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DESIGN CONSIDERATIONS IN THE PRODUCTION
OF STRIPLINE FERRITE CIRCULATORS

Prepared by:
D. E. Kaufman
J. L. Taylor

AEC Distribution: July, 1970
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Prepared for the
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**Kansas City
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ABSTRACT

A theoretical relationship between material parameters (physical and geometrical) and stripline ferrite circulator functions has been developed.

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FOREWORD

Radar duplexers have taken various forms as requirements have evolved through the years. The ferrite circulator approach has only recently come into active consideration. Later, an investigation was initiated for the purpose of determining manufacturing processes for ferrite circulators and defining process controls; included was an effort to establish better correlation among several empirically developed parameters. This report treats the mathematical model developed in this endeavor.

Section I

THE FERROMAGNETIC PROPERTIES OF FERRITES

1.1 Preliminaries

An exact description of ferromagnetism on the microscopic or atomic level can only be obtained by the principles of quantum mechanics. Such an approach could be carried out, but the expenditure of time and energy would be disproportionate to the amount of physical insight gained from the analysis. Instead, the ferrite material properties will be studied on the macroscopic level by simple classical theory.

A second complication in any discussion of ferrite material properties is that no uniform system of magnetic units is employed in the literature, the m-k-s (practical) system and the Gaussian system appearing with nearly equal frequency. A complete discussion of the relationships between these two sets of units is available in Appendix D of Soohoo (1960). In the development of the essential parameters of ferrites in this section, the m-k-s system of units will be employed. However, since most ferrite manufacturers specify ferrite magnetic properties in Gaussian units, the latter system will be used in the design procedure for a circulator.

1.2 The Fundamental Equations of Magneto Dynamics

It can be shown [see for example, Soohoo [1960], Chapter 4] that the magnetic behavior of ferrites is due to unbalanced electron spins among atoms of the iron transition group (iron, nickel, cobalt) in the unit crystal structure of the ferrite. Associated with each of these unbalanced electron spins is an angular momentum vector,

$$\vec{I} = \vec{J} \hbar \quad , \quad (1-1)$$

where

$$|\vec{J}| = \sqrt{J(J+1)} \quad , \quad (1-2)$$

J being the angular momentum quantum number and where

$$\hbar = \frac{h}{2\pi} \quad (1-3)$$

with h as Planck's constant. Classically, the torque acting on each of those spinning electrons is

$$\vec{\tau} = \hbar \frac{d\vec{J}}{dt} \quad (1-4)$$

If the spinning electron is placed in a magnetic field $\vec{H}$, then,

$$\vec{\tau} = \vec{\mu} \times \vec{H}, \quad (1-5)$$

where $\vec{\mu}$ is the spin magnetic moment of the electron given by

$$\vec{\mu} = -g \frac{\mu_0 e \hbar}{2m} \vec{J} \quad (1-6)$$

In equation (1-6), g is the Landau g-factor (g = 2 for a free electron), μ_0 is the free space magnetic permeability, e is the positive electronic charge, and m is the electronic mass.

From equations (1-1) and (1-6) the ratio of spin magnetic moment to angular momentum may be calculated; this ratio γ_e is known as the gyromagnetic ratio,

$$\gamma_e = -\frac{|\vec{\mu}|}{|\vec{I}|} = -g \frac{\mu_0 e}{2m} \quad (1-7)$$

(The negative sign occurs since $\vec{I}$ and $\vec{\mu}$ are oppositely directed.) Numerically, for a free electron and approximately for most ferrite materials (See for example, Soohoo, [1960], pp 68-72),

$$\gamma_e = -2.21 \times 10^5 \text{ (rad/sec)/(amp-turns/meter)}. \quad (1-8)$$

By using the definition of the gyromagnetic ratio, and equating (1-4) and (1-5),

$$\frac{d\vec{\mu}}{dt} = \gamma_e \vec{\mu} \times \vec{H} \quad (1-9)$$

From this microscopic property of the individual electrons, a macroscopic equation may be obtained. Let $\vec{M} = N\vec{\mu}$ be the magnetic moment per unit volume, where N is the number of unbalanced electron spins per unit volume of ferrite material. Thus, the macroscopic magnetodynamic equation will be

$$\frac{d\vec{M}}{dt} = \gamma_e \vec{M} \times \vec{H}. \quad (1-10)$$

The physical interpretation of equation (1-10) is that the individual electron spin magnetic moments $\vec{\mu}$ precess continuously about $\vec{H}$ in the locus of a cone without change in magnitude (Soohoo [1960], pp 60-63). The precession or gyro-magnetic frequency of this process is (Lax and Button [1962], pp 145-147)

$$\omega_0 = -\gamma_e |\vec{H}| \text{ rad/sec.} \quad (1-11)$$

Such a process is rather unrealistic, since $\vec{\mu}$ would never align itself with the applied field to achieve complete magnetization of a ferromagnetic material. A more realistic procedure is to add a damping term to (1-10) so that

$$\frac{d\vec{M}}{dt} = \gamma_e [\vec{M} \times \vec{H} + \frac{\alpha}{|\vec{M}|} \vec{M} \times (\vec{M} \times \vec{H})] \quad (1-12)$$

where α is a dimensionless damping parameter.

If $\alpha < 0.1$, as is usually the case, equation (1-12) may be written:

$$\frac{d\vec{M}}{dt} = \gamma_e [\vec{M} \times \vec{H}] + \frac{\alpha}{|\vec{M}|} \vec{M} \times \frac{d\vec{M}}{dt} \quad (1-13)$$

The net physical effect of the damping term is that under an applied field all spin moments $\vec{\mu}$ precess about $\vec{H}$ in smaller and smaller cones, finally lining up with $\vec{H}$ as they should.

1.3 The Ferrite Permeability Tensor

In this section the behavior of the magnetization $\vec{M}$ is studied when superposed r-f and d-c fields are applied to the ferrite material. Three assumptions will be made in this analysis:

(1) the d-c magnetic fields will be directed along the z-axis, (2) the ferrite material is assumed to be in saturation so that the d-c magnetization along the z-axis is essentially M_0 , the saturated value, and (3) the r-f magnetic fields are very small compared to the d-c magnetic fields. Therefore $\vec{H}$ and $\vec{M}$ in the ferrite will take the form

$$\vec{H} = \vec{a}_z H_i + \vec{h} \text{ EXP } (j\omega t) \quad (1-14)$$

$$\vec{M} = \vec{a}_z M_0 + \vec{m} \text{ EXP } (j\omega t) \quad (1-15)$$

where H_i is the d-c magnetic intensity in the medium, $\vec{h}$ and $\vec{m}$ are the r-f field and magnetization respectively, ω is the angular frequency of the r-f field, and $\vec{a}_z$ is a unit vector along the z-axis.

The substitution of equations (1-14) and (1-15) into (1-13) and the neglect of all terms in $\vec{m}$ and $\vec{h}$ higher than first order gives,

$$j\omega \vec{m} = \gamma_e M_0 (\vec{a}_z \times \vec{h}) + (\omega_0 + j\omega\alpha) (\vec{a}_z \times \vec{m}) \quad (1-16)$$

where $\omega_0 = -\gamma_e H_i$ is the precessional frequency of an electron in the magnetic field H_i .

In terms of unit vectors $\vec{a}_x$, $\vec{a}_y$, $\vec{a}_z$ of the Cartesian coordinate system, the r-f field and magnetization may be written as,

$$\vec{m} = \vec{a}_x m_x + \vec{a}_y m_y + \vec{a}_z m_z \quad (1-17)$$

$$\vec{h} = \vec{a}_x h_x + \vec{a}_y h_y + \vec{a}_z h_z \quad (1-18)$$

If (1-17) and (1-18) are substituted in (1-16), then

$$j\omega m_x = -(\omega_0 + j\omega\alpha) m_y - \gamma_e M_0 h_y \quad (1-19)$$

$$j\omega m_y = \gamma_e M_0 h_x + (\omega_0 + j\omega\alpha) m_x \quad (1-20)$$

$$j\omega m_z = 0 \quad (1-21)$$

It is now a simple matter to solve for m_x and m_y in terms of h_x and h_y ,

$$m_x = (\chi h_x - j\kappa h_y) \mu_0 \quad (1-22)$$

$$m_y = (j\kappa h_x + \chi h_y) \mu_0 \quad (1-23)$$

with

$$\chi = \frac{(\omega_0 + j\omega\alpha) \omega_m}{(\omega_0 + j\omega\alpha)^2 - \omega^2} \quad (1-24)$$

$$\kappa = \frac{-\omega\omega_m}{(\omega_0 + j\omega\alpha)^2 - \omega^2} \quad (1-25)$$

$$\omega_m = -\gamma_e M_0 / \mu_0 \quad (1-26)$$

The usual approach is to put equations (1-22) and (1-23) in matrix form,

$$\begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix} = \mu_0 \begin{bmatrix} \chi & -j\kappa & 0 \\ j\kappa & \chi & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} \quad (1-27)$$

and to define a susceptibility tensor $\overset{\leftrightarrow}{\chi}$ as

$$\overset{\leftrightarrow}{\chi} = \begin{bmatrix} \chi & -j\kappa & 0 \\ j\kappa & \chi & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (1-28)$$

so that equation (1-27) becomes

$$\vec{m} = \overset{\leftrightarrow}{\chi} \vec{H} \mu_0 \quad (1-29)$$

In general, the elements χ and κ occurring in the susceptibility tensor are complex numbers. By rationalizing (1-24) and (1-25) these elements become

$$\chi = \chi' - j\chi'' \quad (1-30)$$

$$\kappa = \kappa' - j\kappa'' \quad (1-31)$$

where,

$$\chi' = \frac{\omega_m \omega_0 [\omega_0^2 + \omega^2 (\alpha^2 - 1)]}{[\omega_0^2 - \omega^2 (1 + \alpha^2)]^2 + 4\omega^2 \omega_0^2 \alpha^2} \quad (1-32)$$

$$\chi'' = \frac{\omega_m \omega \alpha [\omega_0^2 + \omega^2 (1 + \alpha^2)]}{[\omega_0^2 - \omega^2 (1 + \alpha^2)]^2 + 4\omega^2 \omega_0^2 \alpha^2} \quad (1-33)$$

$$\kappa' = \frac{-\omega_m \omega [\omega_0^2 - \omega^2 (1 + \alpha^2)]}{[\omega_0^2 - \omega^2 (1 + \alpha^2)]^2 + 4\omega^2 \omega_0^2 \alpha^2} \quad (1-34)$$

$$\kappa'' = \frac{-2\omega_0 \omega^2 \alpha \omega_m}{[\omega_0^2 - \omega^2 (1 + \alpha^2)]^2 + 4\omega^2 \omega_0^2 \alpha^2} \quad (1-35)$$

In the mks system the r-f magnetic flux density $\vec{b}$ can be written in terms of the Polder permeability tensor, $\overleftrightarrow{\mu}$, as

$$\vec{b} = \overleftrightarrow{\mu} \vec{h} = \vec{m} + \mu_0 \vec{h} \quad (1-36)$$

By the definition of $\vec{m}$, equation (1-29), $\overleftrightarrow{\mu}$ must then be

$$\overleftrightarrow{\mu} = \mu_0 (\overleftrightarrow{\chi} + \overleftrightarrow{I}) \quad (1-37)$$

where $\overleftrightarrow{I}$ is a 3 X 3 unit matrix. In other words, in a cartesian co-ordinate system,

$$\overleftrightarrow{\mu} = \mu_0 \begin{bmatrix} \mu & -j\kappa & 0 \\ j\kappa & \mu & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (1-38)$$

$$\mu = 1 + \chi \quad (1-39)$$

so that by equation (1-36),

$$\begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \mu_0 \begin{bmatrix} \mu & -j\kappa & 0 \\ j\kappa & \mu & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} \quad (1-40)$$

1-4. Magnetic Losses and Ferrite Resonance Line Width.

It can be shown that the presence of the imaginary parts of χ and κ , χ'' and κ'' , results in ferrite magnetic losses. (See, for example, chapters 5 and 7 [Soohoo, 1960]). For a fixed ω_0 or H_i , χ'' and κ'' may be shown by differentiation with respect to ω to attain a maximum value at

$$\omega = \omega_0 = -\gamma_e H_i. \quad (1-41)$$

Thus in order to avoid high magnetic losses, frequencies of operation near the gyromagnetic resonance frequency, ω_0 , should be avoided in the design of circulators.

Furthermore, an examination of χ'' and κ'' , as given by equations (1-33) and (1-35), shows that for $\alpha \ll 1$ (as is usually the case) and ω not in the vicinity of ω_0 ,

$$\chi'' \approx \frac{\omega_m \omega \alpha (\omega_0^2 + \omega^2)}{(\omega_0^2 - \omega^2)^2} \quad (1-42)$$

$$\kappa'' \approx \frac{-2\omega_0 \omega^2 \alpha \omega_m}{(\omega_0^2 - \omega^2)^2} \quad (1-43)$$

The two equations just given state that κ'' and χ'' , and hence the magnetic loss, vary directly with the damping parameter α , and with the saturation magnetization.

Two other physical phenomena associated with the damping parameter are the gyromagnetic relaxation time and the gyromagnetic resonance line width. On the assumption that α is independent of frequency, or nearly so, a gyromagnetic relaxation time τ_r is defined as

$$\tau_r = \frac{1}{\omega \alpha} \quad (1-44)$$

Typically, this relaxation time is of the order of several hundred pico seconds and represents physically an estimate of the minimum time required to magnetize the ferrite.

The gyromagnetic resonance line width ΔH is usually defined as the difference between the magnetic field values at the gyromagnetic resonance frequency, ω_0 , that reduce χ'' to one half its value at resonance. In other words,

$$\chi''(H_i \pm 1/2 \Delta H) = 1/2 \chi''(H_i) \quad \omega = \omega_0 \quad (1-45)$$

This relation is also illustrated in Figure 1.1.

Figure 1.1 Resonance Line Width

In order to calculate ΔH in terms of α , γ_e , and ω_0 , it will be assumed that $\alpha \ll 1$. By defining

$$\omega_0' = -\gamma_e [H_i \pm 1/2 \Delta H] \quad (1-46)$$

and then replacing ω and ω_0 in equation (1-33) by ω_0 and ω_0' respectively, equation (1-45) can be written as

$$\frac{\alpha \omega_0 \omega_m [\omega_0^2 + (\omega_0')^2]}{[(\omega_0')^2 - \omega_0^2] + 4(\omega_0')^2 \alpha^2 \omega_0^2} \approx \frac{1}{2} \frac{\omega_m}{2\omega_0 \alpha} \quad (1-47)$$

Solution of (1-47) gives,

$$\omega_0' = \omega_0 \sqrt{1 \pm 2\alpha} \approx \omega_0 (1 \pm \alpha) \quad (1-48)$$

or

$$\Delta H = - \frac{2\omega_0 \alpha}{\gamma_e} \quad (1-49)$$

by using equation (1-46). Most suppliers of ferrite materials measure the gyromagnetic resonance line width at some specified value of ω_0 ; the necessary value of the damping constant can be inferred from the above equation.

1.5 Dielectric Behavior of Ferrite Materials

A number of mechanisms contribute to the dielectric behavior of ferrite materials in the frequency spectrum from zero to the ultra-violet region. In the microwave region, which is the frequency range of interest in this report, the principal mechanisms are electronic polarization and atomic polarization. Electronic polarization is due to the displacement of electrons with respect to the atomic nuclei in the presence of an applied electric field. Atomic polarization results from the movement of cations and anions within the crystal under the influence of an applied field.

In general, the relative dielectric constant of a ferrite material is a complex number,

$$\epsilon_r = \epsilon' - j\epsilon'' \quad (1-50)$$

Over the microwave region ϵ' for most ferrites, including yttrium-iron garnets, tends to remain relatively constant. On the other hand, ϵ'' , which is proportional to dielectric losses, tends to vary approximately as $1/\omega$, i.e., inversely with frequency.

