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Informal Report

TRANSVERSE COHERENCE DUE TO A TIME VARYING DIPOLE DEFLECTION

M. Month
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Abstract

The effect on one-dimensional coherent transverse betatron motion of a time-dependent dipole perturbation is considered. The method used applies in cases where the tune is not a function of the betatron amplitude, but is a function of either momentum or the betatron amplitude in the other transverse dimension. An equation for the transverse position response of the beam, i.e., the dipole moment per unit length, is derived. The initial value problem is solved. For the case of rf knockout the steady-state solution is obtained as the limit $t \rightarrow \infty$. Some care must be exercised in taking this limit and this subject is gone into. Specifically, the origin of the y -plane integration contour is given. The effect of space charge and image forces on the beam response is derived. The implications of this influence with respect to the use of rf knockout for measuring the parameters of the transverse collective instability are discussed. Finally, the case of a short-pulse perturbation is considered.

1. Introduction

Time varying sinusoidal perturbations can be used to measure accelerator tunes.¹ In addition to the "central" betatron frequency, it is clear that such rf deflecting fields can provide information relating to the distribution of y -values within a circulating beam and we will deal with this subject in detail in a later section. If one includes in the analysis the forces arising from the interaction of the beam with itself as well as with the surrounding vacuum chamber wall, then it has been suggested that certain characteristics of these forces can be determined by using this rf knockout technique. Specifically, it has been pointed out that in this way one could measure the essential parameters of the transverse coherent instability of an intense beam.²

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1. See, for example, E.C. Raka, Brookhaven National Laboratory, Accelerator Dept. Report AGS DIV. 67-5 (1967).
 2. D. Möhl and A.M. Sessler, Proc. 8th Int. Conf. on High-Energy Accelerators, CERN, Geneva, 1971, p. 334.

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Here, we propose to give a detailed account of the coherent effect on a beam of particles of a time-dependent dipole perturbation. The case of rf knockout will be analyzed as an example. We will derive the beam response, as given in Ref. 2, and show clearly the proper integration contour in the ν -plane that is to be taken. First we will treat the case with no space charge or image forces. Then in a later section the influence of these forces on the beam response is included. It will be shown that a conceptually simple experiment can be performed to determine completely the density distribution of ν -values in the beam. The presence of self-forces and image forces introduces some difficulty in the measuring procedure and we will discuss these. Another example considered is the case of a short-pulse perturbation. For the case of no ν -spread we reproduce the results given in Ref. 3. We further obtain the beam response when a spread in ν -values is present.

2. Equation for the Beam Response

A. Motion of a particle:

We consider a system of variables describing the motion of a single particle, as follows. The position, x , and its time derivative, $\dot{x}$, are variables representing the transverse motion of a particle. We denote by ν the betatron wave number which is assumed to be independent of x and $\dot{x}$. The azimuth, θ , defines the actual geometric longitudinal position of the particle, i.e., has the range $0 \leq \theta < 2\pi$.

The variables θ , x , $\dot{x}$ and ν are time dependent, and we consider a system whose time evolution is defined by

$$\begin{aligned}\dot{x} &= \frac{\partial H}{\partial \dot{x}} \quad , \\ \dot{x} &= - \frac{\partial H}{\partial x} \quad , \\ \dot{\theta} &= \Omega \quad , \\ \dot{\nu} &= 0 \quad .\end{aligned}\tag{2.1}$$

3. K. Hubner, CERN Report ISR-TH/69-17 (1969).

Here, Ω is the average angular velocity of a particle with characteristic v ; the v -value of a given particle is time independent; while x and $\dot{x}$ are canonical variables relating to a Hamiltonian function

$$H \equiv H(x, \dot{x}, \theta, v, t) \quad .$$

In particular, we will be considering a Hamiltonian of the form

$$H = \frac{1}{2}\dot{x}^2 + \frac{1}{2}v^2\Omega^2x^2 - xg(\theta, t) \quad , \quad (2.2)$$

where $g(\theta, t)$ is the time varying perturbing dipole.

B. The density distribution of particles:

We introduce a density distribution function $\psi \equiv \psi(x, \dot{x}, \theta, v, t)$ describing a beam of circulating particles. ψ is the number of particles in a unit volume, V , of phase space defined by the coordinates x , $\dot{x}$, θ and v . Since we are dealing with a system where the number of particles in an infinitesimal volume, dV , remains unchanged as the system evolves in time, we have

$$\delta[\psi dV] = 0, \text{ after a time } \delta t \quad . \quad (2.3)$$

It is easy to show that for the system defined in Eq. (2.1), the volume element dV remains invariant. Thus we have that $\delta\psi = 0$, after a time δt . This is simply the Vlasov equation or the collisionless Boltzmann equation, which we write

$$\dot{\psi} = \frac{d\psi}{dt} = 0 = \frac{\partial\psi}{\partial t} + \dot{\theta} \frac{\partial\psi}{\partial\theta} + \dot{x} \frac{\partial\psi}{\partial x} + \dot{\dot{x}} \frac{\partial\psi}{\partial\dot{x}} + \dot{v} \frac{\partial\psi}{\partial v} \quad . \quad (2.4)$$

