RESEARCH ON
ATMOSPHERIC DIFFUSION AND TURBULENCE

S. K. KAO
Principal Investigator

Final Report Prepared for the U. S. Atomic Energy Commission
Under Contract AT(11-1)-1585

Department of Meteorology
University of Utah
Salt Lake City, Utah

July 1972
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SUMMARY OF RESEARCH PROGRESS

The overall objective of the total project is to investigate the characteristics of the large-scale turbulence and diffusion of pollutants in the atmosphere. The investigation is made from both the analytical and numerical approaches with the use of the observed wind data collected over both the Northern and Southern Hemispheres. Some of the results obtained from these studies are reported in twenty published scientific papers listed as follows:


The attached are a summary report of research and reprints of published papers on the large-scale turbulence and diffusion in the atmosphere.
MECHANISM FOR THE LARGE SCALE ATMOSPHERIC DIFFUSION

by

S. K. Kao

University of Utah, Salt Lake City, Utah

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Chapter 1

Spectra and Governing Equations for the Large-scale Atmospheric Motion and Transports in Wavenumber-Frequency Space

1-1. Introduction

1-2. Power- and Cross-Spectra in Wavenumber-Frequency Space


1-4. Governing Equations for the Large-scale Atmospheric Motion and Transports in Physical Space.

1-5. Governing Equations for the Large-scale Atmospheric Motion and Transports in Wavenumber-Frequency Space.
1-1. Introduction

One of the effective ways of studying the large-scale atmospheric motion and its relation to the diffusion process is the analysis of the power- and cross-spectra of the large-scale motion and transport in the atmosphere. The power-spectra, which concern primarily with the kinetic, potential and internal energies, are basic to the understanding of the mechanism of turbulence. The cross-spectra, which deal primarily with the transport and conversion of energies, are fundamental in the maintenance of the general circulation in the atmosphere. In the first three chapters the spectral characteristics of the large-scale motion and transport in the atmosphere, which are related to and provide the mechanism for the dispersion of particles by the large-scale motion, are presented.

Studies of the longitude spectra (Benton and Kahn, 1958; Eliasen, 1958; Saltzman, 1958; Barrett, 1961; Van Mieghem, 1961; Kao, 1954), the Eulerian time spectra (Estoque, 1955; Van der Hoven, 1957; Chiu, 1960), and the Lagrangian time spectra (Kao, 1962, 1965; Kao and Bullock, 1964; Kao and Gain, 1968; Kao and Powell, 1969) have recently been made. These studies have provided a great deal of information regarding the contribution of the large-scale motion due either to the longitude- or time-eddies. In the atmosphere, however, motion at a point is generally affected by waves of various sizes, moving at various speeds. To gain an insight into the structure of the large-scale atmospheric motion, it is necessary to analyze the motion and transport in the wavenumber-frequency space. The most general way of analyzing the spectra of the atmospheric motion is
to expand the velocity field in spherical surface harmonics (Silver-

However, computations of this type for a large number of data needs
a great deal of computer time. Furthermore, in view of the fact
that in the study of the large-scale dispersion of particles we are
primarily interested in the rate of dispersion of the particles in
the zonal and meridional directions, the spectra of the large-scale
atmospheric motion and transports can conveniently be examined in
the longitude-time space. Such an attempt has recently been made
(Kao, 1968a). In this chapter, the spectra of the large-scale atmo-
spheric motion and transports at various altitudes and latitudes
in wavenumber-frequency, wavenumber, and frequency domains will be
presented. Since the data used in earlier investigations are limited
either in the length of time or in the area of coverage, the spectra
presented in this chapter are based on the recent computation (Kao,
Wendell, and Noteboom, 1966; Kao, Sagendorf and Tsay, 1969), made with
the use of the northern hemispheric data analyzed by the National
Meteorological Center.

1-2. **Power- and Co-Spectra in the Wavenumber-Frequency Space**

Let \( q(\lambda, t) \) be a real, single-valued function, which is piecewise
differentiable in a normalized domain: \( 0 \leq \lambda, t \leq 2\pi \). In this study,
\( \lambda \) and \( t \) stand for respectively for longitude and time. \( q(\lambda, t) \) has a
Fourier transform which may be written

\[
Q(k, n) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} q(\lambda, t) e^{-i(k\lambda + nt)} d\lambda dt
\]  

(1-2.1)
where $Q$ is the complex coefficient, $k$ and $n$ are respectively the wave number and frequency. The inverse transform of (1-2.1) gives $q(\lambda,t)$ expressed in terms of its complex coefficient as follows:

$$q(\lambda,t) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} Q(k,n) e^{i(k\lambda+nt)} dn$$

(1-2.2)

Here the summation of the complex coefficient $Q$ with respect to the integer wave numbers is the consequence of the cyclic distribution of $q(\lambda,t)$ along latitude circles.

For convenience of computations in this study, we express

$$Q(k,n) = Q_r(k,n) + iQ_i(k,n)$$

(1-2.3)

where

$$Q_r(k,n) = \frac{1}{4\pi^2} \int_{0}^{2\pi} \int_{0}^{2\pi} q(\lambda,t) \cos (k\lambda+nt) d\lambda dt$$

and

$$Q_i(k,n) = -\frac{1}{4\pi^2} \int_{0}^{2\pi} \int_{0}^{2\pi} q(\lambda,t) \sin (k\lambda+nt) d\lambda dt$$

are respectively the real and imaginary part of $Q(k,n)$.

It can easily be shown that

$$Q^*(\pm k,\mp n) = Q(\mp k,\pm n)$$

$$Q_r(\pm k,\mp n) = Q_r(\mp k,\pm n), \quad Q_i(\pm k,\mp n) = -Q_i(\mp k,\pm n)$$

$$Q(\pm k,\mp n)Q^*(\pm k,\mp n) = Q_r^2(\pm k,\pm n) + Q_i^2(\pm k,\mp n) = |Q(\pm k,\mp n)|^2$$

(1-2.4)

where $Q^*$ denotes the conjugate of $Q$.

Consider the same conditions for another scalar function $s(\lambda,t)$ with a Fourier transform $S(k,n)$. It can be shown that for function $s(\lambda,t)$
and \( q(\lambda, t) \) we have

\[
\frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} s(\lambda, t)q(\lambda, t) e^{-i(k\lambda + nt)} d\lambda \, dt = \sum_{j=-\infty}^{\infty} \int S(j, m)Q(k-j, n-m) dm \tag{1-2.5}
\]

where

\[
S(j, m) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} s(\lambda, t) e^{-i(j\lambda + mt)} d\lambda \, dt
\]

Letting \( k, n \to 0 \), we have the generalized Parseval's formula (Kao, 1968a)

\[
\frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} s(\lambda, t)q(\lambda, t) d\lambda \, dt = \sum_{j=-\infty}^{\infty} \int S(j, m) Q(-j, -m) dm \tag{1-2.6}
\]

It can further be shown that

\[
\frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} s(\lambda, t)q(\lambda, t) d\lambda \, dt = \frac{1}{4} \sum_{k=-\infty}^{\infty} \left[ S(k, n)Q_r(k, n) + S(-k, -n)Q_i(k, n) + S(k, -n)Q_r(k, -n) + S(-k, n)Q_i(k, -n) \right] dn \tag{1-2.7}
\]

In view of the integrand of the right-hand side of the above equation being an even function, equation (1-2.7) may be written

\[
\frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} s(\lambda, t)q(\lambda, t) d\lambda \, dt = \int_0^{\infty} \left[ S_r(o, n)Q_r(o, n) + S_i(o, n)Q_i(o, n) \right] + \sum_{k=1}^{\infty} \left[ S_r(k, n)Q_r(k, n) + S_i(k, -n)Q_i(k, n) \right] dn \tag{1-2.8}
\]

Here relation (1-2.4) has been employed.
Denote the cross-spectrum of \(s(\lambda,t)\) and \(q(\lambda,t)\) due to eddies of wave-number \(k\) and frequency \(n\), moving toward the direction of increasing and decreasing longitude respectively by

\[
E_{sq}(0,\pm n) = S_r(0,\pm n)Q_r(0,\pm n) + S_i(0,\pm n)Q_i(0,\pm n)
\]

\[
E_{sq}(k,\pm n) = 2[S_r(k,\pm n)Q_r(k,\pm n) + S_i(k,\pm n)Q_i(k,\pm n)]
\]  

(1-2.9)

for \(k \neq 0\).

The contribution of \(s(\lambda,t)\) and \(q(\lambda,t)\) integrated over a latitude circle and over a normalized time interval \(2\pi\) may then be expressed in terms of the sum of \(E_{sq}(k,n)\) and \(E_{sq}(k,-n)\) integrated over the frequency and wave-number domain.

\[
\frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} s(\lambda,t)q(\lambda,t) d\lambda dt = \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} [E_{sq}(k,n) + E_{sq}(k,-n)] dn
\]  

(1-2.10)

The above equation may also be expressed

\[
\frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} s(\lambda,t)q(\lambda,t) d\lambda dt = \sum_{k=0}^{\infty} \left\{E_{sq}(k,+) + E_{sq}(k,-)\right\}
\]  

(1-2.11)

or

\[
\frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} s(\lambda,t)q(\lambda,t) d\lambda dt = \int_0^{\infty} \left\{E_{sq}(+n) + E_{sq}(-n)\right\} dn
\]  

(1-2.12)

where

\[
E_{sq}(k,\mp n) = \int_{-\infty}^{\infty} E_{sq}(k,\pm n) dn
\]  

(1-2.9a)

is the cross-spectrum of \(s(\lambda,t)\) and \(q(\lambda,t)\) due to eddies of wave number \(k\) and all frequencies, moving respectively in the direction of increasing
and decreasing longitude, and

\[ E_{sq}(\pm n) = \sum_{k=0}^{+\infty} E_{sq}(k, \pm n) \]  

(1-2.9b)

is the cross-spectrum due to eddies of frequency \( n \) and all wave numbers, moving respectively in the direction of increasing and decreasing longitude.