This loss mechanism is probably due to resonance of ions in the crystal lattice structure at higher microwave frequencies and to migration of free charge carriers in the macroscopic structure of the polycrystalline ferrite material at the lower microwave frequencies. [For more information, see Soohoo (1960), Chapter 3 or Clarricoats (1961), Chapter 2].

An alternative specification of the dielectric loss is the loss tangent:

$$\tan \delta = \frac{\epsilon''}{\epsilon'} \quad (1-51)$$

As might be expected from the above discussion the loss tangent also varies with $\frac{1}{\omega}$ as did ϵ'' .

1.6 The Lossless Permeability Tensor

Many of the basic equations in circulator design can be deduced satisfactorily under the assumption that the ferrite has no magnetic losses, i.e., that $\alpha = 0$. For such a case the elements of the permeability tensor $\vec{\mu}$, equation (1-40), are all real:

$$\mu = 1 + \frac{\omega_m \omega_0}{\omega_0^2 - \omega^2} \quad (1-52)$$

$$\kappa = - \frac{\omega_m \omega}{\omega_0^2 - \omega^2} \quad (1-53)$$

The above assumption that $\alpha = 0$ has validity only if ω , the operating frequency, is not in the vicinity of ω_0 , the gyromagnetic resonance frequency.

Since most manufacturers specify the saturation magnetization in Gaussian units, the above equations (1-52) and (1-53) will be modified to make use of these units. At the same time it is convenient to replace the angular frequencies ω , ω_0 and ω_m by the actual frequencies f , f_0 and f_m :

$$f = \frac{\omega}{2\pi} \quad \text{cycles/sec} \quad (1-54)$$

$$f_0 = \frac{\omega_0}{2\pi} = -\gamma_e H_i \quad \text{cycles/sec} \quad (1-55)$$

$$f_m = \frac{\omega_m}{2\pi} = -4\pi M_s \gamma_e / \mu_0 \quad \text{cycles/sec} \quad (1-56)$$

where the saturation magnetization is now $4\pi M_s$ gauss, the internal d.c. field is H_i oersteds, μ_0 is one gauss per oersted, and the gyromagnetic ratio is

$$\gamma_e = -2.8 \text{ MHz/oersted}. \quad (1-57)$$

Following the example of Fay and Comstock (1965) κ and μ in equations (1-52) and (1-53) can be written as:

$$\mu = 1 + \frac{\sigma \rho}{\sigma^2 - 1} \quad (1-58)$$

$$\kappa = -\frac{\rho}{\sigma^2 - 1} \quad (1-59)$$

with

$$\sigma = \frac{\omega_0}{\omega} = \frac{f_0}{f} = \frac{-\gamma_e H_i}{f} \quad (1-60)$$

$$\rho = \frac{\omega_m}{\omega} = \frac{f_m}{f} = \frac{-4\pi M_s \gamma_e}{\mu_0 f} \quad (1-61)$$

There are two distinct modes of operation possible in the circulator: (a) below resonance operation ($\omega > \omega_0$) and (b) above resonance operation ($\omega < \omega_0$). The former condition of operation is customarily employed for the higher microwave frequencies in order to reduce the size of the external biasing magnet; the latter condition is used for the lower microwave frequencies in order to avoid low field losses in the ferrite. This report will be primarily concerned with above resonance operation. An inspection of equations (1-58) and (1-59) indicates that for $\omega < \omega_0$, $\mu > 1$ and $\kappa < 0$. (The sign associated with κ is contrary to that given in Fay and Comstock, but it should be noted that they have arbitrarily set $\kappa > 0$ for $\omega < \omega_0$, a result in direct contradiction to that obtained in a thorough analysis).

Section II

CIRCULATOR ANALYSIS BY SCATTERING MATRICES

2.1 Preliminaries

The strip line circulator can be analyzed by either of two distinct but mutually complementary approaches, that of scattering matrices or that of field theory (Maxwell's equations). An analysis by scattering matrices yields the external symmetries and general relations that apply to the circulator as a 3-port microwave circuit. This method is a direct extension of the familiar circuit theory approach often applied to 2-port devices such as transmission lines and wave guides.

On the other hand, field theory gives the distribution of microwave fields within the circulator (provided the circulator boundary conditions are known) and provides useful design relations between circulator operations parameters, ferrite material parameters, frequency, and the circulator physical dimensions. A complete description of the field analysis ensues in Chapter 3.

The results of circuit and field approaches can be combined in attacking the problem of broadband operation of the circulator. This procedure will be outlined in Chapter 4 of this report.

2.2 The Scattering Matrix of a 3-port Junction.

At any port p ($p = 1, 2, 3$) of a 3-port microwave device, voltage-like and current-like quantities v_p and i_p can be defined. The voltage-like quantities, v_p , are proportional to the transverse electric field at the port, while the quantities i_p are proportional to the transverse magnetic field. The factors of proportionality are adjusted so that the power flow P_p through port p is given by

$$p_p = 1/2 v_p i_p^* , \quad (2-1)$$

the * symbol in equation (2-1) indicating the operation of complex conjugation.

Furthermore, as in conventional transmission line theory, v_p and i_p can each be written as a sum of forward and backward traveling waves,

$$v_p = v_p^+ + v_p^- \quad (2-2a)$$

$$i_p = i_p^+ - i_p^- , \quad (2-2b)$$

where the plus and minus signs are associated with the forward and backward traveling waves, respectively. Also, without loss of generality, it is convenient to express v_p and i_p in terms of a normalized impedance so that

$$\frac{v_p^+}{i_p^+} = 1 \quad (2-3a)$$

$$\frac{v_p^-}{i_p^-} = -1 \quad (2-3b)$$

(An extended discussion of this procedure is available in Altman [1964], Chapters 1 and 2).

An alternative approach in the application of circuit analysis techniques to microwave devices is to define an incident wave, a_p , and a scattered wave, b_p , at each port,

$$a_p = v_p^+ \quad (2-4a)$$

$$(p = 1, 2, 3)$$

$$b_p = v_p^- \quad (2-4b)$$

With reference to Figure 2.1 it is customary to represent the sets of waves incident upon and scattered from the 3-port structure as column matrices:

Figure 2.1

A glance at equations (2-2) and (2-3) shows that v_p and i_p are related to a_p and b_p at each port by the expressions,

$$v_p = a_p + b_p \quad (2-6a)$$

$$i_p = a_p - b_p \quad (2-6b)$$

or,

$$a_p = 1/2 [v_p + i_p] \quad (2-7a)$$

$$b_p = 1/2 [v_p - i_p] \quad (2-7b)$$

The power incident upon the 3-port structure is given by

$$\begin{aligned} P_i &= 1/2 (a_1 a_1^* + a_2 a_2^* + a_3 a_3^*) \\ &= 1/2 \sum_{p=1}^3 a_p a_p^* \end{aligned} \quad (2-8)$$

Similarly, the power scattered or reflected from all three ports would be

$$P_r = 1/2 \sum_{p=1}^3 b_p b_p^* \quad (2-9)$$

by direct analogy with equation (2-8).

The incoming and outgoing waves are related by the scattering matrix $\underline{S}$, a 3 x 3 matrix:

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad (2-10)$$

Or more compactly in matrix notation, equation (2-10) may be written,

$$\underline{b} = \underline{S} \underline{a} \quad (2-11)$$

As is discussed in chapter 2 of Altman (1964) the scattering matrix description of an n-port microwave junction has a number of advantages over impedance or admittance matrix descriptions. Among these advantages are the following:

- a) Scattering matrices are directly related to the usual microwave measurements of power and VSWR.
- b) Equations (2-8) and (2-9) provide a quick check of the power balance of lossless structures. No such immediate check is possible with impedance or admittance matrices.
- c) For lossless junctions the magnitude of the scattering parameters is invariant under a shift of reference or measurement point. Under the same circumstances, the impedance or admittance parameters vary not only in phase, but also in magnitude.
- d) Finally, under certain conditions of physical symmetry, it is possible to derive the scattering matrix from geometrical considerations only.

2.3 Some Properties of the Scattering Matrix

The following general properties of the scattering matrix are given without proof. Details of the proofs are available in chapters 2 and 3 of Altman (1964), or chapters 5 and 12 of Montgomery, Dicke, and Purcell (1948).

Property 1. For isotropic networks the scattering matrix $\underline{S}$ is symmetric, i.e.,

$$\begin{aligned} S_{ij} &= S_{ji} & (j = 1, 2, 3) \\ & & (2-12) \\ & & (i = 1, 2, 3) \end{aligned}$$

Symmetry of the scattering matrix implies symmetry of the impedance and admittance matrices; the microwave junction with a symmetrical scattering matrix is a reciprocal device. Property 2. If the microwave junction is lossless, then the scattering matrix is a unitary matrix, that is,

$$\underline{S}^+ \underline{S} = \underline{I} \quad (2-13)$$

where $\underline{I}$ is a 3 X 3 unit matrix and + represents a conjugate transpose operation where each element of $\underline{S}$, S_{ij} , is replaced by S_{ji}^* in forming $\underline{S}^+$. This property is simply a restatement of the energy balance between the incoming and outgoing waves and is valid for both reciprocal and non-reciprocal junctions. However, if the junction is both lossless and reciprocal (contains lossless isotropic media) then equation (2-13) reduces to

$$\underline{S}^* \underline{S} = \underline{I} \quad (2-14)$$

where $\underline{S}^*$ is formed from $\underline{S}$ by replacing S_{ij} in $\underline{S}$ by S_{ij}^* .

The fact that $\underline{S}$ is unitary is quite valuable in that it permits a relatively simple relation to be developed between the VSWR at the input port of a circulator and the isolation appearing between the input and isolated ports. This expression will be derived in a later section of this chapter.

2.4 Symmetry Matrices for the 3-port Stripline Circulator

In any structure containing an electromagnetic field, the solution of Maxwell's equations in the structure is determined uniquely by the propagation medium, the frequency, and the boundary conditions. This solution is independent of the coordinate system used or of the position of the structure in space. In particular, the structure may be bodily rotated or translated to a "new" position without affecting the fields inside. With respect to fixed space references, the fields associated with both the "old" and the "new" position are "solutions" (i.e., they can exist).

If the structure has symmetry about certain axes of symmetry, a rotation operation can interchange positions of parts of the structure, while leaving the position of the structure as a whole unchanged. This is demonstrated in Figure 2.2 for a 3-port junction with 120° rotational symmetry, such as a stripline circulator.

Figure 2.2 - Rotational Symmetry of a Stripline Circulator

Similarly, if the structure has symmetry about a plane, a reflection transformation can take place. Such a reflection about one of the planes of bilateral symmetry for a stripline circulator is shown in Figure 2.3. Such reflectional symmetries do not enter as directly into the derivation of the scattering matrix of the stripline circulator as do the rotational symmetries. For this reason, the remainder of this discussion will be concerned with rotational transformations only.

Figure 2.3

It is clear that any rotation operation will interchange symmetrical ports together with the fields associated with them. But the same result will be achieved if only the fields are rotated, while the 3-port junction itself is left intact, since the original and final structures in any rotation were indistinguishable. For example, in Figure 2.4(a) let E_1 , E_2 and E_3 be the fields associated with the 3 ports of a stripline circulator. After counter-clockwise rotation of the fields by 120° , the situation will be as in Figure 2.4(b). If the field distribution (E_1, E_2, E_3) was a permissible field

solution for the structure, then the distribution (E_1', E_2', E_3') is also a solution.

Figure 2.4 - Rotation of Fields

What is more, the two field distributions can be related by a matrix equation:

$$\begin{bmatrix} E_1' \\ E_2' \\ E_3' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \quad (2-15)$$

which can be written more succinctly as,

$$\underline{E}' = \underline{G}_J \underline{E} \quad (2-16)$$

The matrix $\underline{G}_J$ is a rotation matrix describing the cyclic interchange of field quantities under a rotation operation. Two other rotation matrices are possible for the stripline circulator with 120° rotational symmetry. One matrix would describe 120° clockwise rotation of the fields; the other matrix (a unit matrix) would involve the rotation of the fields 360° back upon themselves.

As is discussed in section 3.1 of Altman (1964), any of these rotation matrices $\underline{G}_J$ will commute with the scattering matrix $\underline{S}$ of the circulator,

$$\underline{G}_J \underline{S} = \underline{S} \underline{G}_J. \quad (2-17)$$

In terms of matrix theory $\underline{G}_J$ and $\underline{S}$ are similar matrices. Equation (2-17) is the key to the theoretical derivation of $\underline{S}$ from symmetry considerations.

2.5 The Eigenvalue Problem.

In terms of the incoming and outgoing waves of equations (2-5a) and (2-5b), a 3-port junction can be considered to have a reflection coefficient at each port:

$$\begin{aligned} \Gamma_1 &= \frac{b_1}{a_1} \\ \Gamma_2 &= \frac{b_2}{a_2} \\ \Gamma_3 &= \frac{b_3}{a_3} \end{aligned} \quad (2-18)$$

These reflection coefficients are, in general, not equal. However, for certain choices of the incident waves (a_1 , a_2 , a_3) as provided by the sources driving the device, the same reflection coefficient λ will appear at all ports,

$$\lambda = \Gamma_1 = \Gamma_2 = \Gamma_3 \quad (2-19)$$

For a lossless junction it is necessary that

$$|\lambda| = 1; \quad (2-20)$$

otherwise the total incident power would be more or less than the total scattered power.

Expressed in matrix notation equation (2-19) becomes,

$$\underline{b} = \lambda \underline{a} \quad (2-21)$$

where $\underline{b}$ and $\underline{a}$ are the column vectors of equations (2-5a) and (2-5b).

If equation (2-21) is set equal to equation (2-11), the following important equation results,

$$\underline{S} \underline{a} = \lambda \underline{a}. \quad (2-22)$$

This is the eigenvalue problem for the 3×3 scattering matrix $\underline{S}$. Any value of λ satisfying (2-22) is an eigenvalue of the matrix $\underline{S}$. The column vectors $\underline{a}$ associated with each λ are eigenvectors of $\underline{S}$. Once again, equation (2-22) represents the very special set of circumstances that there be three driving sources (a_1, a_2, a_3) adjusted to the proper amplitude and phase to provide the same reflection coefficient λ at all three ports.

For the case where S is a 3×3 matrix (scattering matrix of a 3-port network), there are, at most, three possible values of $\lambda = \lambda_i$ ($i = 1, 2, 3$). For each value of $\lambda = \lambda_i$ there is a corresponding eigenvector $\underline{a}^i$,

$$\underline{a}^i = \begin{bmatrix} a_1^i \\ a_2^i \\ a_3^i \end{bmatrix} \quad (i = 1, 2, 3) \quad (2-23)$$

Methods for the calculation of eigenvalues and eigenvectors are available in most standard references on the theory of matrices and are outlined briefly in section 3.2 of Altman (1964).

An alternative physical interpretation of the eigenvalue problem is that each eigenvector $\underline{a}^i$ and its associated eigenvalue λ_i specifies a set of incident and reflected waves at the terminals of the 3-port junction and therefore uniquely

describes a solution (eigensolution) to the junction boundary value problem. Since for a lossless junction, $|\lambda_i| = 1$, each eigensolution is then characterized by pure standing waves in the arms of the junction, the amplitudes of the fields in the arms being specified by the elements of the eigenvector $\underline{a}^i$ and the reflection coefficients at the junction terminals by the associated eigenvector.

Any eigenvector $\underline{a}^i$ of $\underline{S}$ when multiplied by an arbitrary constant k remains still an eigenvector. Hence, it is customary to multiply all eigenvectors by appropriate constants so that for all i ,

$$|a_1^i|^2 + |a_2^i|^2 + |a_3^i|^2 = 1. \quad (2-24)$$

Another important relation between $\underline{S}$ and its eigenvalues is that the sum of the eigenvalues is equal to the sum of the elements along the principal diagonal of $\underline{S}$,

$$\lambda_1 + \lambda_2 + \lambda_3 = S_{11} + S_{22} + S_{33} \quad (2-25)$$

Since by equation (2-17) $\underline{S}$ and the rotation matrix $\underline{G}_J$ commute, it will be true that $\underline{S}$ and $\underline{G}_J$ have the same eigenvectors. In other words, if the $\underline{a}^i$ of equation (2-23) satisfy (2-22) for some set of eigenvalues λ_i , they will also satisfy the equation,

$$\underline{G}_J \underline{a} = g \underline{a} \quad (2-26)$$

with possibly g_i a different set of eigenvalues than the λ_i associated with (2-22).

