $\nabla^2 \psi = j_z(\psi)$. For uniform current density j_z within the plasma, we have

$$\psi(x,y) = \psi_0 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right)$$

where $\psi_0 = a^2 b^2 j_z / 2(a^2 + b^2)$ is the total perpendicular flux in the plasma. The flux function $\psi(x,y)$ in the vacuum is not needed in the analysis. The magnetic surfaces within the plasma are similar ellipses. In order that these be coordinate surfaces, we employ coordinates X, θ within the plasma defined by

$$x = aX^{1/2} \cos \theta ; \quad y = bX^{1/2} \sin \theta$$

where $X = \psi/\psi_0$. We note that

$$\begin{aligned} \underline{\nabla} X &= \frac{2X^{1/2}}{ab} (b \cos \theta \underline{e}_x + a \sin \theta \underline{e}_y) , \\ \underline{\nabla} \theta &= \frac{1}{abX^{1/2}} (-b \sin \theta \underline{e}_x + a \cos \theta \underline{e}_y) . \end{aligned} \tag{1}$$

These coordinates are nonorthogonal. Defining length elements $d\ell$ and dn along and normal to the elliptical magnetic surface, we have

$$d\ell = X^{1/2} (a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{1/2} d\theta ,$$

$$dn = \frac{ab dX}{2X^{1/2} (a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{1/2}}$$

so that the element of cross-section area is $dA = d\ell dn$
 $= (1/2) ab d\theta dX$.

To analyse stability, we use the form of the energy principle appropriate to the ordering $B_{\perp}/B_z \sim (p/B_z^2)^{1/2} \ll 1$:

$$\delta W = \delta W^P + \delta W^V,$$

$$2\delta W^P = \int_P [(\nabla \psi_1)^2 + j_z \xi_{\perp} \cdot \nabla \psi_1] dA, \quad (2)$$

$$2\delta W^V = \int_V (\nabla \psi_1)^2 dA. \quad (3)$$

Here, ψ_1 is the perturbed flux function satisfying $\nabla_{\perp} \psi_1 = \mathbf{e}_z \times \nabla \psi_1$; $\xi_{\perp}$ is the plasma displacement which must satisfy the incompressibility constraint $\nabla_{\perp} \cdot \xi_{\perp} = 0$; and P and V refer to plasma and vacuum regions, respectively.

Incompressibility imposes a constraint upon the permitted perturbations ψ_1 within the plasma. For modes with $k_z = 0$, we have

$$\psi_1 = -\xi_{\perp} \cdot \nabla \psi = -\xi_n |\nabla \psi|$$

where ξ_n is the component of $\xi_{\perp}$ normal to the constant- ψ surface. The incompressibility constraint, namely the invariance of the cross-sectional area of the constant- ψ surfaces, may be written

$$\oint \xi_n d\ell = 0.$$

Substituting for ξ_n in terms of ψ_1 , and for $|\nabla\psi|$ and $d\ell$ in terms of θ and $d\theta$, this may be written

$$\oint \psi_1 d\theta = 0$$

the integration being taken around a constant- ψ surface. It follows that the perturbations within the plasma must be of the form

$$\psi_1(X, \theta) = \sum_{m=1}^{\infty} \hat{\psi}_m(X) \begin{vmatrix} \cos m\theta \\ \sin m\theta \end{vmatrix} . \quad (4)$$

The minimization of the vacuum contribution to δW may be carried out explicitly. The minimizing function ψ_1 must satisfy $\nabla^2 \psi_1 = 0$. We use a coordinate system (μ, θ) confocal with the elliptical plasma boundary:

$$x = (b^2 - a^2)^{1/2} \sinh \mu \cos \theta ; \quad y = (b^2 - a^2)^{1/2} \cosh \mu \sin \theta .$$

In terms of the quantities $\hat{\psi}_m(1)$ on the plasma boundary, the solutions vanishing at infinity are

$$\psi_1 = \sum_{m=1}^{\infty} \hat{\psi}_m(1) \exp[-m(\mu - \mu_0)] \begin{vmatrix} \cos m\theta \\ \sin m\theta \end{vmatrix}$$

where $\tanh \mu_0 = a/b$. The minimum for δW^V is then given by an integral around the plasma boundary:

$$2\delta W^V = - \oint \psi_1 (\nabla \psi_1)_n d\ell$$

$$= \pi \sum_{m=1}^{\infty} m \hat{\psi}_m(1)^2$$

where we have used $(\nabla \psi)_n = (a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{-1/2}$.