To compare the longitude-time power- and co-spectra for different latitudes and time intervals, it is convenient to introduce the following normalized cross-spectra:

\[ F_{sq}(k, \pm n) = \frac{2[S_r(k, \pm n)Q_r(k, \pm n) + S_i(k, \pm n)Q_i(k, \pm n)]}{\frac{1}{2}\int \int s(\lambda, t)q(\lambda, t)d\lambda dt} \]

for \( k = 0 \), the factor 2 in numerator should be replaced by 1.

\[ F_{sq}(k, \pm) = \int_{-\infty}^{+\infty} F_{sq}(k, \pm n) dn \]

\[ F_{sq}(\pm) = \sum_{k=0}^{+\infty} F_{sq}(k, \pm n) \]  

(1-2.13)

such that

\[ \sum_{k=0}^{+\infty} [F_{sq}(k, +) + F_{sq}(k, -)] = 1 \]

\[ \int_{-\infty}^{+\infty} [F_{sq}(+n) + F_{sq}(-n)]dn = 1 \]  

(1-2.14)

For the analysis of the power-spectrum of scalar quantity \( q(\lambda, t) \), the quantities \( s(\lambda, t) \) and \( S(k, \pm n) \) in the equations of this section should respectively be replaced by \( q(\lambda, t) \) and \( Q(k, \pm n) \).
Computations of the longitude-time power- and co-spectra of the large-scale atmospheric motion and the meridional transports of angular momentum, and sensible heat for the year 1964 have been made (Kao, Wendell, 1966; Kao, Sagendorf, Tsay, 1969). Analysis of these results, which provide information regarding the mechanism of the large-scale turbulence and transports in the atmosphere, will be presented in the later sections.
1-3. Relationships Between Power Spectra in Wave-Number Frequency Space and those in Wave-Number Space and Frequency Space

We consider an atmospheric quantity with the real single-valued function \( q(X, t) \). If the function has a finite number of discontinuities over a period of time \( T \) which is normalized to \( 2\pi \), \( q(\lambda, t) \) is piecewise differentiable in the domain \( 0 \leq \lambda, t \leq 2\pi \), where \( t = (2\pi / T) t' \). Under these conditions \( q(\lambda, t) \) may be transformed into a wave-number frequency space (Kao, 1968) or into wave-number space for particular values of time (Saltzman, 1957) or into frequency space for specified values of longitude.

The Fourier transforms and their inverse transforms for these three cases are as follows:

In wave-number frequency space

\[
Q(k, n) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} q(\lambda, t) e^{-i(k\lambda + nt)} d\lambda dt \tag{1-3.1}
\]

\[
q(\lambda, t) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} Q(k, n) e^{i(k\lambda + nt)} dn \tag{1-3.2}
\]

In wave-number space only

\[
Q(k, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} q(\lambda, t) e^{-ik\lambda} d\lambda \tag{1-3.3}
\]
\[ q(\lambda,t) = \sum_{k=-\infty}^{\infty} Q(k,t)e^{ik\lambda} \]  

(1-3.4)

In frequency space only:

\[ Q(\lambda,n) = \frac{1}{2\pi} \int_{0}^{2\pi} q(\lambda,t)e^{-int} dt \]  

(1-3.5)

\[ q(\lambda,t) = \int_{-\infty}^{\infty} Q(\lambda,n)e^{int} dn \]  

(1-3.6)

where \( k \) and \( n \) represent wave number and frequency respectively.

The transform pairs presented in (1-3.1) through (1-3.6) are three distinct representations of the function \( q(\lambda,t) \). In wave-number space only, \( q(\lambda,t) \), for a given time, \( t \), is represented by a collection of waves around the latitude circle. Each wave has a distinct length, determined by its wave number, \( k \), and a distinct amplitude and phase angle which may be determined from the complex value of \( Q(k,t) \). In frequency space, for a given longitude, \( \lambda \), and time period, \( T \), \( q(\lambda,t) \) is represented by a collection of oscillations in time. Each oscillation has a distinct period and amplitude which are determined by \( n \) and \( Q(\lambda,n) \). In wave-number frequency space, the function \( q(\lambda,t) \) is represented by a collection of waves around the latitude circle as in the wave-number transforms, but for each value of wave number, there is a distribution of waves over frequency, \( n \).
Since we are concerned with a power spectral analysis of the function \( q(\lambda, t) \) the quantity \( q^2(\lambda, t) \) expressed in terms of the three types of transforms is important. It can be shown that for the three cases we have the following:

In wave-number frequency space

\[
\frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} q^2(\lambda, t) d\lambda dt = \sum_{k=-\infty}^{\infty} \int Q(k,n)Q^*(k,n)dn
\]

\[
= \sum_{k=-\infty}^{\infty} \int |Q(k,n)|^2 dn
\]

(1-3.7)

where \( Q^*(k,n) = Q(-k,-n) \) denotes the complex conjugate of \( Q(k,n) \).

In wave-number space

\[
\frac{1}{2\pi} \int_0^{2\pi} q^2(\lambda, t) d\lambda = \sum_{k=-\infty}^{\infty} |Q(k,t)|^2
\]

\[
= |Q(0,t)|^2 + 2 \sum_{k=1}^{\infty} |Q(k,t)|^2
\]

(1-3.8)

where \( |Q(k,t)|^2 = Q(k,t)Q(-k,t) \).

In frequency space

\[
\frac{1}{2\pi} \int_0^{2\pi} q^2(\lambda, t) dt = \int \int |Q(\lambda,n)|^2 dn = 2 \int |Q(\lambda,n)|^2 dn
\]

(1-3.9)

where \( |Q(\lambda,n)|^2 = Q(\lambda,n)Q(\lambda,-n) \).

It has been shown (Kao, 1968) that (1-3.7) may be written
\[
\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} q^2(\lambda, t) d\lambda dt = \sum_{k=0}^{\infty} \int_0^{2\pi} \left[ E_{qq}(k,n) + E_{qq}(k,-n) \right] dn
\] (1-3.10)

where

\[
E_{qq}(0, \pm n) = |Q(0, \pm n)|^2
\]

\[
E_{qq}(k, \pm n) = 2|Q(k, \pm n)|^2 \quad \text{for } k \neq 0
\] (1-3.11)

are power spectra of the function \(q(\lambda, t)\).

In order to compare (1-3.8) with (1-3.10) we may average (1-3.8) with respect to time and get

\[
\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} q^2(\lambda, t) d\lambda dt = \frac{1}{2\pi} \int_0^{2\pi} |Q(0, t)|^2 dt + \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=1}^{\infty} |Q(k, t)|^2 dt
\] (1-3.12)

From (11) and (13) we see that

\[
\sum_{k=0}^{\infty} \int_0^{2\pi} \left[ E_{qq}(k,n) + E_{qq}(k,-n) \right] dn = \frac{1}{2\pi} \int_0^{2\pi} |Q(0, t)|^2 dt + \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=1}^{\infty} 2|Q(k, t)|^2 dt
\] (1-3.13)

This means that the sum of \(E_{qq}(k,n)\), power spectrum in wave-number frequency space, over wave number and integration over frequency yields the time-averaged sum over wave number of the power spectrum in wave-number space. We would like to show that the time-averaged wave-number spectra for each value of \(k\) can be obtained through an integration of \(E_{qq}(k,n)\) over frequency. If, for a given value of \(k\), \(E_{qq}(k,n)\) is integrated over frequency and the transform definition (1-3.1) is used, the result is
\[ \int_{-\infty}^{\infty} E_{qk}(k,n)dn = 2 \int_{-\infty}^{\infty} Q(k,n)Q^*(k,n)dn \]

\[ = 2 \int_{-\infty}^{\infty} \left\{ \frac{1}{4\pi^2} \int_{0}^{2\pi} \int_{0}^{2\pi} q(\lambda, t)e^{-i(k\lambda \cdot nt)} d\lambda dt \right\} \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} q(-k, t')e^{i(k\lambda' \cdot nt')} d\lambda' dt' \right\} dn \]

\[ = 2 \int_{-\infty}^{\infty} \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} Q(k, t)e^{-int} dt \right\} \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} Q(-k, t')e^{int'} dt' \right\} dn \]

where again the asterisk denotes the complex conjugate.