The proof that $\underline{G}_J$ and $\underline{S}$ have the same eigenvectors is given in section 3.2 of Altman (1969). The only exception to this property occurs when the eigenvalues of $\underline{G}_J$ are degenerate, i.e., not all distinct. This difficulty need not concern us in the case of a 3-port device with 120°

rotational symmetry where the eigenvalues of $\underline{G}_J$ are always distinct.

One important physical implication can be made from the fact that $\underline{S}$ and $\underline{G}_J$ have the same eigenvectors. It is possible to introduce any symmetrical change within the junction without upsetting the eigenvectors $\underline{a}^i$ of $\underline{S}$ and $\underline{G}_J$. Such a symmetrical change will affect only the eigenvalues of $\underline{S}$. But by equation (2-20) the eigenvalues of $\underline{S}$ have for a lossless junction unit magnitude; any symmetric change in such a junction means that the λ_i change only in phase, rotating about on a unit circle.

The transformation matrix $\underline{G}_K$ of $\underline{G}_J$ is defined as the matrix formed with the eigenvectors of $\underline{G}_J$ as columns, the eigenvectors $\underline{a}^i$ being normalized to unit magnitude as in equation (2-24),

$$\underline{G}_K = \begin{bmatrix} a_1^1 & a_1^2 & a_1^3 \\ a_2^1 & a_2^2 & a_2^3 \\ a_3^1 & a_3^2 & a_3^3 \end{bmatrix} \quad (2-27)$$

If $\underline{G}_J$ has distinct eigenvalues, then the scattering matrix $\underline{S}$ can be determined from purely symmetrical considerations by the equation,

$$\underline{S} = \underline{G}_K \underline{S}_d \underline{G}_K^{-1} \quad (2-28)$$

where $\underline{S}_d$ is the diagonal form of $\underline{S}$,

$$\underline{S}_d = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad (2-29)$$

A proof of equation (2-28) is given in Section 3-2 of Altman (1964).

2.6 Derivation of the Scattering Matrix of the 3-port Stripline Circulator.

As discussed in Section 2.4 a rotation matrix $\underline{G}_J$ describing the 120° symmetry of the 3-port stripline circulator is

$$\underline{G}_J = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (2-30)$$

To calculate the eigenvalues of $\underline{G}_J$ the following equation must be solved,

$$\det (\underline{G} - \lambda \underline{I}) = 0 \quad (2-31)$$

or,

$$\lambda^3 - 1 = 0 \quad (2-32)$$

The eigenvalues of $\underline{G}_J$ are then,

$$\lambda_1 = 1, \quad (2-33)$$

$$\lambda_2 = \text{EXP} \left(j \frac{2\pi}{3} \right),$$

$$\lambda_3 = \text{EXP} \left(-j \frac{2\pi}{3} \right)$$

Next, the eigenvectors of $\underline{G}_J$ - and consequently of $\underline{S}$ - are,

$$\begin{aligned}
\underline{a}^1 &= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
\underline{a}^2 &= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ \text{EXP}(-j2\pi/3) \\ \text{EXP}(j2\pi/3) \end{bmatrix} \\
\underline{a}^3 &= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ \text{EXP}(j2\pi/3) \\ \text{EXP}(-j2\pi/3) \end{bmatrix}
\end{aligned} \tag{2-34}$$

where the eigenvectors have been normalized in accordance with equation (2-24). The transformation matrix $\underline{G}_K$, made up of the above eigenvectors, is

$$\underline{G}_K = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \text{EXP}(-j2\pi/3) & \text{EXP}(j2\pi/3) \\ 1 & \text{EXP}(j2\pi/3) & \text{EXP}(-j2\pi/3) \end{bmatrix} \tag{2-35}$$

The scattering matrix of the 3-port stripline circulator may be found by equation (2-28),

$$\underline{S} = \underline{G}_K \underline{S}_d \underline{G}_K^{-1} \tag{2-28)$$

or

$$\underline{S} - \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \text{EXP}(-j2\pi/3) & \text{EXP}(j2\pi/3) \\ 1 & \text{EXP}(j2\pi/3) & \text{EXP}(-j2\pi/3) \end{bmatrix} \begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \text{EXP}(j2\pi/3) & \text{EXP}(-j2\pi/3) \\ 1 & \text{EXP}(-j2\pi/3) & \text{EXP}(j2\pi/3) \end{bmatrix}$$

$$= \begin{bmatrix} \alpha & \gamma & \beta \\ \beta & \alpha & \gamma \\ \gamma & \beta & \alpha \end{bmatrix} \quad (2-36)$$

where

$$\alpha = \frac{1}{3} [S_1 + S_2 + S_3] \quad (2-37)$$

$$\beta = \frac{1}{3} [S_1 + S_2 \text{EXP}(-j2\pi/3) + S_3 \text{EXP}(j2\pi/3)]$$

$$\gamma = \frac{1}{3} [S_1 + S_2 \text{EXP}(j2\pi/3) + S_3 \text{EXP}(-j2\pi/3)]$$

S_1 , S_2 and S_3 being the eigenvalues of $\underline{S}$.

It is interesting to note that if the ferrite were removed from the circulator structure, making the device reciprocal, then according to property (1) of Section 2.3 the scattering matrix must be symmetric, i.e.,

$$\beta = \gamma \quad (2-38)$$

From equation (2-37) it then follows that a reciprocal 3-port junction with 120° symmetry must have two equal eigenvalues,

$$S_2 = S_3 \quad (2-39)$$

and that its scattering matrix is

$$\underline{S} = \begin{bmatrix} \alpha & \gamma & \gamma \\ \gamma & \alpha & \gamma \\ \gamma & \gamma & \alpha \end{bmatrix} \quad (2-40)$$

Furthermore, if the reciprocal 3-port junction with the scattering matrix of (2-40) is terminated at ports 2 and 3 with a matched load, i.e., $a_2 = a_3 = 0$ (Figure 2.5), it is then impossible to obtain a matched condition at the input port ($b_1 = 0$).

Figure 2.5

This can be demonstrated by noting that under such circumstances the following matrix equation would apply:

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \alpha & \gamma & \gamma \\ \gamma & \alpha & \gamma \\ \gamma & \gamma & \alpha \end{bmatrix} \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix} \quad (2-41)$$

If it is desired that $b_1 = 0$, then obviously $\alpha = 0$, so that from equations (2-25) and (2-39)

$$\alpha + \alpha + \alpha = 0 = S_1 + 2S_2. \quad (2-42)$$

This is clearly impossible since from equation (2-20), $|S_i| = 1$ ($i = 1, 2, 3$) so that the three ports of the junction can not be matched simultaneously.

Matters are quite different if the non-reciprocal 3-port junction with scattering matrix given by equation (2-36) is terminated with matched loads at ports 2 and 3. It is possible to obtain matched conditions at port 1 ($b_1 = 0$) in this case since the equation,

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \alpha & \gamma & \beta \\ \beta & \alpha & \gamma \\ \gamma & \beta & \alpha \end{bmatrix} \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix} \quad (2-43)$$

with $b_1 = 0$ still implies that $\alpha = 0$. From equation (2-25) it again follows that

$$\alpha + \alpha + \alpha = 0 = \frac{1}{3} [S_1 + S_2 + S_3], \quad (2-44)$$

a result that is possible under the restriction that $|S_i| = 1$, ($i = 1, 2, 3$). Additionally, it should be noted that the three eigenvalues in this case are separated by phase angles of 120° on the unit circle.

It can be noted that under matched conditions at all three ports, the outgoing waves at ports 2 and 3 are related to the incoming wave at port 1 by,

$$\begin{aligned} b_2 &= \beta a_1 \\ b_3 &= \gamma a_1 \end{aligned} \quad (2-45)$$

The junction is now a power divider and would divide the total input power, $a_1 a_1^*$, between ports 2 and 3, while maintaining matched conditions at all ports.

If clockwise circulator action is required in Figure 2.5, then the three eigenvalues of the device must be adjusted to,

$$\begin{aligned} S_1 &= 1 \\ S_2 &= \text{EXP}(j2\pi/3) \\ S_3 &= \text{EXP}(-j2\pi/3) \end{aligned} \quad (2-46)$$

So that from equation (2-37) the scattering matrix becomes,

$$\underline{S}_C = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (2-47)$$

On the other hand, counterclockwise circulator action would require the following eigenvalues,

$$\begin{aligned} S_1 &= 1 \\ S_2 &= \text{EXP}(-j2\pi/3) \\ S_3 &= \text{EXP}(j2\pi/3) \end{aligned} \quad (2-48)$$

with resultant scattering matrix,

$$\underline{S}_{\text{ccw}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad (2-49)$$

In practice the adjustment of the eigenvalues of the circulator scattering matrix would be accomplished by varying the d.c. magnetic field used to bias the ferrite disk. The two permissible sets of eigenvalues would represent respectively the cases of operation below resonance ($\omega > \omega_0$) or above resonance ($\omega < \omega_0$).

Perfect circulator action, with scattering matrices given by equations (2-47) and (2-49), can take place only at a single frequency. Assuming clockwise circulation and taking port 1 as the input port, the scattering matrix as given by (2-47) yields the following relations among the incoming and outgoing waves of Figure 2.5,

$$\begin{aligned} b_1 &= 0 \\ b_2 &= a_1 \\ b_3 &= 0 \end{aligned} \quad (2-50)$$

This is the case where the VSWR is unity at port 1 and the isolation infinite at port 3.

For a narrow band of frequencies about the frequency where the scattering matrix appears as in equation (2-47), the scattering matrix is best written in the form of equation (2-36),

$$\underline{S}_c = \begin{bmatrix} \alpha & \gamma & \beta \\ \beta & \alpha & \gamma \\ \gamma & \beta & \alpha \end{bmatrix} \quad (2-51)$$

where $|\alpha| \ll 1$. Under these conditions Thaxter and Heller (1960) have pointed out that

$$|\gamma| = |\alpha| \quad (2-52)$$

$$|\beta| = \sqrt{1-2|\alpha|^2} \approx 1 - |\alpha|^2 \quad (2-53)$$

The physical interpretation of equations (2-52) and (2-53) is that the reflection coefficient $|\alpha|$ at port 1 equals the isolation $|\gamma|$ provided at port 3, and that the insertion loss $|\beta|$ to port 2 depends upon the VSWR at port 1 (assuming no ferrite or conductor losses in the circulator).

Other useful relationships that can be noted at this point for the lossless 3-port circulator are,

$$\text{VSWR (port 1)} = \frac{1 + |\alpha|}{1 - |\alpha|} \quad (2-54)$$

$$\text{Db. isolation (port 3)} = 20 \log_{10} \left[\frac{\text{VSWR (port 1)} - 1}{\text{VSWR (port 1)} + 1} \right] \quad (2-55)$$

$$\text{Db. insertion loss (port 2)} = -20 \log_{10} \left[1 - \left(\frac{\text{VSWR (port 1)} - 1}{\text{VSWR (port 1)} + 1} \right)^2 \right] \quad (2-56)$$

2.7 The Relation of the Circulator Eigenvectors to the Internal Field Distribution.

It is now enlightening to consider the r.f. H fields associated with each of the eigenvectors of equation (2-34),

$$\underline{a}^1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \underline{a}^2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ \text{EXP}(-j2\pi/3) \\ \text{EXP}(j2\pi/3) \end{bmatrix}$$

$$\underline{a}^3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ \text{EXP}(j2\pi/3) \\ \text{EXP}(-j2\pi/3) \end{bmatrix} \quad (2-34)$$

As discussed in section 2-2 (see, for example, equations (2-3) and (2-4)) the incident wave a_p ($p = 1, 2, 3$) at each port is proportional to the incident current wave i_p^+ , and thus also proportional to the incident traveling wave of the transverse H field. Thus, each of the eigenvectors above represents a set of transverse H fields incident upon the circulator. The space components of these transverse H fields are shown in Figure 2.6.

Figure 2.6

First, it is obvious that the three incident transverse H fields associated with $\underline{a}^1$, H_1^1 , H_2^1 , H_3^1 , are in phase at the axis of symmetry J, but because of their space relationships, they will add up vectorially to zero at this point.

In Figure 2.7 a graphical construction shows that the three transverse H fields associated with $\underline{a}^2$, because of their phase relations, represent at J a clockwise circularly polarized wave. Similar graphical construction would show that the H fields associated with $\underline{a}^3$ are a counter-clockwise circularly polarized wave.

On the other hand, the scattered waves, b_p , at each port are proportional to the negative of the scattered current wave, $-i_p^-$,

a) Phase of $H_1^2 = 0$

b) Phase of $H_1^2 = 120^\circ$

c) Phase of $H_1^2 = 240^\circ$

Figure 2.7 H Fields Associated With $\underline{a}^2$

which is in turn proportional to the scattered transverse magnetic field. Thus, the vectors $\underline{b}^1$, $\underline{b}^2$, and $\underline{b}^3$ can be associated with sets of transverse H fields scattered from the circulator when the circulator is driven by sets of incident transverse H fields corresponding respectively to the eigenvectors $\underline{a}^1$, $\underline{a}^2$, $\underline{a}^3$. From equations (2-21), (2-22) and (2-46) the scattered wave vectors can be calculated as

$$\begin{aligned}\underline{b}^1 &= S_1 \underline{a}^1 - \underline{a}^1 \\ \underline{b}^2 &= S_2 \underline{a}^2 = [\text{EXP}(j2\pi/3)] \underline{a}^2 \\ \underline{b}^3 &= S_3 \underline{a}^3 = [\text{EXP}(-j2\pi/3)] \underline{a}^3.\end{aligned}\tag{2-57}$$

Since $\underline{b}^1$, $\underline{b}^2$, and $\underline{b}^3$ each differ from their respective incident wave vectors only by a relative phase factor, $\underline{b}^1$, just as did $\underline{a}^1$, they must represent zero magnetic field at the circulator axis, while $\underline{b}^2$ and $\underline{b}^3$ represent respectively clockwise circularly polarized and counter-clockwise circularly polarized fields at J, as did $\underline{a}^2$ and $\underline{a}^3$.

The behavior of the total transverse magnetic field at the circulator axis can be noted by calculating the vector,

$$\underline{i}^i = \underline{a}^i - \underline{b}^i\tag{2-58}$$

for each of the eigenvectors. The components of this vector are proportional to the standing wave of transverse magnetic field at each port (see section 2-2, and equation (2-6)). Thus, for $\underline{a}^1$,

$$\underline{i}^1 = \underline{a}^1 - \underline{b}^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\tag{2-59a}$$

For $\underline{a}^2$,

$$\underline{i}^2 = \underline{a}^2 - \underline{b}^2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 - \text{EXP}(j2\pi/3) \\ \text{EXP}(-j2\pi/3) - 1 \\ \text{EXP}(-j2\pi/3) - \text{EXP}(j2\pi/3) \end{bmatrix} \quad (2-59b)$$

and for $\underline{a}^3$,

$$\underline{i}^3 = \underline{a}^3 - \underline{b}^3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 - \text{EXP}(-j2\pi/3) \\ \text{EXP}(j2\pi/3) - 1 \\ \text{EXP}(j2\pi/3) - \text{EXP}(j2\pi/3) \end{bmatrix} \quad (2-59c)$$

From equation (2-59a), it can be inferred that the standing wave associated with $\underline{a}^1$ has a zero transverse magnetic field at the axis of the circulator. On the other hand, since the components of $\underline{i}^2$ satisfy the relations,

$$\begin{aligned} i_1^2 &= [\text{EXP}(j2\pi/3)] i_2^2 \\ i_2^2 &= [\text{EXP}(j2\pi/3)] i_3^2 \\ i_3^2 &= [\text{EXP}(j2\pi/3)] i_1^2 \end{aligned} \quad (2-60)$$

it can be seen that the relative phases of the total transverse magnetic field at the three ports is the same as the components of $\underline{a}^2$. Hence, this standing wave of transverse magnetic field at the circulator axis is clockwise circularly polarized. In similar manner, it can be shown that the standing wave of transverse magnetic field associated with $\underline{a}^3$ is counter-clockwise circularly polarized.