Since we are concerned with coherent phenomena, we are primarily interested in the position response of the beam as a function of θ and t , or the dipole moment of the beam per unit azimuth, which is given by

$$D(\theta, t) = \int x \psi \, dx \, d\dot{x} \, dv \quad . \quad (2.5)$$

The charge density per unit azimuth is written as

$$\lambda(\theta, t) = \int \psi \, dx \, d\dot{x} \, dv \quad . \quad (2.6)$$

Since the v -value appears explicitly in the particle equation of motion, we will need to work with the quantities $\lambda_v(\theta, t)$ and $D_v(\theta, t)$ which are

related to λ and D by

$$\lambda(\theta, t) = \int \lambda_v(\theta, t) dv, \quad (2.7)$$

and

$$D(\theta, t) = \int D_v(\theta, t) dv. \quad (2.8)$$

The procedure we use is essentially to find an equation for D_v and then obtain D by integrating over v .

Using the fact that the volume element $dx d\dot{x}$ is time invariant and $\dot{\psi} = 0$, we deduce that

$$\dot{D}_v(\theta, t) = \frac{d}{dt} D_v = \int \dot{x} \psi dx d\dot{x}, \quad (2.9)$$

where the total time derivative of a quantity such as λ or D , from our previous definitions, is just given by

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \theta}. \quad (2.10)$$

Differentiating again, we find that

$$\ddot{D}_v = \int \ddot{x} \psi dx d\dot{x}; \quad (2.11)$$

while the time derivatives of λ and λ_v vanish — that is,

$$\dot{\lambda} = \dot{\lambda}_v = 0. \quad (2.12)$$

From the Hamiltonian in Eq. (2.2), we have the particle equation of motion

$$\ddot{x} + v^2 \Omega^2 x = g(\theta, t). \quad (2.13)$$

Substituting this equation into Eq. (2.11) we obtain the equation for D_v ,

$$\ddot{D}_v + v^2 \Omega^2 D_v = \lambda_v g(\theta, t). \quad (2.14)$$

This is a complicated integro-differential equation for ψ . However, a great simplification can be made if we assume that λ_v is separable with respect to its dependence on v and further if we assume that the v -dependence is known. That is, we write

$$\lambda_v(\theta, t) = \lambda(\theta, t) N(v), \quad (2.15)$$

where $N(v)$ is the density of particles per unit v , normalized to unity:

$$\int N(v) dv = 1 \quad . \quad (2.16)$$

Thus, remembering that $\dot{\lambda} = 0$, we can obtain a differential equation for the quantity

$$S_v(\theta, t) = \frac{D_v(\theta, t)}{\lambda(\theta, t)} \quad , \quad (2.17)$$

which is

$$\ddot{S}_v + v^2 \Omega^2 S_v = N(v) g(\theta, t) \quad . \quad (2.18)$$

The solution of this equation can be integrated over v to arrive at

$$S(\theta, t) = \int S_v(\theta, t) dv = \frac{D(\theta, t)}{\lambda(\theta, t)} \quad . \quad (2.19)$$

The quantity $S(\theta, t)$ is just the observable dipole moment per unit charge or the position response of a beam normalized to unit charge. Note that in this formulation, both the question of what azimuthal form for λ to choose and the question of consistency of solution do not enter. Instead a solution for the ratio D/λ is obtained in terms of an assumed functional form for the density function per unit v -value, $N(v)$. The situation becomes much more complicated when including space charge and image forces. When treating these effects, we will here neglect these complications except to point out, as was done in Ref. 2, that the results are easily adapted to the two limiting cases of (1) equally spaced and equally populated bunches, and (2) of decoupled bunches (i.e., sufficiently large bunch to bunch v -spread).

3. Solution of the Initial Value Problem by Fourier Analysis

We write the function $S_v(\theta, t)$ as a Fourier series in θ , since $S_v(\theta, t)$ is periodic in θ with period 2π , and as a Fourier integral in the time - that is,

$$S_v(\theta, t) = \sum_{k=-\infty}^{\infty} \int_C d\omega S_v(\omega, k) e^{i(k\theta - \omega t)} \quad , \quad (3.1)$$

where C represents a contour in the complex ω -plane which is suitable for an initial value problem. We assume that in the finite region of the upper half ω -plane, $S_v(\omega, k)$ is analytic except for poles. An appropriate contour, C , can then be chosen to be a line parallel to the real ω -axis but above the poles of $S_v(\omega, k)$, as shown in Fig. 1.

Figure 1

For $\text{Im}(\omega)$ above the contour C , written $\text{Im}(\omega) > C$, we can then invert the Fourier representation to obtain

$$S_v(\omega, k) = \frac{1}{(2\pi)^2} \int_0^\infty dt \int_0^{2\pi} d\theta S_v(\theta, t) e^{-i(k\theta - \omega t)} \quad (3.2)$$

Note the time integration is from 0 to ∞ , thus requiring a knowledge of the physical quantity $S_v(\theta, t)$ only for times greater than zero, assumed to be the starting point for the time evolution of the system. In performing the inversion, we have only assumed that $S_v(\omega, k)$ goes to zero weakly at ∞ in the upper half ω -plane. Having $S_v(\omega, k)$ for $\text{Im}(\omega) > C$, we can obtain it for $\text{Im}(\omega) \leq C$ by analytic continuation.