If the variable of integration \( t' \) is replaced with \( s \) in the above equation we may write

\[ \int_{-\infty}^{\infty} E_{qk}(k,n)dn = 2 \int_{-\infty}^{\infty} \left\{ \frac{1}{4\pi^2} \int_{0}^{2\pi} \int_{0}^{2\pi} Q(k,t)Q(-k,s)e^{i(s-t)} dtds \right\} dn \]  

(1-3.14)

Interchanging the order of integration results in

\[ \int_{-\infty}^{\infty} E_{qk}(k,n)kn = \frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} Q(kt)Q(-k,s) \left[ \int_{-\infty}^{\infty} e^{i(s-t)} dn \right] dtds \]  

(1-3.15)

The integral in the square bracket in (1-3.15) may be evaluated as follows:

\[ \int_{-\infty}^{\infty} e^{i(s-t)} dn = 2 \left\{ \lim_{N \to \infty} \frac{\sin N(s-t)}{s-t} \right\} \]  

(1-3.16)

Substituting this result into (1-3.15) yields

\[ \int_{-\infty}^{\infty} E_{qk}(k,n)dn = \frac{1}{\pi^2} \int_{0}^{2\pi} Q(k,t) \left\{ \lim_{N \to \infty} \frac{2\pi}{N} \int_{0}^{2\pi} Q(-k,s) \frac{\sin N(s-t)}{s-t} ds \right\} dt \]  

(1-3.17)

Using the transformation \( x/N = s-t \), the quantity in brackets becomes
\[
\lim_{N \to \infty} \int_0^N \frac{Q(-k,s) \sin N(t-s)}{t-s} \, ds = \lim_{N \to \infty} \int_{-Nt}^0 \frac{(2\pi-t)N}{x} \sin \frac{x}{x} \, dx
\]

\[
= Q(-k,t) \int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx
\]

\[
= \pi Q(-k,t) \tag{1.3.18}
\]

Equation (1.3.17) then becomes

\[
\int_{-\infty}^{\infty} E_{qq}(k,n) \, dn = \frac{1}{2\pi} \int_0^{2\pi} B(k)Q(k,t)Q(-k,t) \, dt \tag{1.3.19}
\]

where \(B(k)\) is 1 for \(k=0\) and 2 for \(k\neq0\) according to (1.3.11).

Eq. (1.3.19) indicates that for any wave number the time average of the power spectrum in wave-number space may be obtained from the wave-number frequency spectrum by integrating it over frequency domain for that wave number.

We also want to establish a relationship between the wave-number spectrum and a longitudinally-averaged frequency spectrum. We begin by summing the wave-number frequency spectrum over all wave numbers for the positive and negative values of a particular frequency.

\[
\sum_{k=0}^{\infty} \left( E_{qq}(k,n) + E_{qq}(k,-n) \right) = \sum_{k=0}^{\infty} B(k) \left\{ Q(k,n)Q^*(k,n) + Q(k,-n)Q^*(k,-n) \right\} \tag{1.3.20}
\]

Applying the definitions of the wave-number frequency transforms to (1.3.20) and then extracting those portions which define the frequency transforms yields
\[
\sum_{k=0}^{\infty} \left( E_{qq}(k,n) + E_{qq}(k,-n) \right) = \sum_{k=0}^{\infty} B(k) \left[ \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} Q(\lambda,n)e^{-ik\lambda} d\lambda \right\} \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} Q(\lambda',-n)e^{ik\lambda'} d\lambda' \right\} \\
+ \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} Q(\lambda',-n)e^{-ik\lambda'} d\lambda' \right\} \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} Q(\lambda,n)e^{ik\lambda} d\lambda \right\} \right] \tag{1-3.21}
\]

\[
= \sum_{k=0}^{\infty} \left[ \frac{1}{4\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} Q(\lambda,n)Q(\lambda',-n)e^{ik(\lambda'-\lambda)} d\lambda d\lambda' \\
+ \frac{1}{4\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} Q(\lambda,n)Q(\lambda',-n)e^{-ik(\lambda'-\lambda)} d\lambda d\lambda' \right] \tag{1-3.22}
\]

Combining the integrals, applying Euler's relationship and interchanging the order of the summation and integration results in

\[
\sum_{k=0}^{\infty} \left[ E_{qq}(k,n) + E_{qq}(k,-n) \right] = \frac{1}{2\pi^2} \int_{0}^{2\pi} Q(\lambda,n) \int_{0}^{2\pi} Q(\lambda',n) \left[ \sum_{k=0}^{\infty} B(k) \cos k(\lambda'-\lambda) \right] d\lambda' d\lambda 
\tag{1-3.23}
\]

From the relationship

\[
\sum_{k=1}^{N} \cos kx = \frac{\sin (N+\frac{1}{2})x - \sin \frac{x}{2}}{2 \sin \frac{x}{2}}
\]

and recalling \( B(k) = \begin{cases} 1 & k=0 \\ 0 & k \neq 0 \end{cases} \) we can show that

\[
\sum_{k=0}^{N} B(k) \cos k(\lambda'-\lambda) = \frac{\sin \left( (N+\frac{1}{2})(\lambda'-\lambda) \right)}{\sin \left( \frac{\lambda'-\lambda}{2} \right)} \tag{1-3.24}
\]

Substituting (1-3.24) into (1-3.23) we obtain
\[
\sum_{k=0}^{\infty} \left[ E_{qq}(k,n) + E_{qq}(k,-n) \right] = \frac{1}{2\pi^2} \oint \oint Q(\lambda,n) \left\{ \lim_{N \to \infty} \int Q(\lambda',n) \left[ \frac{\sin \left\{ (N+\frac{1}{2})(\lambda'-\lambda) \right\}}{\sin \left( \frac{\lambda'-\lambda}{2} \right) \lambda} \right] d\lambda' \right\} d\lambda \quad (1-3.25)
\]

If we let \( \lambda' - \lambda = \frac{\xi}{N+\frac{1}{2}} \), the integral in brackets becomes

\[
\lim_{N \to \infty} \int Q(\lambda',n) \left[ \frac{\sin \left\{ (N+\frac{1}{2})(\lambda'-\lambda) \right\}}{\sin \left( \frac{\lambda'-\lambda}{2} \right) \lambda} \right] d\lambda'
\]

\[
= \lim_{N \to \infty} \int_{-(N+\frac{1}{2})}^{(2\pi-\lambda)(N+\frac{1}{2})} Q(\frac{\xi}{N+\frac{1}{2}}, \lambda, n) \left[ \frac{\sin \frac{\xi}{2(N+\frac{1}{2})}}{\sin \frac{\xi}{2(N+\frac{1}{2})}} \right] \frac{d\xi}{N+\frac{1}{2}} \quad (1-3.26)
\]

As \( N \) is allowed to approach infinity, the limit of the denominator of the integrand becomes important. By expanding \( \sin \left( \frac{\xi}{2(N+\frac{1}{2})} \right) \) into a series it can be shown that

\[
\lim_{N \to \infty} (N+\frac{1}{2}) \sin \left( \frac{\xi}{2(N+\frac{1}{2})} \right) = \frac{\xi}{2}
\]

Applying this result to (1-3.26) we obtain

\[
\lim_{N \to \infty} \int_{-(N+\frac{1}{2})}^{(2\pi-\lambda)(N+\frac{1}{2})} Q(\frac{\xi}{N+\frac{1}{2}}, \lambda, -n) \left[ \frac{\sin \frac{\xi}{2(N+\frac{1}{2})}}{\sin \frac{\xi}{2(N+\frac{1}{2})}} \right] \frac{d\xi}{(N+\frac{1}{2})} = Q(\lambda, -n) \int_{-\infty}^{\infty} \frac{\sin \xi}{\xi} d\xi = 2\pi Q(\lambda, -n) \quad (1-3.27)
\]

Applying this result to (1-3.25) yields the result

\[
\sum_{k=0}^{\infty} \left[ E_{qq}(k,n) + E_{qq}(k,-n) \right] = 2 \left[ \frac{1}{2\pi} \oint Q(\lambda,n) Q(\lambda,-n) d\lambda \right] \quad (1-3.28)
\]
This result shows that the sum of the wave-number frequency spectrum over all wave numbers, for the positive and negative values of a given frequency, yields twice the longitudinal average of the frequency spectrum for the same frequency. The factor of two occurs because the power spectrum values for both positive and negative \( n \) are included on the left side of (1-3.28), and as demonstrated in (1-3.9), the frequency spectrum is an even function.

We note from the above discussion that time-averaged wave-number spectra and longitudinally-averaged frequency spectra may be obtained as special cases of wave-number frequency spectra. It is demonstrated that frequency spectra, obtained from wave-number frequency spectra, provide information about the energy of retrogressing waves, which cannot be obtained in the direct computation of the frequency spectra. The primary advantage of wave-number frequency spectra is the detailed information provided about the transient eddies in terms of not only the scale of the eddies, but the speed and direction of the motion of the eddies.
1-4. Equations for the Large-Scale Atmospheric Motion and Transports in the Physical Space.