It now remains to show that circulator operation depends upon the presence of all three standing wave distributions within the stripline junction. This can be accomplished by noting that any set of waves incident upon a 3-port junction can be written as the superposition of the three junction modes or eigenvectors, i.e.,

$$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = c_1 \underline{a}^1 + c_2 \underline{a}^2 + c_3 \underline{a}^3 \quad (2-61)$$

where c_1 , c_2 and c_3 are constants to be determined.

In particular, for a circulator with clockwise rotation as discussed in Section (2.6), $a_2 = a_3 = 0$. Thus, the constants c_1 , c_2 and c_3 can be evaluated from the equations,

$$a_1 = c_1 + c_2 + c_3 \quad (2-62)$$

$$0 = c_1 + c_2 \text{EXP}(-j2\pi/3) + c_3 \text{EXP}(j2\pi/3)$$

$$0 = c_1 + c_2 \text{EXP}(j2\pi/3) + c_3 \text{EXP}(-j2\pi/3)$$

which has solutions,

$$c_1 = c_2 = c_3 = \frac{1}{\sqrt{3}} a_1 \quad (2-63)$$

The decomposition of the set of reflected waves $\underline{b}$ can be carried out by noting that,

$$\begin{aligned} \underline{b} &= \underline{S}_c \underline{a} = \underline{S}_c [c_1 \underline{a}^1 + c_2 \underline{a}^2 + c_3 \underline{a}^3] \\ &= c_1 S_1 \underline{a}^1 + c_2 S_2 \underline{a}^2 + c_3 S_3 \underline{a}^3 \quad (2-64) \\ &= d_1 \underline{a}^1 + d_2 \underline{a}^2 + d_3 \underline{a}^3, \end{aligned}$$

where S_1 , S_2 , S_3 are the eigenvalues of $\underline{S}_c$ as given by equation (2-46). Thus, the constants d_1 , d_2 , and d_3 are given by,

$$d_1 = \frac{1}{\sqrt{3}} a_1$$

$$d_2 = \frac{1}{\sqrt{3}} a_1 \text{EXP}(j2\pi/3)$$

$$d_3 = \frac{1}{\sqrt{3}} a_1 \text{EXP}(-j2\pi/3) \quad (2-65)$$

If equation (2-64) is expanded, the result is,

$$b_1 = (d_1 + d_2 + d_3) = 0 \quad (2-66)$$

$$b_2 = d_1 + d_2 \text{EXP}(-j2\pi/3) + d_3 \text{EXP}(j2\pi/3) = a_1$$

$$b_3 = d_1 + d_2 \text{EXP}(j2\pi/3) + d_3 \text{EXP}(-j2\pi/3) = 0$$

This last result is identical to equation (2-50) of Section 2.6. From equations (2-61), (2-64) and (2-65) it thus follows that all three standing wave distributions (or eigenvectors) are essential for circulator operation.

Section 3

THE ANALYSIS OF THE STRIPLINE Y CIRCULATOR THROUGH FIELD THEORY

3.1 Introduction

In Chapter 2 of this report, circulator action has been analyzed by the application of scattering matrix theory to a nonreciprocal 3-port junction with 120° symmetry. Some of the same conclusions regarding the general properties of the stripline circulator will be deduced in this chapter through the use of field theory (Maxwell's equations). Additionally, a fields approach is important in that it produces useful design relationships between frequency, ferrite material parameters, d.c. biasing field, and ferrite disk size.

The field theory approach can be summarized briefly as follows:

- (a) The basic field equations will be derived from Maxwell's equations.
- (b) Appropriate boundary conditions will be assumed.
- (c) From the solution of the basic field equations, consistent with the assumed boundary conditions, a phenomenological and analytical description of circulator action is generated.
- (d) Such a theoretical picture of circulator behavior should be experimentally verified in order to validate the selection of the boundary condition of (c).

3.2 The Configuration of the 3-part Stripline Circulator

The stripline Y-junction 3-part circulator consists of two ferrite cylinders of radius R filling the space between a metallic center disk and two conducting ground planes. Energy is fed into and from the device by stripline.

(Figure 3.1) The d.c. biasing field H_{DC} , usually provided by a permanent magnet, is applied parallel to the axis of the ferrite cylinders. It should be noted that because of demagnetizing effects in the ferrite disks H_{DC} can be greater than H_i , the internal d.c. field in the ferrite.

Figure 3.1. Configuration of Stripline Circulator

Since the circulator structure exhibits cylindrical symmetry, a cylindrical coordinate system (r, ϕ, z) will be employed in the analysis of the device. The d.c. biasing field

will be taken in the z direction and the three striplines will enter the circulator at azimuths of $\phi = 0^\circ$, 120° and 240° (Figure 3.2).

Figure 3.2. Center Conductor of Circulator

3.3 Derivation of the Holmholtz Wave Equation

For a sinusoidal variation of the applied field, Maxwell's equations within the ferrite are (Atwater, 1962),

$$\nabla \times \vec{E} = -j\omega\vec{B} \quad (3-1)$$

$$\nabla \times \vec{H} = j\omega\epsilon\vec{E} \quad (3-2)$$

$$\nabla \cdot \vec{E} = 0 \quad (3-3)$$

$$\nabla \cdot \vec{B} = 0, \quad (3-4)$$

where it has been assumed that the ferrite has negligible conductivity and that no free charge is contained in the circulator region. All vectors in the above equations have complex components; the time variation of these vectors is given by,

$$\vec{E}(t) = \text{Real} [\vec{E} \text{ EXP}(j\omega t)]$$

$$\vec{H}(t) = \text{Real} [\vec{H} \text{ EXP}(j\omega t)] \quad (3-5)$$

$$\vec{B}(t) = \text{Real} [\vec{B} \text{ EXP}(j\omega t)]$$

with the word Real understood as "the **real** part of".

As was discussed in section 1 the magnetic flux density $\vec{B}$ is related to the magnetic field intensity by a permeability tensor ,

$$\vec{B} = \overset{\leftrightarrow}{\mu} \vec{H} \quad (1-36)$$

In the cylindrical coordinate system $\vec{B}$ and $\vec{H}$ may be written as,

$$\vec{B} = B_r \vec{a}_r + B_\phi \vec{a}_\phi + B_z \vec{a}_z \quad (3-6)$$

$$\vec{H} = H_r \vec{a}_r + H_\phi \vec{a}_\phi + H_z \vec{a}_z \quad (3-7)$$

where $\vec{a}_r$, $\vec{a}_\phi$ and $\vec{a}_z$ are unit vectors in the directions of increasing r , ϕ and z , respectively.

For a cylindrical coordinate system the tensor $\overset{\leftrightarrow}{\mu}$ in equation (1-36) takes the same form as for the Cartesian coordinate system (this can be demonstrated by a transformation of coordinate systems). Thus, in terms of the cylindrical components of equations (3-6) and (3-7), equation (1-36) becomes,

$$\begin{bmatrix} B_r \\ B_\phi \\ B_z \end{bmatrix} = \mu_0 \begin{bmatrix} \mu & -j\kappa & 0 \\ j\kappa & \mu & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} H_r \\ H_\phi \\ H_z \end{bmatrix} \quad (3-8)$$

Carrying out the curl operation of equation (3-1) and equating components yields,

$$-j\omega B_r = \frac{1}{r} \left[\frac{\partial E_z}{\partial \phi} - \frac{\partial (rE_\phi)}{\partial z} \right] \quad (3-9a)$$

$$-j\omega B_\phi = \frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} \quad (3-9b)$$

$$-j\omega B_z = \frac{1}{r} \left[\frac{\partial (rE_\phi)}{\partial r} - \frac{\partial E_r}{\partial \phi} \right] \quad (3-9c)$$

where E_r , E_ϕ and E_z are the cylindrical components of the electric field,

$$\vec{E} = \vec{a}_r E_r + \vec{a}_\phi E_\phi + \vec{a}_z E_z \quad (3-10)$$

At this point it will be assumed that the fields within the circulator have no z dependence and that the only component of the $\vec{E}$ field is in the z direction. Such an assumption has been adequately justified by measurements made by Fay and Comstock (1965) and Bosma (1964). It should also be noted that the fields in the two ferrite disks of the circulator are 180° out of phase (with reference to the z -direction) so that little radiation can occur outside the disks.

Since with no z dependence all partial derivatives with respect to z are zero, and since the electric field has only a z component, equations (3-9) reduce to:

$$-j\omega B_r = \frac{1}{r} \frac{\partial E_z}{\partial \phi} \quad (3-10a)$$

$$j B_\phi = \frac{\partial E_z}{\partial r} \quad (3-10b)$$

$$B_z = 0 \quad (3-10c)$$

By the use of equation (3-8) the components of $\vec{B}$ may be replaced by the components of $\vec{H}$,

$$\frac{1}{r} \frac{\partial E_z}{\partial \phi} = -j\omega\mu_0 [\mu H_r - j\kappa H_\phi] \quad (3-11a)$$

$$\frac{\partial E_z}{\partial r} = -j\omega\mu_0 [j\kappa H_r + \mu H_\phi] \quad (3-11b)$$

If equations (3-11) are solved for H_r and H_ϕ in terms of the electric field, the result is,

$$H_r = \frac{-1}{j\omega\mu_0\mu_{eff}} \left[\frac{1}{r} \frac{\partial E_z}{\partial \phi} - j \frac{\kappa}{\mu} \frac{\partial E_z}{\partial r} \right] \quad (3-12a)$$

$$H_\phi = \frac{1}{j\omega\mu_0\mu_{eff}} \left[\frac{\partial E_z}{\partial r} + j \frac{\kappa}{\mu r} \frac{\partial E_z}{\partial \phi} \right] \quad (3-12b)$$

$$H_z = 0 \quad (3-12c)$$

where μ_{eff} is the effective permeability of the ferrite,

$$\mu_{eff} = \frac{\mu^2 - \kappa^2}{\mu} \quad (3-13)$$

An expansion of equation (3-2) produces,

$$j\omega\epsilon E_z = \frac{1}{r} \left[\frac{\partial (rH_\phi)}{\partial r} - \frac{\partial H_r}{\partial \phi} \right] \quad (3-14)$$

for the only non-zero component of $\vec{E}$ is in the z direction. If next the components of $\vec{H}$ given by (3-13) are substituted in (3-14) the Helmholtz wave equation for E_z appears:

$$\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r} \frac{\partial E_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \phi^2} + \kappa^2 E_z = 0 \quad (3-15)$$

with,

$$k^2 = \omega^2 \epsilon \mu_0 \mu_{\text{eff}} = \omega^2 \epsilon_0 \epsilon_r \mu_0 \mu_{\text{eff}} \quad (3-16)$$

where ϵ_r is the relative dielectric constant of the ferrite, and ϵ_0 the permittivity of free space.

3.4 Solution of the Wave Equation

The wave equation, equation (3-15), will be solved by assuming that

$$E_z(r, \phi) = R(r) \Phi(\phi) \quad (3-17)$$

where $R(r)$ is a function of r alone, and $\Phi(\phi)$ depends only upon ϕ . This approach is the standard "separation of variables" technique.

Substituting $E_z = R\Phi$ into the wave equation yields

$$\frac{\partial^2 R}{\partial r^2} \Phi + \frac{1}{r} \frac{\partial R}{\partial r} \Phi + \frac{1}{r^2} R \frac{\partial^2 \Phi}{\partial \phi^2} + k^2 R\Phi = 0 \quad (3-18)$$

or by division by $R\Phi/r$ and rearrangement,

$$\frac{r^2}{R} \frac{\partial^2 R}{\partial r^2} + \frac{r}{R} \frac{\partial R}{\partial r} + r^2 k^2 = -\frac{1}{\Phi} \frac{\partial^2 \Phi}{\partial \phi^2} \quad (3-19)$$

Since the left side of equation (3-19) is a function of r alone and the right side is dependent only upon ϕ , both sides must be equal to a constant, called the separation constant. For convenience, this constant will be taken as n^2 , so that two ordinary differential equations, one in r and one in ϕ , are produced from equation (3-19).

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + R(k^2 - \frac{n^2}{r^2}) = 0 \quad (3-20)$$

$$\frac{d^2 \Phi}{d\phi^2} + n^2 \Phi = 0 \quad (3-21)$$

(The partial derivatives in (3-19) can be replaced by ordinary derivatives since each side of the equation involves only a single variable). Equation (3-20) is Bessel's differential equation, while equation (3-21) is the differential equation of harmonic motion.

A complete discussion of the solution of (3-20) is available in many standard reference works, such as Moon and Spencer (1961) or Wylie (1960). Equation (3-20) is a second order differential equation so it must have two independent solutions. Both of these solutions are in the form of an infinite series. Since equation (3-20) and its solutions occur quite frequently in applied mathematics, these solutions are termed Bessel functions. Extensive tabulations of these functions are widely available; examples of such tabulations are Jahnke and Emde (1945) or Abramowitz and Stegun (1964).

Of particular interest in the circulator problem are those solutions of equation (3-20) for n integer. In this case a particular solution of equation (3-20) is

$$R(r) = C_1 J_n(kr) + C_2 Y_n(kr) \quad (3-22)$$

where $J_n(kr)$ is a Bessel function of the first kind of order n with argument kr , and $Y_n(kr)$ is a Bessel function of the second kind of order n , also with argument kr . The constants C_1 and C_2 in (3-22) must be evaluated in terms of the circulator boundary conditions.

A particular solution of equation (3-21) is

$$\phi(\phi) = C_3 \text{EXP}(jn\phi) + C_4 \text{EXP}(-jn\phi) \quad (3-23)$$

where again C_3 and C_4 are arbitrary constants set by the boundary conditions on the circulator.

Since the field in the circulator is periodic in ϕ with period of 2π radians, the value of n appearing in (3-22) and (3-23) must be restricted to integer values. Furthermore, $Y_n(kr) \rightarrow \infty$ as $r \rightarrow 0$ so that the solution for $R(r)$ can only be well behaved if $C_2 = 0$. In other words, any physically realizable particular solution for the circulator must take the form,

$$\begin{aligned} E_z(r, \phi) &= R(r) \Phi(\phi) = C_1 J_n(kr) [C_3 \text{EXP}(jn\phi) + C_4 \text{EXP}(-jn\phi)] \\ &= J_n(kr) [a_{+n} \text{EXP}(jn\phi) + a_{-n} \text{EXP}(-jn\phi)] \end{aligned} \quad (3-24)$$

with $n = 0, 1, 2, 3, \dots$ and

$$\begin{aligned} a_{+n} &= C_3 C_1 \\ a_{-n} &= C_4 C_1 \end{aligned} \quad (3-25)$$

The total solution for E_z must be the sum of all possible particular solutions, i.e.,

$$\begin{aligned} E_z(r, \phi) &= \sum_{n=1}^{\infty} a_{-n} J_n(kr) \text{EXP}(-jn\phi) \\ &\quad + a_0 J_0(kr) + \sum_{n=1}^{\infty} a_{+n} J_n(kr) \text{EXP}(jn\phi) \end{aligned} \quad (3-26)$$

where

$$a_0 = a_{+0} + a_{-0} \quad (3-27)$$

Each integer value of n in equation (3-26) represents a mode of the circulator; for $n \neq 0$ each of these modes is split into a pair of complementary modes, one with $\text{EXP}(jn\phi)$ variation (+n mode), the other with $\text{EXP}(-jn\phi)$ variation (-n mode). An exact solution for E_z would require that a doubly infinite set of coefficients be evaluated in terms of the circulator boundary conditions. This is both impossible and unnecessary for it will be shown that certain modes predominate in the circulator when it is adjusted for proper operation.