We now expand $g(\theta, t)$ in the same way as for $S_v(\theta, t)$, assuming that the contour, C , lies above the poles of its Fourier transform $g(\omega, k)$. We have,

$$g(\theta, t) = \sum_{k=-\infty}^{\infty} \int_C d\omega g(\omega, k) e^{i(k\theta - \omega t)} \quad (3.3)$$

and its inverse

$$g(\omega, k) = \frac{1}{(2\pi)^2} \int_0^\infty dt \int_0^{2\pi} d\theta g(\theta, t) e^{-i(k\theta - \omega t)} \quad (3.4)$$

Substituting Eqs. (3.1) and (3.3) into (2.18), we can deduce

$$S_v(\omega, k) = g(\omega, k) \frac{N(v)}{[v^2 \Omega^2 - (k\Omega - \omega)^2]} \quad (3.5)$$

as a particular solution for Eq. (2.18). Thus, if we write

$$I_v(\omega, k) = \frac{N(v)}{[v^2 \Omega^2 - (k\Omega - \omega)^2]} \quad (3.6)$$

the particular solution for $S_v(\theta, t)$ is

$$S_v(\theta, t) = \sum_k \int_C d\omega g(\omega, k) I_v(\omega, k) e^{i(k\theta - \omega t)} \quad (3.7)$$

Assuming the integrand with respect to ω is analytic in the finite region of the lower half plane (i.e., below the contour, C) except for poles, we can close the contour, and noting that $e^{-i\omega t} \rightarrow 0$ in the lower half plane as $\omega \rightarrow \infty$, we obtain

$$S_v(\theta, t) = - 2\pi i \sum_k \sum_{\omega_r} \text{Res}[g(\omega, k) I_v(\omega, k)]_{\omega=\omega_r} e^{i(k\theta - \omega_r t)} \quad (3.8)$$

Since g and I_v have no poles above the contour C , we can see that S_v has zero initial value, i.e.,

$$S_v(\theta, 0) = 0 \quad (3.9)$$

We can also show that

$$\dot{S}_v(\theta, 0) = 0 \quad (3.10)$$

Thus, the initial value aspect to the problem can be considered solely in the context of the homogeneous equation

$$\ddot{S}_v + v^2 \Omega^2 S_v = 0$$

The solution for $S = \int S_v dv$ can be written

$$S(\theta, t) = \int f_v^+(\theta - \Omega t) e^{i\Omega t} dv + \int f_v^-(\theta - \Omega t) e^{i\Omega t} dv,$$

which for reasonable, physical distributions of v will vanish as $t \rightarrow \infty$. We can therefore neglect the transient homogeneous solution and consider the particular solution given in Eqs. (3.8, 9 and 10).

In order to obtain the observable $S(\theta, t)$, we must integrate $S_v(\theta, t)$ over v . One is tempted to interchange the order of integration between ω and v in Eq. (3.7). If we do this, we have

$$S(\theta, t) = \sum_k \int_C d\omega g(\omega, k) I(\omega, k) e^{i(k\theta - \omega t)} \quad (3.11)$$

where

$$I(\omega, k) = \int \frac{N(v)}{[v^2 \Omega^2 - (k\Omega - \omega)^2]} dv \quad (3.12)$$

Now, $I_v(\omega, k)$ is a function of ω with 2 poles, whereas $I(\omega, k)$ may be quite different. For example, if $N(v)$ is Gaussian, then $I(\omega, k)$ is analytic in the entire finite region of the complex plane, but has an essential singularity at infinity. What this means is that the procedure giving Eq. (3.8) would be invalid with I_v replaced by I . In order to avoid this difficulty and retain the residue treatment, we first perform the ω integration to obtain $S_v(\theta, t)$. Integration over v will be done later.

4. RF Knockout - A Sinusoidal Perturbation

To derive an expression for the beam response to an rf deflecting field, we use the form of the solution given in Eq. (3.8). In this case the perturbing field is

$$g(\theta, t) = G \delta(\theta) \sin(\omega_{rf} t + \varphi) , \quad (4.1)$$

where $\delta(\theta)$ is the Dirac delta function, indicating that the perturbation exists at a single azimuthal location, $\theta = 0$; ω_{rf} is the perturbing frequency; φ is a phase factor; and G is the strength of the deflecting field, given by

$$G = \theta_B \Omega^2 R . \quad (4.2)$$

Here, R is the radius of the accelerator, and θ_B is the peak deflection in radians.