In the longitude (λ) latitude (φ), pressure (p) coordinate system, the equations of motion, the hydrostatics equation, the continuity equation, and the energy equation may be written

\[
\left( \frac{\partial}{\partial t} + \frac{u}{a \cos \phi} \frac{\partial}{\partial \lambda} + \frac{v}{a \frac{\partial}{\partial \phi}} + \omega \frac{\partial}{\partial p} \right) u - \left( f + u \frac{\tan \phi}{a} \right) v = - \frac{g}{a \cos \phi} \frac{\partial z}{\partial \lambda} + F_1
\]

(1-4.1)

\[
\left( \frac{\partial}{\partial t} + \frac{u}{a \cos \phi} \frac{\partial}{\partial \lambda} + \frac{v}{a \frac{\partial}{\partial \phi}} + \omega \frac{\partial}{\partial p} \right) v + \left( f + u \frac{\tan \phi}{a} \right) u = - \frac{g}{a \frac{\partial}{\partial \phi}} + F_2
\]

(1-4.2)

\[
\frac{\partial z}{\partial p} + \frac{R}{g} \frac{T}{p} = 0
\]

(1-4.3)

\[
\frac{\partial \omega}{\partial p} + \frac{1}{a \cos \phi} \left( \frac{\partial u}{\partial \lambda} + \frac{\partial v \cos \phi}{\partial \phi} \right) = 0
\]

(1-4.4)

\[
c_p \left( \frac{\partial}{\partial t} + \frac{u}{a \cos \phi} \frac{\partial}{\partial \lambda} + \frac{v}{a \frac{\partial}{\partial \phi}} + \omega \frac{\partial}{\partial p} \right) T = R \frac{\omega T}{p} - h
\]

(1-4.5)

where \( a \) is the radial distance from the center of the earth, \( f \) is the Coriolis parameter, \( g \) is the gravity acceleration, \( z \) is the height of isobaric surfaces, \( \omega \) is the individual rate of change of pressure, \( T \) is the temperature, \( R \) is the gas constant, \( h \) is the rate of heat addition per unit mass, \( u \) and \( v \) are respectively the longitudinal and meridional component of the velocity, and \( F_1 \) and \( F_2 \) are respectively the longitudinal and meridional component of the frictional force. For large-scale atmospheric motion, \( F_1 \) and \( F_2 \) represent the sum of molecular frictional force and the Reynolds stress force due to eddies of high frequencies.
In the study of the maintenance of the general circulation in the atmosphere, we are particularly interested in the local rate of change of the kinetic and internal energies, the rates of the meridional flux of sensible heat and angular momentum. With the use of equations (1-4.1) to (1-4.5), they can be shown to be

\[
\frac{\partial}{\partial t} \left[ \frac{u^2 + v^2}{2} \right] = -\frac{1}{a \cos \varphi} \left\{ \frac{\partial}{\partial \lambda} \left[ u \left( u^2 + v^2 \right) \right] + \frac{\partial}{\partial \varphi} \left[ u \left( u^2 + v^2 \right) \cos \varphi \right] \right\} \\
- \frac{\partial}{\partial \rho} \left[ \omega \left( u^2 + v^2 \right) \right] - \frac{g}{a} \left[ -\frac{u}{\cos \varphi} \frac{\partial z}{\partial \lambda} + v \frac{\partial z}{\partial \varphi} \right] - \left( uF_1 + vF_2 \right) \quad (1-4.6)
\]

\[
\frac{\partial}{\partial t} \left[ c_v T \right] = -\frac{c_v}{a \cos \varphi} \left\{ \frac{\partial}{\partial \lambda} \left[ uT \right] + \frac{\partial}{\partial \varphi} \left[ vT \cos \varphi \right] \right\} + \frac{c_v}{c_p} \left[ \alpha \omega + \frac{c_v}{c_p} h \right] \quad (1-4.7)
\]

\[
\frac{\partial}{\partial t} \left[ c_p vT \right] = -\frac{c_p}{a \cos \varphi} \left\{ \frac{\partial}{\partial \lambda} \left[ uT \right] + \frac{\partial}{\partial \varphi} \left[ vT \cos \varphi \right] \right\} - \frac{c_p}{\rho} \frac{\partial}{\partial \rho} \left( wvT \right) \\
+ \frac{R}{\rho} \left( f + u \tan \varphi \right) c_p uT - \frac{g}{a} c_p T \frac{\partial z}{\partial \varphi} + \nu h + c_p TF_2 \quad (1-4.8)
\]

\[
\frac{\partial}{\partial t} \left( vua \cos \varphi \right) = -\left\{ \frac{\partial}{\partial \lambda} \left[ u^2 v \right] + \frac{\partial}{\partial \varphi} \left[ v^2 u \cos \varphi \right] \right\} - a \cos \varphi \frac{\partial}{\partial \rho} \left( \omega vu \right) \\
- a \cos \varphi \left\{ \left( f + u \tan \varphi \right) \left[ u^2 - v^2 \right] + \frac{g}{a} \left[ \frac{v}{\cos \varphi} \frac{\partial z}{\partial \lambda} + u \frac{\partial z}{\partial \varphi} \right] \right\} \\
- \left( vF_1 + uF_2 \right) \quad (1-4.9)
\]

These equations of transports will be compared with those transformed to the frequency wave-number space in the next section.
1-5. Governing Equations for the Large-Scale Atmospheric Motion and Transports in the Wavenumber-Frequency Space.

To transform the governing equations for the large-scale atmospheric motion and transports to the wave-number frequency space, we introduce the following notations for the Fourier coefficients of the quantities used in this study:

<table>
<thead>
<tr>
<th>q(λ,t,p,φ)</th>
<th>u v w z T h F₁ F₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q(k,n,p,φ)</td>
<td>U V W Z θ H G₁ G₂</td>
</tr>
</tbody>
</table>

Applying the Fourier transform formula (1-2.1) to the zonal component of the equation of motion (1-4.1) we obtain the complex coefficient of the zonal component of the velocity.

\[
U(k,Δn) = \frac{i}{n} \sum_{j=-∞}^{∞} \sum_{m=-∞}^{∞} \left\{ \frac{i}{a \cos φ} U(j,±m) \right. \left. U(k-j,±n±m) \right. \\
+ \frac{1}{a} U(j,±m) V(k-j,±n ± m) + V(j,±m) W(k-j,±n ± m) \\
- \frac{\tan φ}{a} U(j,±m) V(k-j,±n ± m) \left\} \right. \left. \left. \left. \frac{1}{n} \frac{g}{a \cos φ} kZ(k,±n) \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \left. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \r
contribution due to the interaction of the zonal and vertical components of the velocity. The fourth and sixth terms represent respectively the effect of the earth's curvature and rotation, whereas the fifth term represents the contribution of the pressure waves. The last term gives the effect of the frictional force in the atmosphere; for the large-scale atmospheric motion, it represents the complex coefficient of the Fourier transform of the molecular frictional force and the Reynolds stress force due to eddies of high frequency, which is nonlinear in nature.

Equation (1-5.1) may be used to analyze the linear and nonlinear effects of waves of various wavelength and frequency on the zonal component of the velocity. It may be noted that the first and fifth terms of the right-hand side of equation (1-5.1) are weighted by \( k n^{-1} \), whereas the rest of the terms of the right-hand side have a factor \( n^{-1} \). This indicates that waves of small frequency would generally contribute more to the zonal component of the velocity. As a by-product, equation (1-5.1) may be used to evaluate the Fourier coefficient of the frictional forces.

Applying the Fourier transform formula to equations (1-4.2), (1-4.3), (1-4.4) and (1-4.5), we obtain respectively the complex coefficient of the meridional component of the velocity, the height of the pressure surface, the individual rate of change of the pressure, and the temperature.
\[ V(k, \pm n) = \pm \frac{1}{n} \int_{-\infty}^{\infty} \frac{i}{a \cos \phi} j V(j, \pm m) U(k-j, \pm n \mp m) \]
\[ + \frac{\tan \phi}{a} U(j, \pm m) U(k-j, \pm n \mp m) \int \frac{1}{m} \left\{ \frac{1}{a} Z_2(k, \pm n) \right\} \]
\[ + \frac{1}{a} \left\{ \frac{\tan \phi}{a} U(j, \pm m) U(k-j, \pm n \mp m) \int \frac{1}{m} \left\{ \frac{1}{a} Z_2(k, \pm n) \right\} \right\} \]
\[ (1-5.2) \]

\[ Z_p(k, \pm n) = - \frac{R}{c_p} \theta(k, \pm n) \] \[ (1-5.3) \]

\[ W_p(k, \pm n) = - \frac{1}{a \cos \phi} \left\{ ikU(k, \pm n) + V(\pm n) - V(k, \pm n) \sin \phi \right\} \] \[ (1-5.4) \]
and
\[ \theta(k, \pm n) = \pm \frac{1}{n} \int_{-\infty}^{\infty} \frac{i}{a \cos \phi} j \theta(j, \pm m) U(k-j, \pm n \mp m) \]
\[ + \frac{\tan \phi}{a} U(j, \pm m) V(k-j, \pm n \mp m) + \frac{1}{a} \left\{ \theta(j, \pm m) W(k-j, \pm n \mp m) \right\} \int \frac{1}{m} \frac{1}{c_p} H(k, \pm n) \] \[ (1-5.5) \]

The above equations will be used to analyze the kinematic, dynamic, and thermodynamic contributions to velocity and temperature fields in the frequency wave-number space.