It is now a straightforward procedure to calculate the magnetic field components associated with each mode of (3-26). This can be initiated by evaluating the partial derivatives required in equations (3-12a) and (3-12b):

$$\frac{\partial E_z}{\partial \phi} = jnJ_n(kr) [a_{+n} \text{EXP}(jn\phi) - a_{-n} \text{EXP}(-jn\phi)] \quad (3-28)$$

$$\begin{aligned} \frac{\partial E_z}{\partial z} &= [a_{+n} \text{EXP}(jn\phi) + a_{-n} \text{EXP}(-jn\phi)] \frac{\partial}{\partial r} [J_n(kr)] \\ &= [a_{+n} \text{EXP}(jn\phi) + a_{-n} \text{EXP}(-jn\phi)] [KJ_{n-1}(kr) - \frac{n}{r} J_n(kr)] \end{aligned} \quad (3-29)$$

The substitution of these derivatives into equations (3-12a) and (3-12b) yields the magnetic field components for each mode,

$$\begin{aligned} H_r = Y_{\text{eff}} \{ &a_{+n} \text{EXP}(jn\phi) \left[\frac{\kappa}{\mu} J_{n-1}(kr) - \left(1 + \frac{\kappa}{\mu}\right) \frac{nJ_n(kr)}{kr} \right] \right. \\ &\left. + a_{-n} \text{EXP}(-jn\phi) \left[\frac{\kappa}{\mu} J_{n-1}(kr) + \left(1 - \frac{\kappa}{\mu}\right) \frac{nJ_n(kr)}{kr} \right] \right\} \quad (3-30a) \end{aligned}$$

$$\begin{aligned} H_\phi = jY_{\text{eff}} \{ &a_{+n} \text{EXP}(jn\phi) \left[\left(1 + \frac{\kappa}{\mu}\right) \frac{nJ_n(kr)}{kr} - J_{n-1}(kr) \right] \right. \\ &\left. + a_{-n} \text{EXP}(-jn\phi) \left[\left(1 - \frac{\kappa}{\mu}\right) \frac{nJ_n(kr)}{kr} - J_{n-1}(kr) \right] \right\} \quad (3-30b) \end{aligned}$$

where

$$Y_{\text{eff}} = \frac{k}{\omega \mu_0 \mu_{\text{eff}}} = \sqrt{\frac{\epsilon_r \epsilon_0}{\mu_0 \mu_{\text{eff}}}} \quad (3-31)$$

An obvious physical interpretation can be given to the mode fields described by equation (3-24) and (3-30). For $n=0$, there is no variation of E_z and $\vec{H}$ with ϕ ; also, the H fields at $r=0$ (axis of circulator) are identically zero since $J_{-1}(0) \equiv 0$.

This corresponds to the standing wave distribution set up by the eigenvector $\underline{a}^1$ of equation (2-34), where all ports of the circulator are driven by sources of equal amplitude and phase.

When $n \neq 0$, the fields in the circulator are sinusoidal functions of ϕ , with period $\frac{2\pi}{n}$. Since $\text{EXP}(j\omega t)$ time variation has been assumed, the $+n$ modes with their $\text{EXP}(jn\phi)$ variation represent clockwise rotating fields when viewed from the positive z axis (top of the circulator). Conversely, counterclockwise rotating fields are associated with the $-n$ modes.

Thus, the two sets of modes, $\pm n (n \neq 0)$, correspond directly to standing wave distributions excited by the eigenvectors $\underline{a}^2$ and $\underline{a}^3$ of equation (2-34). Also, experiment (Fay and Comstock, 1965) shows that circulator operation requires that two split modes, $+n$ and $-n$, for some n , must predominate in the internal field distribution of the circulator.

3.5 The Normal Modes of the Uncoupled Stripline Circulator

Fay and Comstock (1965) have found experimentally that the stripline circulator has some, but not all of the properties of a low-loss transmission cavity. At its resonant frequency, the structure was well matched but slightly undercoupled, and a standing wave existed in the structure. The maximum isolation occurred almost at the frequency at which the insertion loss was minimum. The above experimental evidence suggests a resonance of the center disk structure as being an essential feature of circulator operation.

With this in mind, it is then of interest to find the character of the normal modes in the uncoupled case ($\Psi = 0$ in Figure 3-2). Even in the uncoupled case, it is difficult to record the exact boundary conditions. Bosma (1964) has argued that the normal modes resulting from the boundary condition,

$$H_{\phi}(r = R) = 0 \quad (3-32)$$

are sufficient for describing the field in the ferrite disks in the uncoupled case. In other words, it will be assumed that H is everywhere zero at the edge of the ferrite disk. The justification for this boundary condition is the rapid exponential falloff of fields outside the disk boundary.

If the boundary condition of equation (3-32) is inserted into equation (3-30b), it is found that H_ϕ can be zero at the edge of the ferrite only if the two following characteristic equations hold,

$$J_{n-1}(kR) - \left(1 + \frac{\kappa}{\mu}\right) \frac{nJ_n(kr)}{kR} = 0 \quad (3-33a)$$

$$J_{n-1}(kR) - \left(1 - \frac{\kappa}{\mu}\right) \frac{nJ_n(kr)}{kR} = 0 \quad (3-33b)$$

Equation (3-33a) applies to the $+n$ or clockwise rotating modes, while (3-33b) deals with the $-n$ or counterclockwise rotating modes. It should be noted that for every n each of the above equations have infinite number of roots. If, for a given n , $(kR)_{m,n+}$ and $(kR)_{m,n-}$ are the m th roots of (3-33a) and (3-33b), respectively, then the possible resonant frequencies of the uncoupled circulator can be calculated from equation (3-16),

$$(kR)_{m,n\pm} = \omega_{m,n\pm} R \sqrt{\epsilon_0 \epsilon_r \mu_0 \mu_{\text{eff}}} \quad (3-34)$$

or

$$\omega_{mn\pm} = \frac{c(kR)_{mn\pm}}{R \sqrt{\epsilon_r \mu_{\text{eff}}}} \quad (3-35)$$

where c is the free space velocity of electromagnetic waves,

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \quad (3-36)$$

For a given m and n , as $\frac{\kappa}{\mu} \rightarrow 0$, it can be seen from equation (3-33) that $(kR)_{m,n+} \rightarrow (kR)_{m,n-}$ and that the split modes

+n and -n degenerate into a single mode with resonant frequency,

$$\omega_{m,n} = \frac{c(kR)_{m,n}}{R\sqrt{\epsilon_r \mu_{eff}}} \quad (3-37)$$

where $(kR)_{m,n}$ is the mth root of,

$$J'_n(kR) = 0 \quad (3-38)$$

The prime sign in the above equation indicates differentiation with respect to r . Bosma (1964) and Fay and Comstock (1965) both point out that circulator operation is theoretically possible in a lossless coupled stripline circulator whenever $\frac{K}{\mu}$ is small, and the applied frequency is near $\omega_{m,n}$, lying somewhere between $\omega_{m,n+}$ and $\omega_{m,n-}$. Experimentally, such circulator operation has been achieved for values of n up to and including $n = 4$.

3.6 Approximate Boundary Condition for the Coupled Stripline Circulator

The most straightforward procedure for establishing design criteria for a stripline circulator is to assume initially that the junction structure is functioning as a circulator and then to find the conditions on the rotating normal modes to satisfy this assumption. As long as the three striplines are not tightly coupled to the circulator, the predominant field structure in the circulator will be standing waves corresponding to the normal modes. The only traveling waves will be ones whose fields are such as to result in net outgoing Poynting flux at port 2, the output port.

The striplines leading into the three ports of the circulator carry TEM waves only. The approximate boundary conditions will be predicated on the assumption that the normal modes of equations (3-24) and (3-30) are driven by the TEM waves on the stripline.

An infinite set of field modes (all values of n) is not essential in the matching of boundary conditions since, as discussed in section 3-5, the circulator is usually operated near the resonant frequency of one pair of split modes so all other field modes besides $+n$, $-n$ (and $n=0$) will be of negligible amplitude.

On the striplines the electric field intensities are perpendicular to the conductors (in the z direction); the magnetic field intensities are parallel to the conductors and perpendicular to the direction of propagation of waves along the line (in the ϕ direction). It will be assumed that the stripline field intensities do not vary over the width of the center strips, and are zero outside these strips. Thus, the fringing fields extending beyond the edges of the strips are neglected. Furthermore, the fields on the two sides of the center strips are 180° out of phase.

As is the case with the fields in the striplines, the electric as well as the magnetic intensities on either side of circulator center conductor are equal in amplitude and oppositely directed at any instant. Hence, the field problem need be solved only for one ferrite disk in association with the fields on the appropriate side of the stripline. At the three ports the ferrite field distribution of a split normal mode will be matched to the TEM fields in the striplines. For all other points on the edge of the ferrite disk the boundary condition is

$$H_\phi(r = R) = 0, \quad (3-39)$$

R being the radius of the ferrite disk (Figure 3.2). Strictly speaking, a non-zero electric field intensity as well as a non-zero value for H_r can exist at the edges of the ferrite disk.

Since both of these fields have at any instant opposite directions in the two ferrite disks, they radiate away no energy and contribute purely a reactive (energy storage) effect. Thus, these last two field components can be ignored in the boundary condition approximation.

At the points where the striplines feed into the circulator structure, the following boundary conditions are assumed to apply (Figure 3.2),

$$\begin{aligned} -\psi < \phi < \psi & \quad H_\phi = H_1 \\ 120^\circ - \psi < \phi < \psi + 120^\circ & \quad H_\phi = H_1 \\ -120^\circ - \psi < \phi < \psi - 120^\circ & \quad H_\phi = 0 \end{aligned} \quad (3-40)$$

and

$$\begin{aligned} \phi = 0 & \quad E_z = E_1 \\ \phi = 120^\circ & \quad E_z = -E_1 \\ \phi = -120^\circ & \quad E_z = 0 \end{aligned} \quad (3-41)$$

The minus sign associated with the field E_z at $\phi = 120^\circ$ arises from the fact that experiments indicate the effective electrical length between the input and output ports is one-half wave length. It is necessary for both E_z and H_ϕ to be zero at the isolated port ($\phi = -120^\circ$) since the boundary condition there is an admittance condition, i.e., the ratio of H_ϕ to E_z at this port must be the wave admittance.

Also, the investigation in this report will be restricted to values of v , stripline width, which are not too large in comparison with R , the ferrite disk radius,

$$v < R \quad (3-42)$$

The smaller the ratio v/R , the more accurate is the assumed distribution of $H_\phi(R, \phi)$ in equation (3-40), but the larger is the disturbing influence of the stray fields of the striplines. Hence, v must not be too small.

If v is too large, the stripline would be tightly coupled to the circulator and the field structure of the rotating normal modes would be severely distorted from that of the uncoupled ferrite disk. In the case then of tight coupling between stripline and circulator, it is a far more difficult task analytically to obtain the design criteria for circulator operation.

3.7 Derivation of the Two Essential Circulation Conditions.

As was discussed in section 3.5, the circulator can operate in the n th mode with an applied frequency near that frequency given by equation (3-37). However, to keep the radius of the ferrite as small as possible and thus minimize field losses in the ferrite, it is most practical to operate the circulator in the $n = 1$ mode. Thus in the following analysis, the boundary condition assumptions of section 3.6 will be matched to the pair of split modes, $n = \pm 1$, in order to obtain design criteria for the circulator. It will also be assumed that the circulator is operating at a d.c. internal field H_1 above that required for gyromagnetic resonance at the applied frequency. Thus, the value of κ in the permeability tensor will be, according to equation (1-57), negative.

If the modes $n = \pm 1$ are assumed to be driven by the TEM waves on the striplines, then from equation (3-24) and by use of the first two boundary conditions of equation (3-41), the following two equations hold,

$$\text{At } \phi = 0, r = R, \quad (3-43a)$$

$$\frac{E_1}{J_1(kR)} = a_{+1} + a_{-1}$$

$$\text{At } \phi = 120^\circ, r = R,$$

$$\frac{-E_1}{J_1(kR)} = a_{+1} \text{EXP}(j\pi/3) + a_{-1} \text{EXP}(-j\pi/3) \quad (3-43b)$$

Solving (3-43) for a_{+1} and a_{-1} gives,

$$a_{+1} = \frac{E_1}{2J_1(kR)} \left(1 + \frac{j}{\sqrt{3}}\right) \quad (3-44a)$$

$$a_{-1} = \frac{E_1}{2J_1(kR)} \left(1 - \frac{j}{\sqrt{3}}\right) \quad (3-44b)$$

It can be easily verified that these values of a_{+1} and a_{-1} make $E_z(R, \phi)$ zero at $\phi = 120^\circ$, the isolated port, so that the last of the three boundary conditions of (3-41) is satisfied.

The substitution of a_{+1} and a_{-1} into equation (3-43) gives the $n = \pm 1$ mode distribution of the electric field within the circulator as a standing wave distribution,

$$E_z = \frac{E_1 J_1(kr)}{J_1(kR)} \left(\cos \phi - \frac{\sin \phi}{\sqrt{3}} \right) \quad (3-45)$$

The magnetic field boundary conditions at the edge of the ferrite disk can be incorporated into the solution by expanding $H_\phi(R, \phi)$ of equation (3-40) into a Fourier series,

$$H_\phi(R, \phi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\phi + b_n \sin n\phi) \quad (3-46)$$

where the coefficients can be evaluated from,

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} H_\phi(R, \phi) d\phi \quad (3-47a)$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} H_\phi(R, \phi) \cos n\phi d\phi \quad (3-47b)$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} H_\phi(R, \phi) \sin n\phi d\phi \quad (3-47c)$$

(See, for example, Wiley (1960), for further discussion of the Fourier series).

Carrying out the integrations of (3-47) permits the magnetic field $H_\phi(R, \phi)$ to be written as,

$$H_\phi(R, \phi) = 2H_1 \left[\frac{\Psi}{\pi} + \sum_{n=1}^{\infty} \frac{\sin n\Psi}{n\pi} \left\{ (1 + \cos \frac{2n\pi}{3}) \cos n\phi + \sin \frac{2n\pi}{3} \sin n\phi \right\} \right] \quad (3-48)$$

for the $n=1$ mode, equation (3-48) can be written as,

$$\begin{aligned} H_{\phi 1} &= \frac{H_1 \sin \Psi}{\pi} [\cos \phi + \sqrt{3} \sin \phi] \\ &= \frac{H_1 \sin \Psi}{2\pi} [(1-j\sqrt{3}) \text{EXP}(j\phi) + (1+j\sqrt{3}) \text{EXP}(-j\phi)] \quad (3-49) \end{aligned}$$

Unfortunately, $H_{\phi 1}$, alone does not satisfy the boundary conditions of equation (3-40). These conditions are satisfied only by the entire set of modes with the amplitudes given by equation (3-48). Nevertheless, the result of (3-49) will be quite useful in establishing procedures for the broadbanding of the stripline circulator.