The Fourier transform of $g(\theta, t)$ is

$$g(\omega, k) = \frac{G}{8\pi^2} \left[\frac{e^{i\varphi}}{\omega + \omega_{rf}} - \frac{e^{-i\varphi}}{\omega - \omega_{rf}} \right] . \quad (4.3)$$

The other term in the residue sum in Eq. (3.8), $I_\nu(\omega, k)$, has two poles and can be written

$$I_\nu(\omega, k) = - \frac{N(\nu)}{(\omega - \omega_+)(\omega - \omega_-)} , \quad (4.4)$$

where,

$$\omega_\pm = \Omega(k \mp \nu) . \quad (4.5)$$

In the case of no ν -spread, simply put $N(\nu) \equiv 1$ and $S_\nu \equiv S$. We can then easily find an expression for $S(\theta, t)$ in terms of ω_{rf} , and further show that in the limit $\omega_{rf} \rightarrow \omega_\pm$, we obtain

$$S \rightarrow t e^{-i\omega_{rf} t} . \quad (4.6)$$

That is, the beam oscillates collectively with frequency ω_{rf} and the amplitude of oscillation grows linearly with time.

When there is v -spread in the beam, we get a form of what has become known as Landau damping.⁴ The unlimited growth is suppressed and the collective oscillation approaches a steady-state condition of finite amplitude.

In order to proceed with the calculation, we require the analyticity properties of the function $I(\omega, k)$ defined in Eq. (3.12). For simplicity in notation, we drop the manifest dependence on k . We then have

$$I(\omega) = \frac{1}{\Omega^2} \int_{-\infty}^{\infty} \frac{N(v)}{v^2 - a^2} dv, \quad (4.7)$$

where

$$a = \frac{\omega}{\Omega} - k. \quad (4.8)$$

Now, we recall that our representation is valid for $\text{Im}(\omega) > 0$. Thus we can trivially continue the function to the real ω -axis [$\text{Im}(\omega) \rightarrow 0$], where we run into a discontinuity. What this means is simply that for $\text{Im}(\omega) < 0$ the representation (4.7) is not the proper function [i.e., it is not the analytic continuation of the function defined for $\text{Im}(\omega) > 0$]. The proper analytic function can be defined as follows:

$$I(\omega) = \frac{1}{\Omega^2} \int_T \frac{N(v)}{v^2 - a^2} dv, \quad (4.9)$$

where T is a contour in the complex v -plane, as given in Fig. 2. We have neglected the dependence of Ω on v . If $\text{Im}(a) > 0$, the contour is simply the real v -axis. As $\text{Im}(a) \rightarrow 0$ and for $\text{Im}(a) < 0$, the contour is deformed and contributions from the poles at $v = \pm a$ must be added to the integral in Eq. (4.7). Thus, as $\text{Im}(\omega) \rightarrow 0$, we have

$$I(\omega) = \frac{1}{\Omega^2} \text{P.V.} \int_{-\infty}^{\infty} \frac{N(v)}{v^2 - a^2} dv + \frac{i\pi}{2a\Omega^2} [N(a) + N(-a)], \quad (4.10)$$

where P.V. indicates that the principal value is to be taken in crossing the poles at $v = \pm a$.

4. J.D. Jackson, J. Nucl. Energy, Part C; Plasma Phys. 1, 171 (1960);
L.J. Laslett, V.K. Neil, and A.M. Sessler, Rev. Sci. Instr. 36, 436 (1965).

ν - PLANE

Figure 2

Let us consider the two pole terms in $g(\omega, k)$ separately. For the pole at $\omega = \omega_{rf}$, we obtain, using Eq. (3.8),

$$S(\theta, t) [\omega = \omega_{rf} \text{ pole}] = \frac{iG}{4\pi} e^{-i\varphi} \sum_k e^{ik\theta} \left\{ I(\omega_{rf}, k) e^{-i\omega_{rf}t} + \right. \\ \left. - \int \frac{N(v) e^{-i\omega_+ t}}{2v\Omega^2 (v + a + i\epsilon)} dv + \right. \\ \left. - \int \frac{N(v) e^{-i\omega_- t}}{2v\Omega^2 (v - a - i\epsilon)} dv \right\} , \quad (4.11)$$

where $\omega_{\pm} = \Omega(k \mp v)$,
 $a = \omega_{rf}/\Omega - k$,

and the $i\epsilon$ factor determines the contour of the integral in the v -plane.

Actually the expression in Eq. (4.11) is continuous in the limit $\epsilon \rightarrow 0$. However, this choice (ω_{rf} having a small positive imaginary part) has the advantage of simplicity in form in that as $t \rightarrow \infty$, the second and third terms vanish. To show this consider the second term, which is proportional to

$$\int \frac{N(v) e^{i\omega_+ t}}{v(v + a + i\epsilon)} dv .$$

If the coefficient of $e^{i\omega_+ t}$ has no singularities along the real axis, then the integral vanishes in the limit $t \rightarrow \infty$. So let us consider only the term which has the nearby pole at $v = -a - i\epsilon$. This can be written

$$\frac{N(a)}{a} \int \frac{e^{i\omega_+ t}}{v + a + i\epsilon} dv .$$

If we write $x = (v + a)t$, this term is then proportional to

$$e^{-iat} \int_{-\infty}^{\infty} \frac{e^{ix}}{x + i\epsilon} dx .$$

A proper integration of this integral in the x -plane shows that its value is zero. In a similar way, the third term in Eq. (4.11) can be shown to vanish as $t \rightarrow \infty$.