One of the objectives of this study is to analyze the contribution of the large-scale atmospheric motion to the kinetic energy, the rate of the meridional transports of sensible heat and angular momentum, and the available potential energy in the atmosphere. To do so, spectra of \( \frac{1}{2}(u^2 + v^2) \), \( vT \), \( u \) \( a \cos \phi \), and \( \omega T \) need to be computed. They can respectively be shown to be
\[ E_{\frac{1}{2}(u^2+v^2)}(k,\pm n) = \frac{i}{n} \sum_{j=\infty}^{\infty} \int_{a \cos \varphi}^{\infty} \frac{i}{a \cos \varphi} j U(k-j,\pm n \pm m) \]

\[ = U(j,\pm m) U(-k,\pm n) + V(j,\pm m) V(-k,\pm n) \]

\[ + \frac{1}{a} V(k-j,\pm n \pm m) \left[ U(\varphi, j, \pm m) U(-k, \pm n) + V(\varphi, j, \pm m) V(-k, \pm n) \right] \]

\[ + W(k-j,\pm n \pm m) \left[ U_p(j, \pm m) U(-k, \pm n) + V_p(j, \pm m) V(-k, \pm n) \right] \]

\[ + \frac{\tan \varphi}{a} U(j,\pm m) \left[ V(-k, \pm n) U(k-j, \pm m) - U(-k, \pm n) V(k-j, \pm m) \right] \]

\[ + \frac{1}{n} \left\{ \frac{g}{a} \left[ Z(\Phi, k, \pm n) U(-k, \pm n) - k Z(k, \pm n) U(-k, \pm n) \right] + \frac{1}{R} \right\} \]

\[ E_{\nu'}(k,\pm n) = \frac{i}{n} \sum_{j=\infty}^{\infty} \int_{a \cos \varphi}^{\infty} \frac{i}{a \cos \varphi} j V(j,\pm m) \theta(-k,\pm n) \]

\[ - \frac{j}{a} V(-k,\pm n) \theta(j,\pm m) + \frac{\tan \varphi}{a} \theta(-k,\pm n) U(j,\pm m) \]

\[ + \frac{1}{a} V(k-j,\pm n \pm m) \left[ V(\varphi, j, \pm m) \theta(-k, \pm n) + V(-k, \pm n) \theta(\varphi, j, \pm m) \right] \]

\[ + W(k-j,\pm n \pm m) \left[ V_p(j, \pm m) \theta(-k, \pm n) + V(-k, \pm n) \theta_p(\varphi, j, \pm m) \right] \]

\[ - \frac{R}{c_p} V(-k, \pm n) \cdot \theta(j, \pm m) \] \[ \frac{d \Phi}{a} \left[ Z(\Phi, k, \pm n) \right] + \frac{1}{c_p} V(-k, \pm n) H(k, \pm n) \] (1-5.6)


\[ E_{\nu U}(k, \pm n) = \pm \frac{i}{n} \sum_{j=\pm \infty}^{\infty} \int_{\pm \infty}^{\infty} \frac{i}{a \cos \phi} jU(k-j, \pm n \mp m) \cdot \]

\[
\left[ V(j, \pm m)U(-k, \pm n) + V(-k, \pm n)U(j, \pm m) \right]
\]

\[ + \frac{1}{a} \left[ W(k-j, \pm n \mp m) \left[ \theta(j, \pm m)U(-k, \pm n) + \theta(-j, \pm m)V(-k, \pm n) \right] \right.\]

\[ + \left. \frac{\tan \phi}{a} U(j, \pm m) \left[ V(-k, \pm n)U(k-j, \pm n \mp m) - V(k, \pm n)V(-k-j, \pm n \mp m) \right] \right\} \, dm \]

\[ + \frac{i}{n} \left[ G_{1}(k, \pm n)V(-k, \pm n) + G_{2}(k, \pm n)U(-k, \pm n) \right] \] (1-5.8)

\[ E_{\omega U}(k, \pm n) = \pm \frac{i}{n} \sum_{j=\pm \infty}^{\infty} \int_{\pm \infty}^{\infty} \frac{i}{a \cos \phi} j \left[ W(k, \pm n)\theta(-j, \pm m)U(-k+j, \pm n \mp m) \right.\]

\[ + \left. W(-k, \pm n)\theta(j, \pm m)U(k-j, \pm n \mp m) \right\} \, dm \]

\[ + \frac{1}{a} \left[ W(-k, \pm n)\theta(j, \pm m)V(k-j, \pm n \mp m) - W(k, \pm n)\theta(-j, \pm m)V(-k+j, \pm n \mp m) \right] \]

\[ + W(-k, \pm m)W(k-j, \pm n \mp m) \left[ \theta_{P}(j, \pm m) - \frac{R \rho}{c_{p} \rho} \theta(j, \pm m) \right] \]

\[- W(k, \pm m)W(-k+j, \pm n \mp m) \left[ \theta_{P}(-j, \pm m) - \frac{R \rho}{c_{p} \rho} \theta(-j, \pm m) \right] \right\} \, dm \]

\[ + \frac{i}{c_{p} n} \left[ W(-k, \pm n)H(k, \pm n) - W(k, \pm n)H(-k, \pm n) \right] \] (1-5.9)

Studies of the linear and nonlinear effects of the velocity, and temperature fields on the kinetic and internal energies, and the meridional transports of sensible heat and angular momentum in the wave-number frequency space are being made.
Chapter 2

Spectral Characteristics of the Large-scale Motion and Potential Temperature in the Atmosphere

2-1. Wavenumber-frequency Spectra of the zonal and meridional components of the velocity.

2-2. Power spectra of the zonal and meridional components of the velocity in Wavenumber and Frequency Domains.

2-3. Mean Kinetic Energy of the zonal and meridional components of the Velocity.


2-5.1 Linear and Nonlinear Contributions to the Large-scale Atmospheric Motion in Wavenumber-Frequency Space.

2-5.2 Partitioning according to Scale and Frequency.

2-5.3 Kinetic Energy in Each Wavenumber-Frequency Category.

2-5.4 Interaction Computations.

2-5.5 Low Frequency Planetary Waves.

2-5.6 Low Frequency Cyclone Waves.

2-5.7 Intermediate Frequency Planetary Waves.

2-5.8 Intermediate Frequency Cyclone Waves.

2-6. Power Spectra of the Potential Temperature and Its Similarity to those of the Velocity.

2-7. Mean Sensible Heat Associated with the Motion in the Atmosphere.
2-1. The Wavenumber-Frequency Spectra of the Zonal and Meridional Components of the Velocity.

In order to investigate the effect of waves of various wavelengths and frequencies on the distribution of the kinetic energy of the large-scale motion in the atmosphere, the wavenumber-frequency spectra of the zonal and meridional components of the velocities at 20°, 40°, 60°, and 80° N, at 100-, 200-, and 500-mb levels for the summer and winter seasons of 1964 have been computed (Kao and Wendell, 1969) with the use of Equations (1-2.1) and (1-2.9). The velocities used in this analysis are derived from the 1964 National Meteorological Center (NMC) stream function analyses at 500- and 200-mb levels, and from the isobaric height field at the 100-mb level. The data intervals used in the integration are of five degrees of longitude and 12 hours except at 100-mb where 24 hours were used for the summer season. The time integrations are carried out over periods of 90 days for the winter and summer seasons of 1964, which are normalized to 2π. Computations of Fourier coefficients are carried out for a wavenumber range of 0 through 21, and for frequency range of 0 through 90 cycles per 90 days.

Because of the similarity in the spectral distribution for the summer and winter only the wave number-frequency spectra of the zonal and meridional components of the velocities at 20°, 40°, 60°, and 80° N, at 100-, and 500-mb levels for the winter season of 1964 are presented in Figures 2.1 through 2.4. In these figures, the vertical axis represents the wavenumber on the latitude circle, and the horizontal axis represents the frequency in unit of cycles per 90 days, the positive and negative frequencies are respectively designated to waves moving from east to west and from west to east.
![Wave number-frequency spectra of the zonal velocity at 100 mb, Winter, 1964](image)

Fig. 2.1 Wave number-frequency spectra of the zonal velocity at 100 mb, Winter, 1964
Fig. 2.2. Wavenumber-frequency spectra of the zonal velocity at 500 mb, Winter, 1964.
Fig. 2.3. Wavenumber-frequency spectra of the meridional velocity at 100 mb Winter, 1964.
Fig. 2.4. Wavenumber-frequency spectra of the meridional velocity at 500 mb, Winter, 1964.
It is seen from Figures 2.1 to 2.4 that there exists a preferred spectral band in the power spectra of the zonal and meridional components of the velocities at various latitudes, which indicates the wavenumber-frequency domain of wave activities. In the middle latitudes (40° to 60° N) the spectral band is oriented in a domain extending from a region of low wavenumbers and frequencies to a region of high wavenumbers and negative frequencies. In the high latitudes (80° N), the spectra are confined to a domain of low wavenumbers and low frequencies. The change of the orientation of the spectral band with latitudes may partially be explained by the increase in the Coriolis effect and the decrease in the length of the latitudinal circle with increasing latitude on the wave motion in the atmosphere.

To examine the characteristics of the wavenumber-frequency spectra of the zonal and meridional components of the velocities, parallel dashed lines are drawn to form an approximate envelope enclosing the band of significant energy values of the spectra of the meridional component of the velocities at 500-mb, 40° N, for the winter, 1964 (Fig. 2.4). The frequency envelopes have been plotted as envelopes phase velocity and are presented by the dashed curves in Figure 2.5.

For the purpose of comparison we consider a particular expression of the phase speed of long planetary waves (Haurwitz, 1940):

\[ C = U - \frac{BL^2/4\pi^2}{1 + L^2/d^2} \]  

(2-1.1)

where \( U \) is the mean zonal velocity, \( L \) the length of the wave, \( \beta \) is the variation of the Coriolis parameter with latitude, and \( d \) is a measure of the latitudinal extent of the wave. The boundary conditions for the wave are that the meridional component of the velocity \( v(d/4) = v(-d/4) = 0 \).
We may note that if \( d \) approaches infinity (2-1.1) reduces to Rossby's (Rossby, 1939) well-known expression for the phase speed of long planetary waves of infinite lateral extent.