For the special case of uniform current density j_z within the plasma, the second term in δW^P may also be expressed as an integral around the plasma boundary. We write $j_z \xi_{\perp} \cdot \nabla \psi_1 = \nabla \cdot (j_z \psi_1 \xi_{\perp})$, so that the second term on the right in Eq. (2) becomes

$$\oint j_z \psi_1 \xi_n d\ell = - \oint j_z \psi_1^2 d\ell / |\nabla \psi|$$

$$= - \frac{\pi(a^2+b^2)}{ab} \sum_{m=1}^{\infty} \hat{\psi}_m(1)^2.$$

Thus:

$$2\delta W = \int_P (\nabla \psi_1)^2 dA + \pi \sum_{m=1}^{\infty} (m - \frac{a^2+b^2}{ab}) \hat{\psi}_m(1)^2.$$

We must minimize this with respect to functions ψ_1 of the form given in Eq. (4). Using Eqs. (1), we have

$$\begin{aligned} (\nabla \psi_1)^2 &= (\nabla X)^2 \left(\frac{\partial \psi_1}{\partial X}\right)^2 + 2 \nabla X \cdot \nabla \theta \frac{\partial \psi_1}{\partial X} \frac{\partial \psi_1}{\partial \theta} + (\nabla \theta)^2 \left(\frac{\partial \psi_1}{\partial \theta}\right)^2 \\ &= \frac{1}{2a^2b^2X} \{ (a^2+b^2) [4X^2 \left(\frac{\partial \psi_1}{\partial X}\right)^2 + \left(\frac{\partial \psi_1}{\partial \theta}\right)^2] \\ &\quad + (b^2-a^2) [4X^2 \left(\frac{\partial \psi_1}{\partial X}\right)^2 \cos 2\theta \\ &\quad - 4X \frac{\partial \psi_1}{\partial X} \frac{\partial \psi_1}{\partial \theta} \sin 2\theta - \left(\frac{\partial \psi_1}{\partial \theta}\right)^2 \cos 2\theta] \}. \end{aligned} \quad (5)$$

From symmetry considerations, possible modes will divide into symmetric and antisymmetric modes about the x-axis [given by the $\cos m\theta$ and $\sin m\theta$ series in Eq. (4), respectively], and will divide further into symmetric and antisymmetric modes about the y-axis. Choosing first the symmetric modes in both senses, we have only the even terms in the $\cos m\theta$ series in Eq. (4):

$$\psi_1 = \sum_{\text{even } m=2}^{\infty} \hat{\psi}_m(X) \cos m\theta.$$

Substituting into Eq. (5) and carrying out the θ -integrations, we have

$$\begin{aligned} \int_P (\nabla \psi_1)^2 dA &= \frac{\pi}{ab} \sum_{\text{even } m=2}^{\infty} \int_0^1 \frac{dX}{X} \{ (a^2 + b^2) (X^2 \hat{\psi}_m'^2 + \frac{m^2}{4} \hat{\psi}_m^2) \\ &+ (b^2 - a^2) [X^2 \hat{\psi}_m' \hat{\psi}_{m+2}' - \frac{m(m+2)}{4} \hat{\psi}_m \hat{\psi}_{m+2}] \\ &+ \frac{X(m+2)}{2} \hat{\psi}_m' \hat{\psi}_{m+2} - \frac{Xm}{2} \hat{\psi}_m \hat{\psi}_{m+2}' \} . \end{aligned} \quad (6)$$

where prime denotes a differentiation with respect to X . The integral of the terms quadratic in $\hat{\psi}_m^2$ is stationary with respect to variations of the function $\hat{\psi}_m(X)$, holding $\hat{\psi}_m(1)$ fixed, if

$$X(X\hat{\psi}_m')' - \frac{1}{4} m^2 \hat{\psi}_m = 0$$

i.e. if

$$\hat{\psi}_m = \hat{\psi}_m(1) X^{m/2} . \quad (7)$$

The integral of the mixed terms in $\hat{\psi}_m \hat{\psi}_{m+2}$ is stationary with respect to variations of the function $\hat{\psi}_{m+2}(X)$ if

$$X(X\hat{\psi}_m')' + \frac{1}{4} m(m+2) \hat{\psi}_m - X(m+1) \hat{\psi}_m' = 0 .$$