Thus, we have derived the basic response of the beam to rf knockout. This is the result used in Ref. 2. It is simply that the response of the beam

to a knockout term

$$G e^{-i\omega_{rf}t} \delta(\theta)$$

is simply

$$\frac{1}{2\pi} \sum_k I(\omega_{rf}, k) e^{i(k\theta - \omega_{rf}t)}$$

where the response function I is defined in Eq. (4.10).

Using this result, we can write Eq. (4.11) as

$$S(\theta, t) [\omega = \omega_{rf} \text{ pole}] = \frac{iG}{4\pi} \sum_k e^{i(k\theta - \omega_{rf}t - \varphi)} I(\omega_{rf}, k) \quad (4.12)$$

Or, using Eq. (4.10), we arrive at

$$S(\theta, t) [\omega = \omega_{rf} \text{ pole}] = \frac{iG}{4\pi\Omega^2} \sum_k e^{i(k\theta - \omega_{rf}t - \varphi)} \left\{ \text{P.V.} \int \frac{N(v)}{v^2 - a^2} dv + \frac{i\pi}{2a} [N(a) + N(-a)] \right\} \quad (4.13)$$

To obtain the contribution of the pole at $\omega = -\omega_{rf}$, we must change the sign of the entire term, put $\omega_{rf} \rightarrow -\omega_{rf}$, and put $\varphi \rightarrow -\varphi$. If we further put $k \rightarrow -k$ in the sum, then $a \rightarrow -a$. Thus, there results

$$S(\theta, t) [\omega = -\omega_{rf} \text{ pole}] = -\frac{iG}{4\pi\Omega^2} \sum_k e^{-i(k\theta - \omega_{rf}t - \varphi)} \left\{ \text{P.V.} \int \frac{N(v)}{v^2 - a^2} dv - \frac{i\pi}{2a} [N(a) + N(-a)] \right\} \quad (4.14)$$

The beam response is just the sum of these two terms. Adding, we thus obtain

$$S(\theta, t) = \frac{G}{2\Omega^2} \sum_k \left\{ \frac{1}{\pi} \text{P.V.} \int \frac{N(v)}{v^2 - a^2} dv \sin(\omega_{rf}t + \varphi - k\theta) - \frac{[N(a) + N(-a)]}{2a} \cos(\omega_{rf}t + \varphi - k\theta) \right\} \quad (4.15)$$

For a sufficiently small v -spread, only one k -mode contributes provided we are not too close to the half-integer in v . Let the v -values lie between the integers k_- and $k_+ = k_- + 1$. Then for

$$k_- < v < k_- + \frac{1}{2} \quad ,$$

we have

$$S(\theta, t) \approx \frac{\theta_B R}{4a_-} \left\{ \frac{1}{\pi} \text{P.V.} \int \frac{N(\nu)}{\nu - a_-} \sin [\omega_{rf} t + \varphi + k_- \theta] - N(a_-) \cos (\omega_{rf} t + \varphi + k_- \theta) \right\}, \quad (4.16)$$

where $a_- = \omega_{rf}/\Omega + k_-$, and we have replaced G by $\theta_B \Omega^2 R$. If, however, $k_+ - \frac{1}{2} < \nu < k_+$, then

$$S(\theta, t) \approx - \frac{\theta_B R}{4a_+} \left\{ \frac{1}{\pi} \text{P.V.} \int \frac{N(\nu)}{\nu - a_+} \sin [\omega_{rf} t + \varphi - k_+ \theta] + N(a_+) \cos (\omega_{rf} t + \varphi - k_+ \theta) \right\}, \quad (4.17)$$

where $a_+ = k_+ - \omega_{rf}/\Omega$. Thus, if one were to apply such an rf perturbation and measure the amplitude of beam response as well as the phase difference between the applied field and the response at some azimuth as a function of ω_{rf} , then one obtains the density function $N(\nu)$ as a function of ν . The measurement of the principal value integral can be used as a consistency check by comparing this with the appropriate integral over the measured $N(\nu)$.

5. RF Knockout Including Self-Forces and Image Forces

If we include the forces arising from the effect of the beam on itself, Eq. (2.18) must be altered to give⁵

$$\ddot{S}_\nu + \nu^2 \Omega^2 S_\nu = N(\nu) \left\{ \frac{\lambda e}{m} U S + \frac{\lambda e}{m} \frac{W}{\sqrt{\pi}} \int_0^t \frac{S(\theta, t')}{\sqrt{t-t'}} dt' \right\} + N(\nu) g(\theta, t). \quad (5.1)$$

Here, we have assumed a coasting beam, i.e., constant linear charge density, λ . The constants U and W are related to the characteristics of the beam and its environment. If we take account of a finite wall resistivity, we have for a circular geometry,

5. E.D. Courant and A.M. Sessler, Rev. Sci. Instr. 37, 1579 (1966).

$$U = - \frac{2}{\gamma^2} \left(\frac{1}{a^2} - \frac{1}{b^2} \right) , \quad (5.2)$$

$$W = \frac{4c\beta^2}{b^3} \frac{1}{\sqrt{4\pi\sigma}} ; \quad (5.3)$$

where

a = beam radius,

b = chamber radius,

γ = energy in units of proton rest mass,

β = particle velocity in units of c ,

σ = chamber conductivity in units 1/time.