The solid curve in Figure 1.5 represents the phase velocity computed with the use of (2-1.1) for \( U = 18.5 \text{ m sec}^{-1} \) which was computed from \( E_{uu}(0,0) \). The value of \( d \) was 10,000 km. Figure 1.5 indicates that wave motion in the atmosphere is essentially of Rossby type and generally agrees with the planetary wave formula (2-1.1) of which \( U, B, \) and \( L \) are the main parameters. This is important in the later discussion of the power spectra of the zonal and meridional components of the velocity in the wavenumber and frequency domains.

It may be noted that the zonal component of the waves is most active in the low and medium wavenumber ranges, whereas the meridional component of the motion is most active in the medium wavenumber range. The wave activities are most pronounced near the tropopause, associating with the maximum mean zonal velocity. The intensity of the wave activities in the summer is about 50% of that in the winter.

It may also be noted that there is a definite shift of the region of wave activities from the troposphere to the stratosphere. In the troposphere, the maximum wave activity occurs in the middle latitudes, whereas in the stratosphere, it occurs in the low latitudes. This characteristic distribution of the power spectra holds for both the zonal and meridional components of the velocities and for both the summer and winter seasons.
Fig. 2.5. Comparison of phase-speed wavenumber relationships from the wavenumber frequency spectra with a planetary wave model. The dashed curves correspond to the envelops in the wavenumber frequency spectra.

The distribution of the power spectra of the zonal and meridional components of the velocity in the frequency domain at 100-, 200-, and 500-mb levels, at 20°, 40°, 60°, and 80° N for the summer and winter of 1964 are computed with the use of (1-2.9b). Because of the similarity in the spectral distribution at the above mentioned levels, only the spectra at the 200-mb levels are shown in Figures 2.6 to 2.9. In these figures, the solid curves are the frequency spectra of the velocity contributed by waves moving from west to east, and the dashed curves are those contributed by waves moving from east to west.

It is seen from Figures 2.6 and 2.7 that the power spectra of the zonal component of the velocity generally decrease with increasing frequency, indicating that more kinetic energy is associated with the slowly moving waves than fast moving waves. Figures 2.6 and 2.7 also indicate that there are more wave activities in the winter than in the summer. It may be noted that there is more kinetic energy involved in waves moving from west to east than waves moving from east to west, except in the low frequency range of the spectra and at the low latitudes in the summer.

The distribution of the power spectra of the meridional component of the velocity at 200-mb in the frequency domain, shown in Figures 2.8 and 2.9, indicate the similar characteristics as those of the zonal component of the velocity. However, the power spectra of the meridional component of the velocity of the waves moving from west to east show energy peaks in the frequency range from 4 to 20 cycles per 90 days.
Fig. 2.6 Frequency spectra of the zonal velocity at 200 mb, Summer 1964.
Fig. 2.7 Frequency spectra of the zonal velocity at 200 mb, Winter 1964.
Fig. 2.8 Frequency spectra of the meridional velocity at 200 mb, Summer 1964.
Fig. 2.9 Frequency spectra of the meridional velocity at 200 mb, Winter 1964.
In the high frequency range of the spectra, the kinetic energy of the zonal and meridional components of the velocity is approximately proportional to the minus first power of the frequency in the low latitudes but are proportional to the minus second power of the frequency in the high latitudes.

The distributions of the power spectra of the zonal and meridional velocities at the 200-mb level in the wavenumber domain is shown in Figs. 2.10 to 2.13. In these figures, as a visual aid the points of line spectra are connected by solid and dashed lines. The solid curves represent the contribution made by the stationary and nonstationary waves, whereas the dashed curves represent the contribution made by the stationary waves. It is seen from these figures that most of the kinetic energy is contributed by the moving waves, and that the kinetic energy of the motion in the winter is greater than that in the summer. In the high wavenumber range, the energy spectra of both the zonal and meridional velocities are approximately proportional to the minus third power of the wavenumber.

In their studies of the two-dimensional turbulence of an incompressible, nonrotating, viscous fluid, Kraichnan (1967), Leith (1968) and Lilly (1969) have found that the enstrophy of the motion is transferred to the higher wavenumbers through a $k^{-3}$ spectrum, whereas the kinetic energy of the motion is transferred to the lower wavenumbers through a $k^{-5/3}$ spectrum. The spectral behavior in the high wavenumber range is very similar to those shown in Figs. 2.10 to 2.13. This similarity may be attributed to the fact that in the high wavenumber range the effect of the latitudinal variation of the Coriolis parameter becomes small, the enstrophy of the motion in a rotating system becomes same as in a nonrotating system.
The spectra of the meridional component of the velocity (Figs. 2.12 and 2.13) show an energy peak in the wavenumber range, $k = 4$ to 10. It may be noted that there is no kinetic energy involved in the power spectra at zero wavenumber, since the zonal mean of the meridional velocity computed from the stream function is zero.

The distributions of the frequency and wavenumber spectra of the zonal and meridional components of the velocity at 100- and 500-mb levels are similar to those at 200-mb level, except that at 200-mb the kinetic energy is greater than that at 100- and 500-mb.
Fig. 2.10 Wave-number spectra of the zonal velocity at 200 mb, Summer 1964.
Fig. 2.11 Wave-number spectra of the zonal velocity at 200 mb, Winter 1964.
Fig. 2.12  Wave-number spectra of the meridional velocity at 200 mb, Summer 1964.
Fig. 2.13 Wave-number spectra of the meridional velocity at 200 mb, Winter 1964.
Fig. 2.14 Mean kinetic energy of the zonal component of the motion.

Fig. 2.15 Kinetic energy of the zonal component of the stationary zonal mean motion.
Fig. 2.16 Kinetic energy of the zonal component of the nonstationary zonal mean motion.

Fig. 2.17 Kinetic energy of the zonal component of the stationary waves.

Fig. 2.18 Kinetic energy of the zonal component of the moving waves.
Fig. 2.19 Mean kinetic energy of the meridional component of the motion.

Fig. 2.20 Kinetic energy of the meridional component of the stationary waves.

Fig. 2.21 Kinetic energy of the meridional component of the moving waves.
about two-thirds of the kinetic energy of the zonal component of the total motion.

The kinetic energy of the zonal component of the nonstationary zonal mean motion (Fig. 2.16) is small as compared with the other terms in the equation (2-3.1) However, its maximum kinetic energy occurs near 80° N.

The distribution of the kinetic energy of the zonal component of the stationary wave motion in the winter (Fig. 2.17) is similar to that of the zonal component of the total motion. However, in the summer the distribution is quite different; the maximum of the kinetic energy in the stratosphere and at the tropopause occurs near 20° N.

The distribution of the kinetic energy of the zonal component of the moving waves (Fig. 2.18) is rather different from that of the total zonal motion. In the troposphere, the maximum kinetic energy of the moving waves for both the summer and winter seasons occurs near 60° N, however, in the stratosphere it occurs near 40° N at 200 mb and near 20° N at 100 mb, indicating that the region of activity of moving waves shifts from high latitudes in the troposphere to low latitudes in the stratosphere. It may be noted that moving waves are most active near the jet stream core. However, the kinetic energy of the moving waves maintains about the same level for the summer and winter, although in the winter it is slightly greater than that in the summer.

The distribution of the kinetic energy of the meridional component of the total motion in the summer as shown in Figure 2.19 is similar to that of the moving waves in the summer (Fig. 2.21), since the kinetic energy involved in the stationary wave is very small (Fig. 2.20). It
may be noted that the kinetic energy of the meridional velocity of the moving waves is very similar to that of the zonal velocity of the moving waves; the maximum of the kinetic energy of both the zonal and meridional velocities occurs near 20° N at 100 mb, near 40° N at 200 mb, and near 60° N at 500-mb level.

The distribution of the kinetic energy of the meridional component of the motion in the winter (Fig. 2.19) indicates that the maximum of the kinetic energy of the meridional motion occurs near 60° N at 500 mb, near 40° N at 200 mb; at 100 mb there are two maximums, the primary maximum occurring near 80° N and the secondary one near 20° N. The abundant kinetic energy occurring at the high latitudes in the winter stratosphere is probably due to the winter instability of the polar vortex in the stratosphere. It may be noted that most of the kinetic energy of the meridional component of the motion is due to the activity of the moving waves, only a small portion of the kinetic energy of the meridional motion is involved in the stationary waves. It may also be noted that moving waves are most active near the jet stream core.
2-4. **The Power Spectra of the Vertical Component of the Velocity**

The wavenumber-frequency spectra of the individual rate of change of pressure, which is proportional to the negative vertical velocity, at 500-mb, 40° N, winter, 1964, are computed† and shown in Figure 1.22. Here again, a preferred spectral band appears in the spectra of the vertical velocity. There is a striking similarity between the spectral distribution of the kinetic energy of the vertical velocity and those of the horizontal velocity, indicating that a close relationship between the vertical and the horizontal components of the velocity exists.

The frequency spectra of the individual rate of change of the pressure are shown in Figure 2.23. In this figure, the solid curves represent the spectra contributed by the waves moving from west to east, and the dashed curves are those contributed by the waves moving from east to west. Figure 2.23 shows that the spectra of the kinetic energy of the vertical velocity have a smaller slope as compared with those of the horizontal velocity in the frequency range between 1 and 30 cycles per 90 days, and that there is a sharp drop in the spectra for the frequency higher than 45 cycles/90 days. The latter may be the results of the folding frequency of the 24 hour interval data of observation.