Another solution for $H_{\phi 1}(R, \phi)$ can be obtained by setting $n=1$ in equation (3-30b) and using the values of a_{+1} and a_{-1} given by equation (3-44). After appropriate rearrangement, this solution appears as,

$$H_{\phi 1} = \frac{Y_{\text{eff}} E_1}{2\sqrt{3} J_1(kR)} \{ (-1 + j\sqrt{3}) [(1 + \frac{\kappa}{\mu}) \frac{J_1(kR)}{kR} - J_0(kR)] \text{EXP}(j\phi) + (1 + j\sqrt{3}) [(1 - \frac{\kappa}{\mu}) \frac{J_1(kR)}{kR} - J_0(kR)] \text{EXP}(-j\phi) \} \quad (3-50)$$

Since equations (3-49) and (3-50) both represent for the $n=1$ mode the ϕ component of the magnetic field at the edge of the ferrite disk, a comparison of the two equations yields immediately two circulation conditions,

$$H_1 \frac{\sin \psi}{2\pi} = - \frac{Y_{\text{eff}} E_1}{2\sqrt{3} kR} \frac{\kappa}{\mu} \quad (3-51)$$

$$-[(1 + \frac{\kappa}{\mu}) \frac{J_1(kR)}{kR} - J_0(kR)] = (1 - \frac{\kappa}{\mu}) \frac{J_1(kR)}{kR} - J_0(kR) \quad (3-52)$$

Equation (3-52) is equivalent to the condition,

$$J_0(kR) - \frac{J_1(kR)}{kR} = J_1'(kR) = 0 \quad (3-53)$$

of which the first root is

$$(kR)_{1,1} = 1.84 \quad (3-54)$$

so that from equation (3-37) the operating frequency of the circulator in the $n=1$ mode is

$$\omega_c = \frac{1.84c}{R\sqrt{\epsilon_r \mu_{\text{eff}}}} \quad (3-55)$$

It should be noted that ω_c lies between the uncoupled resonant frequencies of the split modes $n=\pm 1$, $\omega_{1,1+}$ and $\omega_{1,1-}$, as given by equation (3-35). For the case where the circulator is biased above ferromagnetic resonance, $\kappa < 0$, and the following relation holds between the various resonant frequencies,

$$\omega_{1,1+} > \omega_c > \omega_{1,1-} \quad (3-56)$$

The circulation condition of equation (3-51) states that the input wave admittance at port 1 and the output wave admittance at port 2 must be adjusted to match the admittance on the stripline terminals. The required wave admittance is

$$Y_w = G_R = \left| \frac{H_1}{E_1} \right| = \frac{\left| \frac{\kappa}{\mu} \right| Y_{eff} \pi}{(kR)_{1,1} \sqrt{3} \sin \psi} \approx \frac{\left| \frac{\kappa}{\mu} \right| Y_{eff}}{\sin \psi} \quad (3-57)$$

as,

$$\frac{\pi}{(kR)_{1,1} \sqrt{3}} = 0.986 \approx 1.0 \quad (3-58)$$

3.8 Confirmation of the Analysis Through Experiment

In view of the many approximations and assumptions utilized in the field theory analysis of the circulator, it is only reasonable that experimental confirmation be sought for the various results of this analysis. A scan of the literature on the operation of the Y-junction stripline circulator indicates that useful experimental results have been published regarding the circulator electric field distribution as well as the relation of ferrite disk radius to the parameters of frequency and the gyrotropic constants κ and μ .

One of the crucial assumptions of this analysis of the Y-stripline circulator is that the device is operating near resonance for the $n = \pm 1$ modes so that these modal fields predominate in the circulator. A second crucial assumption is that the transverse electric field at the three input ports has the distribution given by equation (3-41). When these two assumptions are combined with the solution of the wave equation, E_z at the ferrite disk circumference should have the following sinusoidal distribution with ϕ ,

$$E_z(R_1\phi) = E_1 \left(\cos \phi - \frac{\sin \phi}{\sqrt{3}} \right) \quad (3-59)$$

In order to verify these two assumptions, Fay and Comstock (1965) have probed the electric field at points around the circumference of the ferrite disk and found it to follow essentially a sinusoidal variation with ϕ , as given by equation (3-59). The more lightly the striplines are coupled to the circulator, the more nearly sinusoidal is the distribution of E_z at the ferrite circumference. The only distortion in the sinusoidal pattern occurred at the points (0° , 120° , -120°) where the high impedance striplines were connected. Equation (3-59) predicts that the maximum value of E_z should appear at $\phi = -30^\circ$ and $\phi = 150^\circ$ and minimum value at 60° and 240° ; this indeed was the case.

When the circulator was tightly coupled to the circulator by means of low impedance stripline (Ψ of figure 3.2 was set to 22°), Fay and Comstock no longer found E_z to follow a sinusoidal distribution at $r=R$. Much greater distortion of the E_z pattern occurred at the stripline connections to the circulator, and the minimums of E_z were no longer so distinct, especially at $\phi = 60^\circ$.

From this it can be inferred that the $n=\pm 1$ mode does not predominate as much in the heavily loaded circulator as in one lightly loaded. For the heavily loaded circulator, then, greater amplitudes are required for the modes $n \neq \pm 1$, particularly the $n=0$ mode, in order that the observed E_z field pattern be produced.

Bosma (1964) has also measured the field of the stripline circulator, not only at the disk circumference, but inside the ferrite disk as well. Along the disk circumference he found almost the same sinusoidal distribution for E_z is small, i.e., at $\phi=60^\circ$ and $\phi=-120^\circ$. By the heroic procedure of having an electronic computer match the first 18 modes of equation (3-24) to the assumed boundary conditions at the striplines, extremely good correspondance between experiment and theory for E_z was obtained at all points within the circulator. For the case of the lightly coupled circulator discussed in this report this extreme accuracy of matching boundary conditions is not necessary, since Bosma was working with rather wide striplines ($v/R = 0.48$ or $\Psi = 12.8^\circ$) where it would be expected that modes $n \neq \pm 1$ would have some contribution to the overall internal field structure of the ferrite disk. However, such a device as computer matching of boundary conditions might very well be necessary for an accurate design of heavily loaded stripline circulators.

Fay and Comstock (1965) have designed an experimental L-band circulator operating in the $n = \pm 1$ mode; the experimental data discussed above was taken on this circulator since such a circulator had dimensions large enough to allow accurate probing of the electric field at the edge of the ferrite

disk. By using equation (3-55) to determine the relation between ω_c , the operating frequency, and R, the ferrite disk radius, such an L-band circulator was found to operate at 1300GHz with a biasing field of 20 oersteds (below resonance operation) or at 1130GHz with a biasing field of 2000 oersteds (above resonance operation). In both cases $|\frac{K}{\mu}|$ was adjusted a value somewhere between 0.25 and 0.5, the exact value depending upon bandwidth requirements, as will be outlined in following sections.

Section 4

CIRCULATOR LOSSES AND THE PROBLEM OF BROADBANDING

4.1 The Equivalent Circuit of the Circulator

Both the calculation of circulator losses and the problem of broadbanding are best approached by considering the strip-line Y-circulator as a lightly coupled resonator structure. At the input port, $\phi=0$, the frequency behavior of the input wave admittance (or impedance) at small deviations from the operating frequency can be expressed in terms of a lumped circuit model. If as assumed in Section 3, the fields due to the mode $n = \pm 1$ predominate within the ferrite disk, then from equations (3-24) and (3-44) the electric field E_z at the input port ($r = R, \phi = 0^\circ$) may be written as

$$E_z(R, 0) = E_1 = E^+ + E^- \quad (4-1)$$

where

$$E^+ = \frac{E_1}{2} \left(1 + \frac{j}{\sqrt{3}}\right) \quad (4-2a)$$

$$E^- = \frac{E_1}{2} \left(1 - \frac{j}{\sqrt{3}}\right) \quad (4-2b)$$

When E_z of equation (4-1) is divided by H_1 , the driving magnetic field at the input port stripline, then the input wave impedance z is obtained,

$$z = \frac{E_z(R, 0)}{H_1} = z^+ + z^- = \frac{1}{G_R} \quad (4-3)$$

with

$$z^+ = \frac{1}{2G_R} \left(1 + \frac{j}{\sqrt{3}} \right) \quad (4-4a)$$

$$z^- = \frac{1}{2G_R} \left(1 - \frac{j}{\sqrt{3}} \right) \quad (4-4b)$$

and where $G_R = |H_1/E_1|$, as given by equation (3-57).

These impedances z^+ and z^- can be represented by the lumped constant resonators of Figure 4-1. The resonant frequencies (ω^+) of the resonators are given by equation (3-35),

$$\omega^+ = \omega_{1, 1^+}, \quad \omega^- = \omega_{1, 1^-} \quad (4-5)$$

Figure 4-1. Equivalent Circuit for Input Wave Impedance of Stripline Circulator.

If $\frac{\kappa}{\mu}$ is small, then Bosma (1964) has demonstrated that ω_c , the circulator operating frequency as given by equation (3-55), lies approximately midway between ω^+ and ω^- ,

$$\omega_c \approx \frac{\omega^+ + \omega^-}{2} \quad (4-6)$$

Also, by equation (3-56),

$$\omega^+ > \omega_c > \omega^- \quad (4-7)$$

It also should be noted from equation (4-4) that the + and - resonators have phase angles of $\pm 30^\circ$ at the operating frequency, ω_c , so that the input wave impedance (or admittance) is real at this frequency.

Physically, the impedances z^+ and z^- are related to the clockwise and counterclockwise rotating field patterns in the ferrite disk. Their sum z , the input wave impedance, is related to the standing wave distribution or stationary resonance of the ferrite disk. This stationary resonance will have the same Q as each of its components, and will be the one to be considered when the loaded Q and the broadbanding of the circulator are discussed. By a procedure such as that outlined in Chapter 5 of Atwater (1962), the equivalent circuit of the stripline circulator as a resonator can be reduced in the vicinity of band-center to the single-shunt resonator shown in Figure 4-2.

Figure 4.2. Lumped Element Equivalent Circuit of the Stripline Circulator as a Resonator.

4.2 Loss Calculations for the Stripline Circulator

If the stripline circulator is lightly coupled at the input port and no energy is extracted from the output port, the only power entering the input port will be that power P_{Loss} required to supply all loss mechanisms within the circulator. For this condition an unloaded Q , Q_0 , may be defined,

$$Q_0 = \frac{\omega_c U}{P_{\text{Loss}}} \quad (4-8)$$

where ω_c is the operating frequency, and U is the maximum energy stored in the ferrite cylinders.

Similarly, if power P_{out} is taken from the output port of the circulator and if it is assumed $P_{\text{out}} \gg P_{\text{Loss}}$, then a loaded Q , Q_L , may be defined

$$Q_L = \frac{\omega_c U}{P_{\text{out}}} \quad (4-9)$$

It is convenient to express the insertion loss between the input and output ports of the circulator in terms of Q_0 and Q_L . If P_{in} is the power entering the input port, then the fractional insertion loss would be

$$\frac{P_{out}}{P_{in}} = 1 - \frac{P_{Loss}}{P_{in}} \quad (4-10)$$

since $P_{out} = P_{in} - P_{Loss}$. On the assumption that $P_{Loss} \ll P_{in}$, it follows that

$$\begin{aligned} \frac{P_{out}}{P_{in}} &\approx 1 - \frac{P_{Loss}}{P_{out}} \\ &= 1 - \frac{\frac{\omega_c U}{Q_0}}{\frac{\omega_c U}{Q_L}} = 1 - \frac{Q_L}{Q_0} \end{aligned} \quad (4-11)$$

A calculation of the loaded Q , Q_L , can be made directly from the definition of equation (4-9), assuming that the $n = \pm 1$ modes predominate in the circulator. According to Ghose (1963), p. 168, the maximum stored energy U in a cavity resonator of any arbitrary shape for a sinusoidally time-varying field is given by

$$U = \frac{\epsilon_r \epsilon_0}{2} \int_V \vec{E} \cdot \vec{E}^* dV \quad (4-12)$$

where V denotes the volume of the cavity, and where $\vec{E} = E_z \vec{a}_z$, the electric field within the ferrite disks, is defined by equation (3-45):

$$E_z = \frac{J_1(kr)}{J_1(kR)} \left[\cos \phi - \frac{\sin \phi}{\sqrt{3}} \right] \quad (3-45)$$

The integration can be carried out in cylindrical co-ordinates by noting that

$$dV = r dr d\phi dz \quad (4-13)$$

and by letting $\phi' = \phi + \frac{\pi}{6}$ so that

$$E_z = E_{\max} \frac{J_1(kr)}{J_1(kR)} \cos \phi' \quad (4-14)$$

where

$$E_{\max} = \frac{2E_1}{\sqrt{3}} \quad (4-15)$$

the maximum value of E_z at the ferrite disk perimeter. Thus, if each disk has a thickness of d ,

$$\begin{aligned} U &= \frac{\epsilon_r \epsilon_0 E_{\max}^2}{2 J_1^2(kR)} \int_0^{2\pi} \int_0^R \int_{-d}^d J_1^2(kr) (\cos^2 \phi') r dr d\phi' dz \\ &= \frac{\pi \epsilon_r \epsilon_0 d E_{\max}^2}{J_1^2(kR)} \int_0^R J_1^2(kr) r dr \end{aligned} \quad (4-16)$$

$$U = \frac{R^2 \pi \epsilon_r \epsilon_0 d E_{\max}^2}{2 J_1^2(kR)} \{ J_1^2(kR) - J_0(kR) J_2(kR) \}$$

where the last integration was carried out by an identity given by Moon and Spencer (1961), p 175.

Through use of the identities,

$$x J_1'(x) = -J_1(x) + x J_0(x) \quad (4-17)$$

$$x J_1'(x) = J_1(x) - x J_2(x)$$

and the application of the conditions for first order resonance, equations (3-53) and (3-54),

$$\chi = kR = 1.84 \quad (3-54)$$

$$J_1'(1.84) = 0 \quad (3-53)$$

it follows that

$$J_0(1.84) = J_2(1.84) = \frac{J_1(1.84)}{1.84} \quad (4-18)$$

so that the energy stored in the ferrite circulator at the operating frequency can be expressed as

$$U = \frac{\epsilon_r \epsilon_0 d E_{\max}^2 R^2}{2} \left[1 - \frac{1}{(1.84)^2} \right] \approx 1.11 E_{\max}^2 R^2 \epsilon_r \epsilon_0 d \quad (4-19)$$

The power taken from the output port, P_{out} , can be calculated by applying Poynting's theorem to the transverse electromagnetic fields at either the output port or the input port, since it has been assumed that P_{Loss} is much smaller than P_{out} . Thus,

$$P_{\text{out}} = \frac{1}{2} \text{Real} \int_S \vec{E} \times \vec{H} \cdot d\vec{s} \quad (4-20)$$

where S is the stripline cross section at the input port ($\phi = 0, r = R$). From equation (3-45), the E field at $r = R$ is

$$\vec{E}(r = R) = E_1 \left(\cos \phi - \frac{\sin \phi}{\sqrt{3}} \right) \vec{a}_z, \quad (4-21)$$

and from the assumed boundary conditions of equation (3-40) the H field at $-\Psi < \phi < \Psi$ is

$$\vec{H}(r = R, -\Psi < \phi < \Psi) = H_1 \vec{a}_\phi = \pm G_R E_1 \vec{a}_\phi \quad (4-22)$$

In the preceding equation, G_R is the input admittance as defined by equation (3-57). Since power is assumed to flow into the input port, the plus sign will be associated with G_R in the following calculation. Also, the surface differential element in the cylindrical co-ordinate system is given by

$$d\vec{s} = R d\phi dz \vec{a}_r \quad (4-23)$$

It should be noted in equations (4-21), (4-22) and (4-23) that $\vec{a}_z$, $\vec{a}_\phi$, and $\vec{a}_r$ are unit vectors in the direction of increasing z , ϕ and r , respectively. The substitution of $\vec{E}$, $\vec{H}$ and $d\vec{s}$ in the Poynting integral gives P_{out} as

$$P_{out} = \frac{G_R E^2 R}{2} \int_{-d}^d \int_{-\Psi}^{\Psi} \left(\cos \phi - \frac{\sin \phi}{\sqrt{3}} \right) d\phi dz$$

$$P_{out} = 2Rd G_R E^2 \max \cos^2 30^\circ \sin \Psi \quad (4-24a)$$

The limits of integration can be ascertained by reference to figure (4-3)

Experimental evidence [Rosenkranz (1969), Fay & Comstock (1965)] indicates that the value of output power given by equation (4-24a) is too small. A more accurate value of output power would be calculated by also considering the stripline fringing fields at the input of the circulator. Such a calculation would be extremely involved and would

Figure 4.3. Limits of Integration for Calculation of P_{out}

yield little physical insight in the overall design procedure. In order to simplify matters, it would be assumed that these fringing fields are significant only in the region $2\Psi < \phi < -2\Psi$, a fact demonstrated by experiment (Rosenkranz, 1969). Also it will be assumed that the Poynting vector $\vec{E} \times \vec{H}^*$ maintains the same value over the region $2\Psi < \phi < -2\Psi$ as over the region $\Psi < \phi < -\Psi$. This last assumption is a reasonable compromise between ignoring the fields outside $2\Psi < \phi < -2\Psi$ and ignoring the fact that the fields inside $2\Psi < \phi < -2\Psi$ fall off exponentially with distance from the stripline center conductor. With these assumptions, the power output of the circulator is

$$P_{out} = 2Rd G_R E_{max}^2 \cos^2 30^\circ \sin 2\Psi$$

$$\approx 4Rd G_R E_{max}^2 \cos^2 30^\circ \sin \Psi \quad (4-24b)$$

The substitution of expressions (4-24b) and (4-19) into the definition of loaded Q, equation (4-9), gives another fundamental circulator relationship,

$$Q_L = \frac{(kR - \frac{1}{kR}) \omega_c R \epsilon_r \epsilon_0}{2\sqrt{3} Y_{eff} |\frac{k}{\mu}|} = 0.69 |\frac{k}{\mu}| \quad (4-25)$$

In reducing equation (4-25), use has been made of the following expressions,

$$G_R = \frac{\pi |\frac{k}{\mu}| Y_{eff}}{\sqrt{3} \sin \psi} \quad (3-57)$$

$$Y_{eff} = \sqrt{\frac{\epsilon_r \epsilon_0}{\mu_0 \mu_{eff}}} \quad (3-31)$$

$$k^2 = \omega_0^2 \epsilon_0 \epsilon_r \mu_0 \mu_{eff} \quad (3-16)$$

$$kR = 1.84 \quad (3-54)$$

At this point it should be noted that Fay and Comstock (1965) give after a number of additional approximations, the relation

$$Q_L \approx 0.71 |\frac{\mu}{k}| \quad (4-26)$$

whereas equation (4-25) has been shown to follow directly from the assumed boundary conditions and internal field distribution for the circulator.