If we solve Eq. (5.1) with the methods previously used, we have the solution to the initial value problem, given for $\text{Im}(\omega) > 0$ by

$$\begin{aligned} S_v(\omega, k) &= \frac{N(v)}{\sqrt{2}\Omega^2 - (\omega - k\Omega)^2} \frac{\lambda e}{m} \left[U + W \left(\frac{1}{\omega} \right)^{\frac{1}{2}} \right] S(\omega, k) = \\ &= \frac{1}{\sqrt{2}\Omega^2 - (\omega - k\Omega)^2} \left[\Phi_v(\omega, k) + N(v) g(\omega, k) \right] ; \end{aligned} \quad (5.4)$$

where $(1/\omega)^{\frac{1}{2}}$ is defined so that its real part is positive, and the initial conditions are included in the quantity $\Phi_v(\omega, k)$, which is

$$\begin{aligned} \Phi_v(\omega, k) &= \frac{1}{(2\pi)^2} \left\{ i(\Omega k - \omega) \int_0^{2\pi} d\theta S_v(\theta, 0) e^{-ik\theta} + \right. \\ &\quad \left. + \int_0^{2\pi} d\theta \dot{S}_v(\theta, 0) e^{-ik\theta} \right\} . \end{aligned} \quad (5.5a)$$

For S_v initially independent of θ , only the $k = 0$ term is non-zero,

$$\Phi_v(\omega, 0) = \frac{1}{2\pi} \{ -i\omega S_v(0, 0) + \dot{S}_v(0, 0) \} . \quad (5.5b)$$

Thus, we can write for $S(\theta, t)$,

$$\begin{aligned} S(\theta, t) &= \sum_k \int_C d\omega e^{i(k\theta - \omega t)} \frac{\Phi(\omega, k)}{H(\omega, k)} + \\ &\quad + \int dv N(v) \sum_k \int_C d\omega \frac{I_v(\omega, k) g(\omega, k)}{H(\omega, k)} e^{i(k\theta - \omega t)} , \end{aligned} \quad (5.6)$$

where

$$H(\omega, k) = 1 - \left[U + W \left(\frac{1}{\omega} \right)^{\frac{1}{2}} \right] \frac{\lambda e}{m} I(\omega, k) \quad , \quad (5.7)$$

and

$$\Phi(\omega, k) = \int d\nu \frac{\tilde{\Phi}_\nu(\omega, k)}{\nu^2 \Omega^2 - (\omega - k\Omega)^2} \quad . \quad (5.8)$$

The function I is defined for $\text{Im}(\omega) > 0$. To obtain values for $\text{Im}(\omega) < 0$, we use analytic continuation. $I(\omega, k)$ is thus the function previously defined in Eq. (4.9).

If we define the function I in this way, then we can show in a similar way to that used in Section 4 [see discussion following Eq. (4.11)] that in the large t limit,

$$\begin{aligned} S(\theta, t) \approx \sum_k \tau(\omega_c, k) e^{i(k\theta - \omega_c t)} + \frac{iG}{4\pi} \sum_k \frac{I(\omega_{rf}, k)}{H(\omega_{rf}, k)} e^{i[k\theta - \omega_{rf} t - \phi]} + \\ + 2\pi i \sum_k I(\omega_c, k) g(\omega_c, k) \text{Res} \left[\frac{1}{H(\omega, k)} \right]_{\omega=\omega_c} e^{i(k\theta - \omega_c t)} + \text{c.c.} \quad , \quad (5.9a) \end{aligned}$$

where c.c. refers to the complex conjugate, and ω_c is a function of k and satisfies the equation,

$$H(\omega_c, k) = 0 \quad . \quad (5.9b)$$

Here, τ is dependent on the initial conditions

$$\tau(\omega_c, k) = 2\pi i \tilde{\Phi}(\omega_c, k) \text{Res} \left[\frac{1}{H(\omega, k)} \right]_{\omega=\omega_c} \quad , \quad (5.10)$$

and we have assumed $\tilde{\Phi}(\omega, k)$ has no poles in the complex plane. The complex conjugate term arises from the fact that if a solution to $H(\omega, k) = 0$ exists for $\omega = \omega_c$, then it can easily be shown that

$$H(-\omega_c^*, -k) = 0 \quad .$$

Under unstable conditions, i.e., $\text{Im}(\omega_c) > 0$, the first term represents spontaneous growth of the beam, while the last term represents beam growth stimulated by the rf knockout procedure. If $\text{Im}(\omega_c) < 0$, the beam is stable and the first and third terms will damp in the long time limit. Thus, we are left with the second term, the beam response to rf knockout. This is just the

result in Ref. 2 that the beam response (for a particular mode k) is determined by the function

$$F(\omega_{rf}, k) = \frac{I(\omega_{rf}, k)}{H(\omega_{rf}, k)} \quad (5.11)$$

If we write

$$B = - \left[U + \left(\frac{1}{\omega_{rf}} \right)^{\frac{1}{2}} W \right] \frac{\lambda e}{m} \quad (5.12)$$