The wavenumber spectra of the individual rate of change of pressure are shown in Figure 2.24. It shows that there is an energy peak of the vertical velocity, occurring near $k = 10$, which also occurs for the spectra of the meridional component of the velocity. This indicates that

†In view of the small magnitude of the vertical velocity, only the spectra at 500-mb, 40° N, are computed.
the large-scale vertical motion is closely associated with the meridional motion. Again, it may be noted that in the high wavenumber portion the power spectra of the vertical velocity is approximately proportional to $k^{-3}$. 
Fig. 2.22 Wavenumber-frequency spectra of the individual rate of change of pressure at 500 mb, 40°N, Winter, 1964.
Fig. 2.23 Frequency spectra of the individual rate of change of pressure at 500 mb, 40°N, Winter, 1964.

Fig. 2.24 Wavenumber spectra of the individual rate of change of pressure at 500 mb, 40°N, Winter, 1964.
2-5.1 **Linear and Nonlinear Contributions to the Large-scale Atmospheric Motion in Wavenumber-frequency Space.**

The nonlinear interaction of the large-scale atmospheric motion has been studied by Wendell (Wendell, 1969). He considered the kinetic energy equations for the zonal and meridional components of the velocity

\[
\frac{1}{2} \frac{\partial u^2}{\partial t} = - \frac{u^2}{a \cos \phi} \frac{\partial u}{\partial \lambda} - \frac{uv}{a} \frac{\partial u}{\partial \phi} + \tan \frac{\phi}{a} u^2 v + fu(v - v_g) + uF_1
\]

\[(u1)\quad (u2)\quad (u3)\quad (u4)\quad (u5)\quad (2-5.1)\]

\[
\frac{1}{2} \frac{\partial v^2}{\partial t} = - \frac{uv}{a \cos \phi} \frac{\partial v}{\partial \lambda} - \frac{v^2}{a} \frac{\partial v}{\partial \phi} - \tan \frac{\phi}{a} u^2 v + fv(u - u_g) + uF_2
\]

\[(v1)\quad (v2)\quad (v3)\quad (v4)\quad (v5)\quad (2-5.2)\]

where \(u_g\) and \(v_g\) are the zonal and meridional components of the geostrophic wind. The numbers below the terms in the equations are for future reference.

The corresponding equations in wave-number frequency space for a given value of \(k\) and \(n\) are

\[
|U(k, n)|^2 = \frac{T}{2\pi} \left[ U(-k, -n) \frac{1}{n} \sum_j \sum_m \frac{1}{\cos \phi} jU(j, m)U(k-j, n-m) \right]
\]

\[(U1)\]

\[
+ U(-k, -n) \frac{1}{na} \sum_j \sum_m U(j, m)V(k-j, n-m)
\]

\[(U2)\]

\[- U(-k, -n) \frac{i}{n} \tan \frac{\phi}{na} \sum_j \sum_m U(j, m)V(k-j, n-m) \]

\[(U3)\]

\[
- U(-k, -n) \frac{i}{n} \left\{ V(k, n) - V_g(k, n) \right\} - \frac{i}{n} U(-k, -n)G_{1}(k, n)
\]

\[(U4)\quad (U5)\quad (2-5.3)\]
\[ |V(k,n)|^2 = \frac{T}{2\pi} \left[ V(-k,-n) \sum_{j} \sum_{m} \frac{i}{n} \cos \phi \ j V(j,m) U(k-j, n-m) \right] \]

\hspace{1cm} (V1)

\[ + V(-k,-n) \sum_{j} \sum_{m} \frac{i}{n} \cos \phi \ j V(j,m) V(k-j, n-m) \]

\hspace{1cm} (V2)

\[ + V(-k,-n) \sum_{j} \sum_{m} \frac{i}{n} \cos \phi \ j V(j,m) U(k-j, n-m) \]

\hspace{1cm} (V3)

\[ + V(-k,-n) \sum_{j} \sum_{m} \frac{i}{n} f \left\{ U(k,n) - U_g(k,n) \right\} - \frac{i}{n} V(-k,-n) G_2(k,n) \]

\hspace{1cm} (V4)

\[ + V(-k,-n) \sum_{j} \sum_{m} \frac{i}{n} \cos \phi \ j V(j,m) U(k-j, n-m) \]

\hspace{1cm} (V5) \hspace{5cm} (2-5.4)

where \( U_g(k,n) \) and \( V_g(k,n) \) are the transforms of the zonal and meridional components of the geostrophic wind. Again the symbols below the terms on the right side of the equations are for future reference.

The summations over the index \( j \) are from \(-j_s\) to \(+j_s\) where \( j_s \) depends on the number of discrete points on the latitude circle. The summations over \( m \) are from \(-m_s\) to \(+m_s\) depending on whether frequency \( n \) is positive or negative, where \( m_s \) depends on the number of discrete time data points used.
2-5.2 Partitioning According to Scale and Frequency

The transformation of the nonlinear terms of the equations of motion into wave-number frequency space results in terms which involve sums of products of transforms for various waves and frequencies. For example, the transformed zonal advection term of the u component equation is

\[ \sum_{j=-j}^{j} \sum_{m=-m}^{m} a \cos \phi \frac{1}{n} U(j,m) U(k-j,n-m) \]  

(2-5.5)

For the computations over 90-day seasons the upper and lower limits of wave number and frequency used are respectively \( j_s = \pm 20 \) and \( m_s = \pm 90 \). It may be noted from (2-5.5) that depending on the values of \( k \) and \( n \) the term will involve from 1800 to 7200 products of transforms. Each product may be considered an interaction between the scales and frequencies represented by \((j,m)\) and \((k-j, n-m)\). When (2-5.5) is multiplied by \( \frac{T}{2\pi} U(-k,-n) \) it becomes the term denoted by \( U_l \) in the energy equation (2-5.3) and represents a contribution to the spectral energy \( |U(k,n)|^2 \), through nonlinear interaction of various scales and frequencies of motion. If \( j \) and \( m \) are allowed to vary over their entire range, the term \( U_l \) represents the total contribution from all possible interactions. Because of the extremely large number of possible interactions, it is necessary for analysis purposes to make use of a system of classification which places the energy over specified groups of wave numbers and frequencies into particular scale-frequency categories.
Table 1. Scale partitioning and designation for 40° N.

<table>
<thead>
<tr>
<th>Wave Number (k)</th>
<th>Wave Length (L) (km)</th>
<th>Classification</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>not defined</td>
<td>Zonally-averaged flow</td>
<td>M</td>
</tr>
<tr>
<td>1 - 5</td>
<td>30,700 - 6,140</td>
<td>Planetary waves</td>
<td>P</td>
</tr>
<tr>
<td>6 - 9</td>
<td>5,110 - 3,410</td>
<td>Cyclone waves</td>
<td>C</td>
</tr>
<tr>
<td>10 - 20</td>
<td>3,070 - 1,535</td>
<td>Short waves</td>
<td>S</td>
</tr>
</tbody>
</table>

Table 2. Partition classification according to frequency

<table>
<thead>
<tr>
<th>Frequency (n) (cycles per 90 days)</th>
<th>Period (days)</th>
<th>Definition (Frequency)</th>
<th>Direction of wave motion</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>-90 to -31</td>
<td>1-3</td>
<td>High</td>
<td>West to East</td>
<td>H</td>
</tr>
<tr>
<td>-30 to -11</td>
<td>3-9</td>
<td>Intermediate</td>
<td>West to East</td>
<td>I</td>
</tr>
<tr>
<td>-10 to -1</td>
<td>9-90</td>
<td>Low</td>
<td>West to East</td>
<td>L</td>
</tr>
<tr>
<td>0</td>
<td>9-90</td>
<td>Zero</td>
<td>Stationary</td>
<td>0</td>
</tr>
<tr>
<td>1 to 10</td>
<td>9-90</td>
<td>Low</td>
<td>East to West</td>
<td>L</td>
</tr>
<tr>
<td>11 to 30</td>
<td>3-9</td>
<td>Intermediate</td>
<td>East to West</td>
<td>I</td>
</tr>
<tr>
<td>31 to 90</td>
<td>1-3</td>
<td>High</td>
<td>East to West</td>
<td>H</td>
</tr>
</tbody>
</table>

From the classifications in Tables 1 and 2, according to scale and frequency, it may be noted that there are 28 categories which may interact with each other and themselves for a total of 406 possible interaction combinations. Each category is designated by combining the symbols used for the wave number and frequency classifications, i.e. the category of planetary waves moving from west to east with low frequency would be represented by (P, -L). The interaction combinations are designated by placing the scale-frequency category symbols side by side with an asterisk separating them, (P, -L)*(C, -I).
2-5.3 **Kinetic Energy in Each Wave-Number Frequency Category**

In order to determine the importance of the average eddy kinetic energy and in particular the transient eddy kinetic energy, the values of $E_{uu}(k,n)$ and $E_{vv}(k,n)$ are summed over each of the defined categories for the winter and summer seasons of 1964 and are shown in Tables 3 through 6. The percentages listed indicate the comparison of the amount of energy in each category compared with the energy summed over all the categories. The sums over all categories are shown in the lower left corner of each table. It may be observed from these tables, that for both the summer and winter seasons at $40^\circ$N, the category $(M,0)$ contains far more energy than any other single category. This category represents the time-averaged mean zonal flow.