Remarkably, this is also satisfied by the choice (7), including the variation with respect to $\hat{\psi}_{m+2}(1)$; the other solution is

$$\hat{\psi}_m = \hat{\psi}_m(1) X^{(m/2)+1} . \quad (8)$$

An important result then follows: a minimizing choice for the functions $\hat{\psi}_m(X)$ is given by

$$\hat{\psi}_2(X) = \hat{\psi}_2(1) X; \quad \hat{\psi}_4(X) = \hat{\psi}_6(X) = \dots = 0 .$$

For (6) is then trivially stationary with respect to variations of $\hat{\psi}_6, \hat{\psi}_8, \dots$; furthermore, from the general results above, it is also stationary with respect to variations of $\hat{\psi}_2$ (the terms in $\hat{\psi}_2^2$ being the only nontrivial ones) and with respect to variations of $\hat{\psi}_4$ (the terms in $\hat{\psi}_2 \hat{\psi}_4$ being the only nontrivial ones). The stationary value of the expression (6) is $\pi(a^2 + b^2) \hat{\psi}_2(1)^2 / ab$, so that the minimum value of δW is $\pi \hat{\psi}_2(1)^2$. The minimum being positive, it follows that the plasma is stable to this mode: this result is exact.

A similar technique suffices for the next mode of the same symmetry-type: another minimizing choice for the $\hat{\psi}_m(X)$ is

$$\hat{\psi}_2(X) = c_2(X^2 - X); \quad \hat{\psi}_4(X) = \hat{\psi}_4(1)X^2; \quad \hat{\psi}_6(X) = \dots = 0.$$

Stationarity with respect to variations of $\hat{\psi}_8 \dots$ is trivial; stationarity with respect to variations of $\hat{\psi}_6$ (terms in $\hat{\psi}_4 \hat{\psi}_6$ being nontrivial) and with respect to variations of $\hat{\psi}_4$ (terms in $\hat{\psi}_4^2$ and $\hat{\psi}_2 \hat{\psi}_4$ being nontrivial) is assured by the general results above. Stationarity with respect to variations of $\hat{\psi}_2(X)$ and, of the whole expression for δW , with respect to variations of $\psi_2(1)$, require

$$2(a^2 + b^2)[-X(X\hat{\psi}_2')' + \hat{\psi}_2] + (b^2 - a^2)[-X(X\hat{\psi}_4')' - 2\hat{\psi}_4 - 3X\hat{\psi}_4'] = 0,$$

$$(a^2 + b^2)\hat{\psi}_2'(1)/ab + (b^2 - a^2)[\hat{\psi}_4'(1) + 2\hat{\psi}_4(1)]/2ab = 0.$$

These are both satisfied by our functions $\hat{\psi}_2(X)$ and $\hat{\psi}_4(X)$ if

$$(a^2 + b^2)c_2 + 2(b^2 - a^2)\hat{\psi}_4(1) = 0.$$

Additional terms $\hat{\psi}_2 = \hat{\psi}_2(1)X$ lead to no $\hat{\psi}_2(1)\hat{\psi}_4(1)$ terms in (6). For this mode the stationary value of (6) is found to be $\pi\hat{\psi}_4(1)^2[a^2 + b^2 + 4a^2b^2/(a^2 + b^2)]/ab$, so that $\delta W_{\min} = 2\pi\hat{\psi}_4(1)^2(a^2 + b^2 + ab)/(a^2 + b^2)$. We conclude that this mode is also stable.

Consider next the modes which are symmetric about the x-axis, but antisymmetric about the y-axis. These have

$$\psi_1(X) = \sum_{\text{odd } m=1}^{\infty} \hat{\psi}_m(X) \cos m\theta.$$

Substituting into Eq. (5) and carrying out the θ -integrations, we have

$$\begin{aligned} \int_P (\nabla \psi_1)^2 dA &= \frac{\pi}{ab} \sum_{\text{odd } m=1}^{\infty} \int_0^1 \frac{dX}{X} [\text{terms as in Eq. (6)}] \\ &+ \frac{\pi(b^2 - a^2)}{2ab} \int_0^1 \frac{dX}{X} (X^2 \hat{\psi}_1'^2 + X \hat{\psi}_1' \hat{\psi}_1 + \frac{1}{4} \hat{\psi}_1^2). \end{aligned} \quad (9)$$

The analysis which follows Eq. (6) and runs through Eq. (8) is also applicable here, for variations of the terms inside the summation. The extra term in Eq. (9) is stationary if $\hat{\psi}_1(X) = \hat{\psi}_1(1)X^{1/2}$. It follows, exactly as before, that a minimizing choice for the functions $\hat{\psi}_m(X)$ is given by:

$$\hat{\psi}_1(X) = \hat{\psi}_1(1)X^{1/2}; \quad \hat{\psi}_3(X) = \dots = 0.$$

The stationary value of (9) is $\pi\hat{\psi}_1(1)^2 b/a$, so that $\delta W_{\min} = \pi\hat{\psi}_1(1)^2 (b-a)/2b$. Thus, for our vertically elongated ellipses ($b > a$) this mode (which corresponds to a horizontal displacement) is also stable.