that is,

$$H(\omega_{rf}, k) = 1 + B I(\omega_{rf}, k) \quad ; \quad (5.13)$$

then F becomes

$$F(\omega_{rf}, k) = \left(\frac{1}{1 + B} \right)^{-1} \quad (5.14)$$

A knowledge of the functional form of $H(\omega, k)$ has thus allowed us to "observe" the parameters of the transverse collective instability, $\text{Re}(B)$ and $\text{Im}(B)$.² By measuring $(1/F)$ as a function of ω_{rf} , we can therefore obtain the parameters $\text{Re}(B)$, $\text{Im}(B)$, as well as the function $N(v)$. It is important not to make any a priori assumptions about the form of the function $N(v)$. For, in fact, the form of $N(v)$ may be current dependent, becoming skewed, for example, just as B is also increasing. Consider then the following sequence of steps:

1) Measure $1/F$ as a function of ω_{rf} , admitting a particular mode, k . Thus we have as a function of ω_{rf} , the quantities

$$\text{Re}(1/F) = \frac{I_R}{I_R^2 + I_I^2} + B_R \quad , \quad (5.15)$$

and

$$\text{Im}(1/F) = - \frac{I_I}{I_R^2 + I_I^2} + B_I \quad , \quad (5.16)$$

where the subscripts R, I refer to real and imaginary parts respectively.

2) Since outside the range of v -values, $I_I \approx 0$, the limiting value of $\text{Im}(1/F)$ gives the quantity B_I . Actually B_I is a slowly varying function of ω_{rf} ; but since this functional dependence has been chosen, this can be accounted for.

3) Outside the range, an asymptotic expansion of I_R can be made. For example, define the variable x by

$$x = k - v_0 - \frac{\omega_{rf}}{\Omega} \quad , \quad (5.17)$$

where v_0 is the center of the range of $N(v)$, the distribution in v . Then, the real function I can be expanded in inverse powers of x ,

$$I = \frac{1}{2v_0\Omega^2} \left[\frac{1}{x} - \frac{\alpha}{x^2} + \frac{\sigma^2 + \alpha^2}{x^3} - \frac{\gamma}{x^4} + \dots \right] \quad , \quad (5.18)$$

where α , σ^2 , and γ are measures of the mean, the dispersion, and the skewing of the v -distribution, given by

$$\alpha = \int (v - v_0) N(v) dv \quad , \quad (5.19)$$

$$\sigma^2 = \int (v - v_0 - \alpha)^2 N(v) dv \quad , \quad (5.20)$$

and

$$\gamma = \int (v - v_0)^3 N(v) dv \quad . \quad (5.21)$$

We can thus obtain these parameters together with B_R .

4) B_R and B_I have been determined from measurements of knockout with frequency ω_{rf} corresponding to v -values outside the range of $N(v)$. Knowledge of these quantities can then be used to determine the function $N(v)$ directly. From the measured $1/F$ values, compute the quantity

$$\left(\frac{1}{F} - B \right)^{-1} \quad ,$$

which, since

$$\text{Im} \left\{ \left(\frac{1}{F} - B \right)^{-1} \right\} = \text{Im}(I) = \frac{\pi}{2v_0\Omega^2} N \left(k - \frac{\omega_{rf}}{\Omega} \right) \quad , \quad (5.22)$$

gives us the density function $N(v)$.

5) From the function N we can compute the principal value integral and compare it with the measured $\text{Re}(I)$. We can also compute the mean, dispersion, and skewing parameter for the distribution and compare it with values obtained by the asymptotic expansion. We thus have some consistency tests for the measurements.

As pointed out in Ref. 2, the measurements must be performed in a stable intensity range where the first and third terms of Eq. (5.9) are damped ($\text{Im}(\omega_c) < 0$). This is generally a necessary requirement, thus leading to the need to extrapolate from low current to high current results. However, if the conditions are such that the "spontaneous growth" term is not significant, one might be able to approach sufficiently close to the region of instability to allow direct measurement of the complex frequency ω_c near the instability threshold. This can be done by looking at the third term of Eq. (5.9), that corresponding to "stimulated growth" arising from the application of rf knockout. Of course it should be emphasized that the conditions for this measurement to be possible are severe. It is required that both spontaneous growth and stimulated growth through forces other than through the applied rf knockout are small compared to the applied knockout term and further that all growing modes provide a beam life long compared to the "lifetime" of all transients.