The calculation of the unloaded Q , Q_0 , can be facilitated by noting that P_{Loss} is due primarily to three loss mechanisms: (1) magnetic loss P_μ , (2) dielectric loss P_ϵ , and (3) conductor skin effect losses P_s so that

$$P_{\text{Loss}} = P_{\mu} + P_{\epsilon} + P_s \quad (4-27)$$

Thus, it follows that three separate Q's can be defined, one for each loss mechanism,

$$\frac{1}{Q_0} = \frac{1}{Q_{\mu}} + \frac{1}{Q_{\epsilon}} + \frac{1}{Q_c} \quad (4-28)$$

where,

$$Q_{\mu} = \frac{\omega_c U}{P_{\mu}} \quad (4-29a)$$

$$Q_{\epsilon} = \frac{\omega_c U}{P_{\epsilon}} \quad (4-29b)$$

$$Q_c = \frac{\omega_c U}{P_s} \quad (4-29c)$$

The calculation of Q_{μ} and Q_{ϵ} can be made by a slight modification of μ_{eff} and ϵ as they appear in equation (3-16). Both of these will now be assumed complex with small but non-zero imaginary parts,

$$\mu_{\text{eff}} = \mu'_{\text{eff}} - j\mu''_{\text{eff}} \quad \epsilon = \epsilon' - j\epsilon'' \quad (4-30)$$

It can be shown [see for example, Ghose (1963), pp 407 - 411] that the imaginary parts of μ_{eff} and ϵ are proportional to the energy dissipated per cycle in the magnetic and dielectric losses, respectively, while the real parts of μ_{eff} and ϵ are proportional to the energy stored in the magnetic and electric fields. Thus, from the definitions of Q_{μ} and Q_{ϵ} , it follows that

$$Q_{\mu} = \frac{\mu'_{\text{eff}}}{\mu''_{\text{eff}}} \quad (4-31)$$

$$Q_{\epsilon} = \frac{\epsilon'}{\epsilon''} = \frac{1}{\tan \delta} \quad (4-32)$$

where $\tan \delta$ is the dielectric loss tangent defined in equation (1-51).

The real and imaginary parts of μ_{eff} may be calculated by reference to the following equations,

$$\mu_{\text{eff}} = \frac{\mu^2 - \kappa^2}{\mu} \quad (3-13)$$

$$\mu = 1 + \chi \quad (1-39)$$

$$\chi = \frac{(\omega_0 + j\omega_c \alpha) \omega_m}{(\omega_0 + j\omega_c \alpha)^2 - \omega_c^2} \quad (1-24)$$

If, as in chapter 1

$$\sigma = \frac{\omega_0}{\omega_c} \quad (1-60)$$

$$\rho = \frac{\omega_m}{\omega_c} \quad (1-61)$$

then

$$\chi = \frac{(\sigma + j\alpha) \rho}{(\sigma + j\alpha)^2 \rho - 1} \quad (4-33)$$

$$\kappa = \frac{-\rho}{(\sigma + j\alpha)^2 - 1} \quad (4-34)$$

so that μ_{eff} can be written as

$$\mu_{\text{eff}} = \frac{1 - (\rho + \sigma + j\alpha)^2}{1 - (\sigma + j\alpha)(\rho + \sigma + j\alpha)} \quad (4-35)$$

When equation (4-35) is rationalized so that its real and imaginary parts can be separated, it is found after neglecting terms in α^2 and α^4 that the magnetic Q is given by

$$Q_{\mu} = \frac{(\rho\sigma + \sigma^2 - 1) [(\rho + \sigma)^2 - 1]}{\alpha\rho[(\rho + \sigma)^2 + 1]} \quad (4-36)$$

The calculation of Q_c is somewhat more involved; the procedure will only be briefly sketched here. By reference to Atwater (1962), p. 78, the skin effect losses on the conductors of the circulator can be calculated from

$$P_s = \frac{R_s}{2} \int_A H_t^2 dA \quad (4-37a)$$

where H_t is the tangential magnetic field at the conducting surfaces adjacent to the ferrite disks, and R_s is the surface resistance of the conductors,

$$R_s = \sqrt{\frac{\pi f_c \mu_0}{\eta}} \quad (4-37b)$$

In the last expression η is the conductivity of the ground plane and center strip conductors. The region of integration A must be taken over the areas of all conductors that are in contact with the fields in the ferrite disk. It should be noted that equation (4-37a) does not take into account the conductor losses of the striplines feeding the circulator.

The tangential field required in equation (4-37) may be written as

$$H_t^2 = \vec{H}_t \cdot \vec{H}_t^* = |H_{\phi}|^2 + |H_r|^2 \quad (4-38)$$

where

$$\vec{H}_t = H_\phi(r, \phi) \vec{a}_\phi + H_r(r, \phi) \vec{a}_r \quad (4-39)$$

The components H_ϕ and H_r for the $n=1$ mode can be calculated directly from equation (3-30) using the value of $a_{\pm 1}$ given by equation (3-44). After the transformation of the variable ϕ to the variable $\phi' = \phi + \frac{\pi}{6}$, these components can be written in the form,

$$H_\phi(r, \phi) = \frac{-E_{\max} Y_{\text{eff}}}{J_1(kR)} \left\{ \frac{\kappa}{\mu} \frac{J_1(kr)}{kr} \sin \phi' + J_1(kr) \cos \phi' \right\} \quad (4-40a)$$

$$H_r(r, \phi) = \frac{E_{\max} Y_{\text{eff}}}{J_1(kR)} \left\{ \frac{\kappa}{\mu} J_1'(kr) \cos \phi' - j \frac{J_1(kr)}{kr} \sin \phi' \right\}$$

In consideration of the fact that the ferrite disks are bounded by four conductors, the two groundplanes and the two sides of the center conductor, the integral of (4-36) may be written as

$$P_s = 2R_s \int_{r=0}^{r=R} \int_{\phi'=0}^{\phi'=2\pi} (|H_\phi|^2 + |H_r|^2) r dr d\phi' \quad (4-41)$$

The integration is rather tedious but can be performed with the assistance of appropriate identities involving Bessel functions to yield

$$P_s = \pi R_s Y_{\text{eff}}^2 R^2 \left[1 + \left(\frac{\kappa}{\mu} \right)^2 \right] \left[1 - \frac{1}{(kR)^2} \right]$$

where $kR = 1.84$.

(4-42)

Thus, from equation (4-29c) the value of Q_c will be given by

$$Q_c = \frac{\omega_c U}{P_s} = \frac{\omega_c d \epsilon_r \epsilon_0}{2 R_s Y_{eff}^2 (1 + \frac{\kappa^2}{\mu^2})} \quad (4-43)$$

By the use of equations (3-13), (3-31), and the definition of R_s , the above expression can be reduced to

$$\begin{aligned} Q_c &= d \sqrt{\pi f_c \eta \mu_0} \left[\frac{\mu (\mu^2 - \kappa^2)}{\mu^2 + \kappa^2} \right] \\ &= \left(\frac{d}{\tau} \right) \mu \left(\frac{\mu^2 - \kappa^2}{\mu^2 + \kappa^2} \right) \end{aligned} \quad (4-44)$$

where $\tau = \sqrt{\pi f_c \eta \mu_0}$, the skin depth of the conductors [Atwater (1962), p. 787]

4.3 Matching the Circulator to the Stripline

The circulator as a resonator structure will be matched to the connecting striplines by means of a quarter-wave transformer. Other matching methods are possible, and although the analysis may be more involved, the basic design philosophy will be much the same as is followed in this report.

This approach to the design of the appropriate matching transformer begins with standard transmission line theory. Figure 4.4.

Figure 4.4. Matching Transformer as a Transmission Line Section

represents a length s of transmission line of characteristic impedance z_{0Q} terminated in the arbitrary impedance z_r . This length of transmission line represents the quarter-wave matching transformer, and z_r is the equivalent circuit impedance appearing at the input port of the circulator. (Since circuit theory is now being employed, z_r should not be confused with the wave impedance z of equation 4-3). z_i , the driving point impedance of this section at terminated line, may be expressed as

$$z_i = z_{0Q} \frac{1 + \rho \text{EXP}(-2j\beta_s)}{1 - \rho \text{EXP}(-2j\beta_s)} \quad (4-45)$$

where β is the propagation constant given by

$$\beta = \frac{2\pi f}{v} = \frac{2\pi}{\lambda} \quad (4-46)$$

and where ρ is the reflection coefficient at the end of line, $x=0$.

$$\rho = \frac{z_r - z_{0Q}}{z_r + z_{0Q}} \quad (4-47)$$

In equation (4-46), v is the velocity of propagation along the line, and λ the wavelength. The development of these equations is given in Atwater, Section I.

If the stripline connected to the circulator has a characteristic impedance of z_0 , matching is achieved if $z_i = z_0$ at the resonant frequency f_c and if z_0 is made pure resistive at $f = f_c (1 \pm \Delta)$, where Δ is the fractional bandwidth of the circulator,

$$\Delta = \frac{f_h - f_c}{f_c} = \frac{f_c - f_L}{f_c} = \frac{f_h - f_L}{2f_c} \quad (4-48)$$

where f_h and f_e are respectively the high and low frequencies of the circulator passband. (This assumes that the bandwidth

characteristics are symmetric about f_c .) Also, it is required that the VSWR (or isolation) at $f = f_c (1 \pm \Delta)$ be within prescribed limits. This latter requirement can be satisfied by appropriate adjustment of the circulator d.c. biasing field.

Now, at resonance,

$$s = -\frac{\lambda}{4} = -\frac{v}{4f_c} = -\frac{\pi}{2\beta_0} \quad (4-49)$$

where, $\beta_0 = \frac{2\pi f_c}{v}$, is the transmission line propagation constant at the resonant frequency. It also follows that at resonance the argument of the exponentials in equation (4-45) becomes,

$$j2\beta_0 s = -j\pi \quad (4-50)$$

On the other hand, at the edge of the circulator passband, $f = f_c(1 \pm \Delta)$, so that argument of the exponentials in (4-45) can be calculated from

$$\beta(\pm \Delta) = \frac{2\pi f_c(1 \pm \Delta)}{v} \quad (4-51)$$

$$\beta(\pm \Delta)s = \frac{-4\pi f_c(1 \pm \Delta)}{v} \cdot \frac{v}{4f_c} = -\pi(1 \pm \Delta)$$

Since it has been assumed that the bandwidth behavior is symmetric about the resonant frequency f_c , the analysis will continue by dealing only with the one bandedge, $f = f_c(1 + \Delta)$.

The exponentials of equation (4-46) may be approximated at $f = f_c(1 + \Delta)$ by noting that

$$\begin{aligned}
\text{EXP}[-j2\beta (1 + \Delta)s] &= \text{EXP } j\pi (1 + \Delta) \\
&= \cos [\pi(1 + \Delta)] + j \sin[\pi (1 + \Delta)] \\
&= \cos \pi \cos \Delta - \sin \pi \sin \Delta \\
&\quad + j(\sin \pi \cos \Delta + \cos \pi \sin \Delta) \approx -1 - j\Delta
\end{aligned}$$

since

(4-52)

$$\begin{aligned}
\cos \Delta &\approx 1 \\
\sin \Delta &\approx \Delta
\end{aligned}
\quad \text{for small } \Delta.$$

For convenience, z_i , z_r , and ρ will now be normalized with respect to z_{0Q} , i.e.,

$$z_i = \frac{z_i}{z_{0Q}} \quad z_r = \frac{z_r}{z_{0Q}} \quad 4-53a)$$

$$\rho = \frac{z_r - 1}{z_r + 1} = \frac{1 - Y_r}{1 + Y_r} \quad (4-53b)$$

where

$$Y_r = \frac{1}{z_r} \quad (4-54)$$

With these normalized values, and by using the approximations of (4-52), equation (4-45) takes the form,

$$z_r = \frac{1 - \rho - j\Delta}{1 + \rho + j\Delta} \quad (4-55)$$

As discussed in Atwater, Chapter 4, the wave impedances and admittances appearing at a waveguide or resonator port are proportional to equivalent circuit impedances and

admittances. Thus, the equivalent circuit admittance y_r , equation (4-54), appearing at the input port of the circulator at resonance can be written in terms of the input wave admittance at resonance as

$$y_r(f_c) = y_{ro} = KG_R \quad (4-56)$$

where G_R is given by equation (3-57) and K is a factor of proportionality to be determined.