For the next mode of this symmetry-type, it follows exactly as before that a minimizing choice for the $\hat{\psi}_m(X)$ is:

$$\hat{\psi}_1(X) = c_1(X^{3/2} - X^{1/2}); \quad \hat{\psi}_3(X) = \hat{\psi}_3(1)X^{3/2}; \quad \hat{\psi}_5(X) = \dots = 0.$$

Stationarity with respect to variations of $\hat{\psi}_7 \dots$ is trivial, and with respect to variations of $\hat{\psi}_3$ and $\hat{\psi}_5$ is assured by our general results. Stationarity with respect to variations of $\hat{\psi}_1$ requires:

$$(a^2 + 3b^2)[-X(X\hat{\psi}_1')] + \frac{1}{4}\hat{\psi}_1 + (b^2 - a^2)[-X(X\hat{\psi}_3')] - \frac{3}{4}\hat{\psi}_3 - 2X\hat{\psi}_3' = 0$$

which is satisfied by our functions $\hat{\psi}_1(X)$ and $\hat{\psi}_3(X)$ if :

$$(a^2 + 3b^2)c_1 + 3(b^2 - a^2)\hat{\psi}_3(1) = 0.$$

The stationary value of (9) is found to be $3\pi\hat{\psi}_3(1)^2b(3a^2 + b^2)/a(a^2 + 3b^2)$, so that $\delta W_{\min} = \pi\hat{\psi}_3(1)^2[3b - a + 8b^2a/(a^2 + 3b^2)]/2b$. We conclude that for vertically elongated ellipses ($b > a$) this mode is also stable.

Consider next the modes which are antisymmetric about the x-axis, but symmetric about the y-axis. These have

$$\psi_1(X) = \sum_{\text{odd } m=1}^{\infty} \hat{\psi}_m(X) \sin m\theta,$$

and correspond to a 90° rotation of the modes just considered. Accordingly, results for the lowest two modes of this symmetry-type can be obtained by interchanging a and b in the expressions for $\delta W_{\min}$ obtained in the previous two paragraphs. For the lowest mode, which has only $\hat{\psi}_1$, we have $\delta W_{\min} = \pi\hat{\psi}_1(1)^2(a - b)/2a$. Thus, for vertically elongated ellipses ($b > a$) this mode (which corresponds to a vertical displacement) is always unstable. For the next mode of this type, which has $\hat{\psi}_1$ and $\hat{\psi}_3$, we have $\delta W_{\min} = \pi\hat{\psi}_3(1)^2[3a - b + 8a^2b/(b^2 + 3a^2)]/2a$ which is negative, corresponding to instability if $b/a > 4.5$.

Consider finally the modes which are antisymmetric about both x and y-axes. These have

$$\hat{\psi}_1(X) = \sum_{\text{even } m=2}^{\infty} \hat{\psi}_m(X) \sin m\theta.$$

Substituting into (5) and carrying out the θ -integrations, we obtain an expression identical with that given in Eq. (6). Thus, the analysis of these modes is identical with that of the symmetric

modes (with $\cos 2\theta$, $\cos 4\theta$, etc.). There is a lowest mode, which has only $\hat{\psi}_2$ and corresponds roughly to a rotation of the ellipse, for which $\delta W_{\min} = \pi \hat{\psi}_2(1)^2$. The next mode, with $\hat{\psi}_2$ and $\hat{\psi}_4$, has $\delta W_{\min} = 2\pi \hat{\psi}_4(1)^2 (a^2 + b^2 + ab)/(a^2 + b^2)$. Both modes are always stable.

Summarizing our results: the only unstable modes found are vertical displacement ($m = 1$) whenever $b > a$, and triangulation ($m = 3$) when $b/a > 4.5$; all $m = 2, 4$ modes are stable. Comparing with Ref. 2: the symmetric $m=2$ instability found therein (for a confocal system within the plasma) must violate incompressibility; furthermore, modes antisymmetric about the y-axis are improperly treated in a confocal system since they involve double valuedness on the line between foci: our $m = 1, 3$ instabilities are, however, symmetric about the y-axis, and indeed our results for these are identical with those of Ref. 2.

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