6. Excitation of a Coherent Betatron Oscillation by a Deflecting Pulse of Short Duration

The problem here is just slightly different from the rf knockout problem as far as formalism is concerned. Since here we have no steady state solution, the excitation of coherency, as long as there is a nonzero spread in v -values, will decay. We will derive an expression for the time dependence of the coherent oscillation. We will assume that at $t = 0$, no coherent motion exists. Thus the relevant expression for the beam response, $S_v(\theta, t)$, is given by Eq. (3.7):

$$S_v(\theta, t) = \sum_k \int_C d\omega g(\omega, k) I_v(\omega, k) e^{i(k\theta - \omega t)} \quad .$$

If τ is the duration of the pulse, $g(\omega, k)$ will have a term of the form $e^{i\omega\tau}$. Thus with the restriction $t > \tau$, we can close the contour of integration in the lower half of the ω -plane, which thus gives us Eq. (3.8) for $S_v(\theta, t)$. However, because $g(\omega, k)$ is analytic in the finite region of the ω -plane, $g(\omega, k)$ does not contribute to the residue sum, and we obtain

$$S_v(\theta, t) = - 2\pi i \sum_k \sum_{\omega_r} \text{Res}[I_v(\omega, k)]_{\omega=\omega_r} g(\omega_r, k) e^{i(k\theta - \omega_r t)} \quad . \quad (6.1)$$

Since

$$I_{\nu}(\omega, k) = - \frac{N(\nu)}{(\omega - \omega_+) (\omega - \omega_-)} ,$$

$S_{\nu}(\theta, t)$ can be written

$$S_{\nu}(\theta, t) = - \frac{\pi i}{\Omega \nu} N(\nu) \sum_k e^{ik\theta} [g(\omega_+, k) e^{-i\omega_+ t} - g(\omega_-, k) e^{-i\omega_- t}] , \quad (6.2)$$

where we recall that

$$\omega_{\pm} = \Omega(k \mp \nu) .$$

Consider the perturbation

$$g(\theta, t) = G \delta(\theta) h(t) , \quad (6.3)$$

with

$$G = \theta_B R \Omega^2 ,$$

and

$$\begin{aligned} h(t) &= \sin \pi t / \tau , \quad 0 < t < \tau \\ h(t) &= 0 , \quad \text{otherwise} . \end{aligned} \quad (6.4)$$

In this case we have

$$g(\omega, k) = \frac{G\tau}{2\pi^3} \frac{e^{i\omega\tau/2} \cos \frac{\omega\tau}{2}}{1 - \left(\frac{\omega\tau}{\pi} \right)^2} , \quad (6.5)$$

where we have assumed that $\tau < T$, the period of revolution. If we put $k \rightarrow -k$ in the second sum of Eq. (5.2), $S_{\nu}(\theta, t)$ becomes

$$\begin{aligned} S_{\nu}(\theta, t) &= - \frac{\pi i}{\Omega \nu} N(\nu) \frac{G\tau}{2\pi^3} \sum_k \left\{ \frac{\cos \frac{\Omega\tau}{2} (k - \nu)}{1 - \left[\frac{\Omega\tau}{\pi} (k - \nu) \right]^2} \right\} \times \\ &\times \left\{ e^{i[k\theta - \Omega(t - \tau/2)(k - \nu)]} - e^{-i[k\theta + \Omega(t - \tau/2)(k - \nu)]} \right\} . \end{aligned} \quad (6.6)$$

Writing

$$\Omega = \frac{2\pi}{T} , \quad (6.7)$$

and

$$H(\tau/T) = \frac{2}{\pi} \frac{\tau}{T} \frac{\cos \frac{\pi\tau}{T} (k - \nu)}{1 - 4(\tau/T)^2 (k - \nu)^2} , \quad (6.8)$$

we have,

$$S_{\nu}(\theta, t) = \theta_B \frac{R}{\nu} N(\nu) \sum_k H(\tau/T) \sin \{k\theta + (\nu - k) \Omega(t - \tau/2)\} , \quad (6.9)$$

which is essentially Hubner's result³ for the half-sinusoidal pulse deflection. Because of the form of H , only a couple of k -modes contribute significantly to the sum. Thus with the appropriate use of electronic filters, this response function, S_{ν} , can be effectively used as a means of measuring betatron tunes.⁶

To obtain an expression for the beam response at a position sensitive electrode, we integrate over ν to obtain

$$S(\theta, t) = \sum_k \int d\nu N(\nu) \theta_B \frac{R}{\nu} H(\tau/T) \sin \{k\theta + (\nu - k) \Omega(t - \tau/2)\} . \quad (6.10)$$

Thus as $t \rightarrow \infty$, $S \rightarrow 0$ for any reasonable function $N(\nu)$ and the decay rate of the coherence is a function which depends on the particular form of $N(\nu)$.

7. Conclusions

We have obtained an equation for the dipole moment per unit charge $S(\theta, t)$. We have also derived an explicit solution for the case of a time varying deflecting field assuming that the particle density function for betatron frequencies is known and separable from the general particle density function. A simplifying assumption throughout has been the decoupling of ν from the canonical variables x and $\dot{x}$. This led to the equations for the time evolution of the system given in (2.1). To remove this decoupling constraint would require a set of such equations in which ν is dependent on x and $\dot{x}$ and thus $\dot{\nu} \neq 0$.

Fourier analysis suitable for the solution of the initial value problem was employed and particular solutions to the problems of (1) rf knockout and (2) a short-pulse perturbation were obtained.

With regard to the rf knockout problem, we have treated both the case without the inclusion of self forces and image forces and the case including them. Some of the difficulties involved in measuring the parameters of the transverse collective instability have been discussed.

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6. S. Hansen, Proc. 8th Intern. Conf. on High Energy Accelerators, CERN, Geneva, 1971, p. 448