Since the matching section is a quarter-wave at resonance, the characteristic impedance of the matching section must be (Atwater, p. 19),

$$z_{oQ} = \sqrt{z_o z_{ro}} \quad (4-57)$$

where $z_{ro} = z_r(f_c)$, the equivalent circuit impedance at the input port of the circulator at resonance. Thus, it follows that

$$Y_{ro} = \frac{1}{z_{ro}} = \frac{z_o}{z_{oQ}^2} \quad (4-58)$$

The normalization of Y_{ro} with respect to z_{oQ} gives

$$Y_{ro} = \frac{z_o}{z_{oQ}} \quad (4-59)$$

From equations (4-56) and (4-59), the factor of proportionality K can be evaluated as

$$K = y_{ro} \frac{1}{G_R} = \frac{z_o}{z_{oQ}} \frac{1}{G_R} \quad (4-60)$$

If it is assumed that the real part of wave admittance G_R varies but slightly in the neighborhood of resonance, the equivalent normalized circuit admittance at the input port of the circulator for $f = f_c(1 + \Delta)$ may be written as

$$y_r = \frac{y_{ro}}{G_R} [G_R + jB_R] \quad (4-61)$$

with B_R the wave susceptance at $f = f_c(1 + \Delta)$. If the phase angle θ of the wave admittance is introduced, i.e.,

$$\theta = \tan^{-1} \frac{B_R}{G_R} \quad (4-62)$$

then (4-61) can be expressed as

$$y_r = y_{ro} (1 + j \tan \theta) \quad (4-63)$$

The substitution at the value of y_r given by (4-63) into (4-53b) gives the value of the reflection coefficient at $f = f_c(1 + \Delta)$ as

$$\rho(+\Delta) = \frac{1 - y_{ro}^2 \sec^2 \theta - 2 y_{ro} \cos \theta}{1 + 2 y_{ro} + y_{ro}^2 \sec^2 \theta} \quad (4-64)$$

Placing this value of ρ into equation (4-55) produces the normalized driving point impedance at the input of the matching section,

$$z_i(+\Delta) = \frac{A + jC}{D - jC} \quad (4-65a)$$

$$A = 2y_{ro} (1 + y_{ro}^2 \sec^2 \theta) \quad (4-65b)$$

$$D = 2 (1 + y_{ro}) \quad (4-65c)$$

$$C = 2y_{ro} \tan \theta - \Delta(1 + 2y_{ro} + y_{ro}^2 \sec^2 \theta) \quad (4-65d)$$

Thus, the requirement that Z_i be pure resistive (or z_i pure real) at $f = f_c(1 + \Delta)$ yields one basic design relation between y_{ro} , Δ , and θ .

$$2y_{ro} \tan \theta = \Delta (1 + 2y_{ro} + y_{ro}^2 \sec^2 \theta) \quad (4-66)$$

A second basic design relation for the quarter wave matching transformer can be deduced from considering the stripline VSWR at $f = f_c(1 + \Delta)$. From equation (2-55), it can be seen that the procedure is equivalent to considering the minimum isolation desired within the passband of the circulator. In terms of the reflection coefficient ρ' at the input to the quarter-wave transformer, the VSWR on the input stripline is

$$\text{VSWR} = \frac{1 + |\rho'|}{1 - |\rho'|} \quad (4-67)$$

If condition (4-66) is satisfied, then z_i for $f = f_c(1 + \Delta)$ can be written as

$$z_i = \frac{Z_i}{Z_{OQ}} = \frac{y_{ro}(1 + y_{ro} \sec^2 \theta)}{y_{ro} + 1} \quad (4-68)$$

In turn, ρ' may be written as

$$\rho' = \frac{z_i - 1}{z_i + 1} \quad (4-69)$$

with $z'_i = Z_i/Z_o$, i.e., z_i is now normalized to the stripline characteristic impedance. Now z'_i is easily related to z_i by noting that

$$z'_i = \frac{Z_i}{Z_o} = \frac{Z_i}{Z_{OQ}} \frac{Z_{OQ}}{Z_o} = \frac{z_i}{y_{ro}} \quad (4-70)$$

Thus, in terms of z_i , ρ' can be expressed as

$$\rho' = \frac{z_i - y_{ro}}{z_i + y_{ro}} \quad (4-71)$$

or by using the value of z_i in (4-68),

$$\rho' = \frac{y_{ro} \tan^2 \theta}{2 + y_{ro} (1 + \sec^2 \theta)} \quad (4-72)$$

After the insertion of the above value of ρ' into (4-67), plus some algebraic manipulation, a second basic relation between y_{ro} , θ , and VSWR is obtained:

$$\text{VSWR} = 1 + \frac{y_{ro} \tan^2 \theta}{y_{ro} + 1} \quad (4-73)$$

It should be noted that if y_{ro} is large, i.e., the circulator rather heavily loaded, then (4-73) reduces to the expression given by Faye and Comstock (1965),

$$\text{VSWR} \approx \sec^2 \theta \quad (4-74)$$

A more convenient form of equation (4-73) is obtained by solving for y_{ro} in terms of the VSWR,

$$y_{ro} = \frac{1 - V}{V - 1 - \tan^2 \theta} \quad (4-75)$$

where V is the VSWR on the stripline at $f = f_c(1 \pm \Delta)$.

4-4 Adjustment of Circulator d.c. Biasing Field for Suitable VSWR and Isolation.

Equation (4-75) specifies the necessary value of characteristic impedance for the quarter-wave matching transformer for a given VSWR and $\tan \theta$. The following analysis will show that $\tan \theta$ is dependent upon the adjustment of the circulator d.c. biasing field.

From the equivalent circuit of Figure 4-3, it can be seen that the wave admittance at the input port of the circulator has a positive susceptance B_R for $f > f_c$. Furthermore, if B_R is expanded in a power series in δ , where

$$\delta = \frac{f - f_c}{f_c} \quad (4-76)$$

and if δ is sufficiently small so that all terms in δ higher than first degree can be neglected, then

$$B_R = K_1 \delta \quad (4-77)$$

The constant K_1 can be evaluated by noting that at $f = f_c(1 + \Delta)$, $\delta = \Delta$, and

$$Y_R = G_R + jB_R = G_R(1 + j \tan \theta) \quad (4-78)$$

so that $K_1 = G_R \frac{\tan \theta}{\Delta}$, and

$$Y_R(\delta) \approx G_R [1 + j \frac{\tan \theta}{\Delta} \delta] \quad (4-79)$$

Since the circulator as a resonator is represented by a shunt R-L-C circuit, the loaded Q of this circuit can be defined as (Atwater, Chapter 5),

$$Q_L = \frac{f_c}{2(f_{\frac{1}{2}} - f_c)} \quad (4-80)$$

where $f = f_{\frac{1}{2}}$ is the frequency where

$$|B_R| = G_R \quad (4-81)$$

or from (4-79),

$$\frac{\tan \theta}{\Delta} \Delta_Q = 1 \quad (4-82)$$

with

$$\Delta_Q = \frac{f_{\frac{1}{2}} - f_c}{f_c} \quad (4-83)$$

Thus, Q_L can be written as

$$Q_L = \frac{1}{2\Delta_Q} = \frac{\tan \theta}{2\Delta} \quad (4-84)$$

since from (4-71), $\Delta_Q = \Delta \cot \theta$.

Now by equation (4-25), Q_L is determined by the gyrotropic ratio κ/μ , i.e.,

$$Q_L = 0.69 \left| \frac{\mu}{\kappa} \right| \quad (4-25)$$

so that by proper adjustment of κ/μ through the d.c. biasing field, the desired value of $\tan \theta$ can be obtained. This desired value of $\tan \theta$ can be calculated by eliminating y_{ro} from equations (4-66) and (4-73). A cubic equation in $X = \tan \theta$ results,

$$X^3 - \frac{2FX^2}{\Delta} + F^2X + \frac{2F^2}{\Delta} = 0 \quad (4-85)$$

where F is VSWR - 1. The correct value of $X = \tan \theta$ is the smallest positive root of this cubic equation.

Section 5
A COMPLETE DESIGN PROCEDURE FOR
A STRIPLINE CIRCULATOR

The concluding section of this report will specify a complete design procedure for a Y-junction stripline circulator with quarter-wave matching. Following each step of the procedure, comments relating that step to the appropriate part of the circulator analysis will be made. The input parameters to this procedure are the isolation, isolation bandwidth, operating frequency, and 4π Ms of the ferrite.

- 1.) Select an isolation factor,

$$\gamma = \frac{|E_3|}{|E_1|} \quad (5-1)$$

at the following frequencies $f = f_c(1 \pm \Delta)$, where f_c is the center frequency and Δ the fractional bandwidth of the circulator as given by equation (4-48). E_1 and E_3 are transverse stripline fields at the input and isolated port, respectively.

- 2.) The VSWR for the input port is given by,

$$\text{VSWR} = \frac{1 + \gamma}{1 - \gamma} \quad \text{or} \quad \gamma = \frac{\text{VSWR} - 1}{\text{VSWR} + 1} \quad (5-2)$$

Comment: Equation (5-2) is a consequence of scattering matrix theory, and follows as a result of the discussion of Section 2.6.

- 3.) The phase angle θ between the real and imaginary parts of the circulator input wave admittance may be found by solving for the smallest positive root of the following cubic equation,

$$X^3 - \frac{2FX^2}{\Delta} + F^2X + \frac{2F^2}{\Delta} = 0 \quad (4-85)$$

where,

$$X = \tan \theta \quad (5-3)$$

and

$$F = \text{VSWR} - 1 \quad (5-4)$$

Comment: Equation (4-85) results from treating the circulator near its operating frequency as a lightly coupled resonator, represented by a shunt R-L-C circuit. Complete details of the derivation of (4-85) appear in Section 4.

- 4.) The loaded Q of the ferrite is found from

$$Q_L = \frac{\tan \theta}{2\Delta} \quad (4-83)$$

Comment: Equation (4-83) is, as was equation (4-48) above, a consequence of treating the circulator as a resonator.

- 5.) In turn, the loaded Q may be related to the gyrotropic ratio κ/μ by

$$\left| \frac{\kappa}{\mu} \right| = \frac{0.69}{Q_L} \quad (5-5)$$

Reasonable values of $|\frac{\kappa}{\mu}|$ fall between 0.2 and 0.6. It should be noted that $\kappa < 0$ for above resonance operation, and $\kappa > 0$ for below resonance operation. Comment: The quantities κ and μ are elements of the ferrite Polder tensor, derived in Section 1.3. As such, they are dependent upon the circulator d.c. biasing field, as discussed in Section 1.6.

Equation (5-5) follows from the calculation of the ratio of the energy stored per cycle in the circulator to the power transferred through the circulator; this derivation is carried out in Section 4.2.

Large values of $|\frac{\kappa}{\mu}|$ indicate operation too near gyromagnetic resonance with corresponding large magnetic losses; small values of $|\frac{\kappa}{\mu}|$ lead to restrictive circulator bandwidth. (Fay & Comstock, 1965).

- 6.) The normalized saturation magnetization is defined as

$$\rho = \frac{-4\pi M_s \gamma_e / \mu_0}{f_e} \quad (1-61)$$

with $4\pi M_s / \mu_0$ the saturation magnetization of the ferrite in oersteds, γ_e the gyromagnetic constant,

$$\gamma_e = -2.8 \text{ MHz/oersted} \quad (1-57)$$

and f_c the operating frequency of the circulator.

Comment: Equations (1-57) and (1-61), relating to the material parameters of the ferrite, are discussed in Sections 1.3 and 1.6.

7.) The normalized internal biasing field,

$$\gamma = \frac{-\gamma_e H_i}{f_c} \quad (1-60)$$

where H_i in oersteds is the internal biasing field, may be determined from,

$$\sigma = \frac{1}{2} [-\rho + \sqrt{\rho^2 + 4(1 - \rho \frac{\mu}{K})}] \quad (5-6)$$

Values of σ in the range $0.75 < \sigma < 1.15$ should be avoided because of high circulator insertion loss near gyromagnetic resonance.

One restriction appears when applying this design equation to below resonance operation, i.e., since $\kappa > 0$ for below resonance operation, then

$$\rho^2 \geq 4(\rho \frac{\kappa}{\mu} - 1)$$

or,

$$\rho \geq 2[\frac{\mu}{K} + \sqrt{(\frac{\mu}{K})^2 - 1}]$$

This inequality implies that below resonance operation is only satisfactory for a low operating frequency, high saturation magnetization, or large gyrotropic ratio.

Comment: The gyrotropic ratio $\frac{\kappa}{\mu}$ may be calculated in terms of $4\pi M_s$ and the internal biasing field from equations (1-58) and (1-59), then the resultant expression solved for σ in order to produce equation (5-5).

- 8.) The internal biasing field H_i is related to the the flux density just external to the ferrite disc, B_{ext} , by

$$H_i \approx (B_{ext} - 4\pi M_s) / \mu_0 \quad (5-7)$$

B_{ext} being given in gauss.

Comment: Equation (5-7) is merely a reflection of the fact that the magnetic flux density must be continuous at the ferrite boundary so that

$$B_{int} = B_{ext} = \mu_0 H_i + 4\pi M$$

where M is magnetic moment per unit volume.

Since the ferrite disc is biased with a strong magnetic field, M is equal to the saturation magnetic moment per unit volume (M_s) and equation (5-7) follows.

- 9.) The effective r.f. permeability may be calculated from

$$\mu_{eff} = \frac{1 - (\rho + \sigma)^2}{1 - \sigma(\rho + \sigma)} \quad (5-8)$$

Comment: The effective r.f. permeability is developed in Section 3.3, equation (3-13), and is calculated through use of equations (1-58) and (1-59).

- 10.) For first order mode operation in the ferrite disk, which gives the smallest disk radius and lowest loss, the radius R is given by

$$R = \frac{1.84}{k} \quad (5-9)$$

where,

$$k = \frac{2\pi f_c \sqrt{\epsilon_r \mu_{eff}}}{c} \quad (5-10)$$

with ϵ_r the relative dielectric constant of the ferrite disk, and c the free space velocity of electromagnetic waves, $c = 3 \times 10^8$ meter/second.

Comment: The equation for the ferrite radius is developed in Section 3-7, equation 3-54, whereas k is defined by equation 3-16 of Section 3-3.

- 11.) The necessary characteristic impedance Z_{OQ} for the quarterwave matching transformer is determined from

$$\frac{Z_O}{Z_{OQ}} = \frac{-F}{F - \tan^2 \theta} \quad (5-11)$$

where F is defined by equation (5-4) and Z_O is the characteristic impedance of the striplines to the circulator.

Comment: Equation (5-11) is a consequence of the analysis in Section 4-3 and follows directly from equation (4-75).

- 12.) It will be assumed that the stripline width v and the quarter-wave matching transformer stripline width are identical. Then if ϵ_{ir} is the relative dielectric constant associated with Z_O , the required relative dielectric constant for Z_{OQ} is

$$\epsilon_{Qr} = \epsilon_{ir} \left(\frac{Z_O}{Z_{OQ}} \right)^2 \quad (5-15)$$

Comment: Equation (5-12) results because the characteristic impedance of a line is proportional to $C^{-\frac{1}{2}}$, C the capacitance per unit length, while in turn C is directly proportional to the dielectric constant.

- 13.) The design procedure just outlined will be valid only if

$$V < R \quad (3-42)$$

Comment: Equation (3-42) implies that the circulator is lightly coupled to the stripline, a circumstance that vastly simplifies the analysis.

The previous thirteen steps are the basic design procedure for a lightly coupled Y-junction stripline circulator. The remaining steps of the procedure are essentially calculations that yield a lower bound on the circulator insertion loss. Certainly the actual loss will be greater, since in the derivation of these calculations no attempt has been made to include such factors as stripline and matching transformer losses, fringing field losses, and the effect of any circulator asymmetries.

- 14.) The lower bound of the circulator loss can be calculated from

$$I.L. = 10 \log_{10} \left[1 - \frac{Q_L}{Q_0} \right] \quad (5-13)$$

where

$$\frac{1}{Q_0} = \frac{1}{Q_\mu} + \frac{1}{Q_\epsilon} + \frac{1}{Q_c} \quad (4-28)$$

Comment: The derivation of equation (5-13) is discussed in Section 4.2 of this report, as are the calculations of Q_μ , Q_ϵ , and Q_c which follow.

- 15.) Q_μ is associated with the magnetic losses in the ferrite and is given by

$$Q_\mu = \frac{[\rho\sigma + \sigma^2 - 1] [(\rho + \sigma)^2 - 1]}{\alpha\rho [(\rho + \sigma)^2 + 1]}$$

Here α is the gyrotropic damping constant defined in Section 1.4:

$$\alpha = - \frac{\gamma_e \Delta H}{2f_T} \quad (5-14)$$

ΔH being the gyromagnetic resonance bandwidth at the test frequency f_T .

Comment: Equation (5-14) follows from equation (1-49), with ΔH being in oersteds and γ_e given by equation (1-57).

- 16.) Q_ϵ is associated with the ferrite dielectric losses,

$$Q_\epsilon = \frac{1}{\tan \delta} \quad (4-32)$$

where $\tan \delta$ is the dielectric loss tangent of the ferrite material.

- 17.) Q_c is associated with the losses in the conductors bounding the ferrite disk,

$$Q_c = \frac{d}{\tau} \mu \frac{\mu^2 - \kappa^2}{\mu^2 + \kappa^2} \quad (4-44)$$

with,

$$\mu = 1 + \frac{\rho\sigma}{\sigma^2 - 1} \quad (1-58)$$

$$\kappa = \frac{-\rho}{\sigma^2 - 1} \quad (1-59)$$

In equation (4-44) d is the thickness of one ferrite disk, and τ the conductor skin depth at the operating frequency,

$$\tau = (\pi f_c \mu_0 \eta)^{-\frac{1}{2}} \quad (5-15)$$

with μ_0 the free space permeability and η the conductivity of the conductor.

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