The importance of the eddy kinetic energy may be seen by combining the energy contained in all of the other categories of $E_{uu}(k,n)$ and all the categories of $E_{vv}(k,n)$. We find that for the winter season the average eddy kinetic energy is $169.2 \times 10^4 \text{ cm}^2 \text{ sec}^{-2}$, of which 72% is transient. During the summer season the average eddy kinetic energy is $72.3 \times 10^4 \text{ cm}^2 \text{ sec}^{-2}$, of which 88% is transient.

Because of the significantly greater amount of transient eddy kinetic energy during the winter, the evaluation of the terms of the energy equations will be carried out for this season. Also because of the expense of the interaction computations, evaluation of the interaction terms for all categories is prohibitive. Six categories which contain collectively about 70% of the transient eddy kinetic energy are selected for the interaction computations. These categories are $(P,+L)$, $(C,+L)$, $(P,-L)$, $(C,-L)$, $(P,-H)$, $(C,-H)$. Because there is no
### Table 3. $E_{uu}(k,n)$ (units: $10^4 \text{ cm}^2 \text{ sec}^{-2}$) for $40^\circ \text{ N}$, Winter 1964.

<table>
<thead>
<tr>
<th>k</th>
<th>n</th>
<th>M (0)</th>
<th>P (1-5)</th>
<th>C (6-9)</th>
<th>S (10-20)</th>
<th>Sum Over k</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>(31 to 90)</td>
<td>.14</td>
<td>.114</td>
<td>.48</td>
<td>.57</td>
<td>2.33</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(.05%)</td>
<td>(.4%)</td>
<td>(.2%)</td>
<td>(.2%)</td>
<td>(.9%)</td>
</tr>
<tr>
<td>I</td>
<td>(11 to 30)</td>
<td>.25</td>
<td>1.74</td>
<td>.40</td>
<td>.26</td>
<td>2.64</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(.1%)</td>
<td>(.6%)</td>
<td>(.2%)</td>
<td>(.1%)</td>
<td>(1.0%)</td>
</tr>
<tr>
<td>L</td>
<td>(1 to 10)</td>
<td>2.05</td>
<td>13.39</td>
<td>1.12</td>
<td>.28</td>
<td>16.83</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(.8%)</td>
<td>(5.0%)</td>
<td>(.4%)</td>
<td>(.1%)</td>
<td>(6.2%)</td>
</tr>
<tr>
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<td>(0)</td>
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<td>.11</td>
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<td>(13.8%)</td>
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<td>(.04%)</td>
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<td>-L</td>
<td>(-1 to -10)</td>
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<td></td>
<td>(.8%)</td>
<td>(6.2%)</td>
<td>(1.0%)</td>
<td>(.1%)</td>
<td>(8.1%)</td>
</tr>
<tr>
<td>-I</td>
<td>(-11 to -30)</td>
<td>.25</td>
<td>7.12</td>
<td>3.35</td>
<td>.80</td>
<td>11.52</td>
</tr>
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<td></td>
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<td>(2.6%)</td>
<td>(1.2%)</td>
<td>(.3%)</td>
<td>(4.3%)</td>
</tr>
<tr>
<td>-H</td>
<td>(-31 to -90)</td>
<td>.14</td>
<td>1.99</td>
<td>2.72</td>
<td>1.80</td>
<td>6.65</td>
</tr>
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<td></td>
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<td>(.05%)</td>
<td>(.7%)</td>
<td>(1.0%)</td>
<td>(.7%)</td>
<td>(2.5%)</td>
</tr>
<tr>
<td>Sum over n</td>
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<td>175.50</td>
<td>79.46</td>
<td>10.91</td>
<td>4.14</td>
<td>270.02</td>
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<td></td>
<td>(65.0%)</td>
<td>(29.4%)</td>
<td>(4.0%)</td>
<td>(1.5%)</td>
<td>(100.0%)</td>
</tr>
</tbody>
</table>

### Table 4. $E_{vv}(k,n)$ (units: $10^4 \text{ cm}^2 \text{ sec}^{-2}$) for $40^\circ \text{ N}$, Winter 1964.

<table>
<thead>
<tr>
<th>k</th>
<th>n</th>
<th>M (0)</th>
<th>P (1-5)</th>
<th>C (6-9)</th>
<th>S (10-20)</th>
<th>Sum Over k</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>(31 to 90)</td>
<td>.00</td>
<td>.18</td>
<td>.35</td>
<td>1.28</td>
<td>1.80</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(.0%)</td>
<td>(.3%)</td>
<td>(.5%)</td>
<td>(1.8%)</td>
<td>(2.6%)</td>
</tr>
<tr>
<td>I</td>
<td>(11 to 30)</td>
<td>.00</td>
<td>.61</td>
<td>.33</td>
<td>.57</td>
<td>1.51</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(.0%)</td>
<td>(.9%)</td>
<td>(.5%)</td>
<td>(.8%)</td>
<td>(2.2%)</td>
</tr>
<tr>
<td>L</td>
<td>(1 to 10)</td>
<td>.00</td>
<td>7.85</td>
<td>1.52</td>
<td>.71</td>
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<tr>
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<td>(1.0%)</td>
<td>(14.4%)</td>
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<tr>
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<td>.26</td>
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<td>7.31</td>
<td>5.80</td>
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<td>(10.5%)</td>
<td>(8.3%)</td>
<td>(1.4%)</td>
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<td>(3.6%)</td>
<td>(28.9%)</td>
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<td>(.6%)</td>
<td>(6.6%)</td>
<td>(11.0%)</td>
<td>(18.3%)</td>
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<td>27.99</td>
<td>14.05</td>
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<td>(.0%)</td>
<td>(39.8%)</td>
<td>(40.1%)</td>
<td>(20.1%)</td>
<td>(100.0%)</td>
</tr>
</tbody>
</table>
### Table 5. \( E_{uu}(k,n) \) (units: \( 10^4 \text{ cm}^2 \text{ sec}^{-2} \)) for 40° N, Summer 1964.

<table>
<thead>
<tr>
<th>n</th>
<th>H (31 to 90)</th>
<th>I (11 to 30)</th>
<th>L (1 to 10)</th>
<th>0 (0)</th>
<th>-L (-1 to -10)</th>
<th>-I (-11 to -30)</th>
<th>-H (-31 to -90)</th>
<th>Sum over n</th>
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<tr>
<td></td>
<td>M (0)</td>
<td>P (1-5)</td>
<td>C (6-9)</td>
<td>S (10-20)</td>
<td>Over k</td>
<td></td>
<td></td>
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</tr>
<tr>
<td></td>
<td>.07 (.1%)</td>
<td>.59 (.6%)</td>
<td>.31 (.3%)</td>
<td>.32 (.3%)</td>
<td>1.28 (.1%)</td>
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<td>1.16 (1.2%)</td>
<td>.42 (.4%)</td>
<td>.20 (.2%)</td>
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<tr>
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<td>1.08 (1.1%)</td>
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<td>.5699</td>
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<td>.35 (.4%)</td>
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<td>.77 (.8%)</td>
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<td></td>
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<tr>
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<td>.00 (0%)</td>
<td>1.04 (3.1%)</td>
<td>2.39 (7.2%)</td>
<td>.34 (1.0%)</td>
<td>3.76 (11.3%)</td>
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<tr>
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<td>2.14 (7.5%)</td>
<td>6.04 (18.1%)</td>
<td>1.75 (5.2%)</td>
<td>10.27 (30.8%)</td>
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<td></td>
<td>.00 (0%)</td>
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<td>.57 (1.7%)</td>
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<tr>
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<td>9.30 (27.9%)</td>
<td>15.57 (46.7%)</td>
<td>8.47 (25.4%)</td>
<td>33.34 (100.0%)</td>
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</tbody>
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### Table 6. \( E_{vv}(k,n) \) (units: \( 10^4 \text{ cm}^2 \text{ sec}^{-2} \)) for 40° N, Summer 1964.

<table>
<thead>
<tr>
<th>n</th>
<th>H (31 to 90)</th>
<th>I (11 to 30)</th>
<th>L (1 to 10)</th>
<th>0 (0)</th>
<th>-L (-1 to -10)</th>
<th>-I (-11 to -30)</th>
<th>-H (-31 to -90)</th>
<th>Sum over n</th>
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<tbody>
<tr>
<td></td>
<td>M (0)</td>
<td>P (1-5)</td>
<td>C (6-9)</td>
<td>S (10-20)</td>
<td>Over k</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>.00 (.0%)</td>
<td>.11 (.3%)</td>
<td>.19 (.6%)</td>
<td>.56 (1.7%)</td>
<td>.86 (2.6%)</td>
<td></td>
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<tr>
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<td>.00 (.0%)</td>
<td>.56 (1.7%)</td>
<td>.33 (1.0%)</td>
<td>.30 (.9%)</td>
<td>1.19 (3.6%)</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>.00 (.0%)</td>
<td>4.32 (12.9%)</td>
<td>2.29 (6.9%)</td>
<td>.46 (1.4%)</td>
<td>7.07 (21.2%)</td>
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<tr>
<td></td>
<td>.00 (.0%)</td>
<td>1.04 (3.1%)</td>
<td>2.39 (7.2%)</td>
<td>.34 (1.0%)</td>
<td>3.76 (11.3%)</td>
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<td></td>
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<tr>
<td></td>
<td>.00 (.0%)</td>
<td>2.49 (7.5%)</td>
<td>6.04 (18.1%)</td>
<td>1.75 (5.2%)</td>
<td>10.27 (30.8%)</td>
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<tr>
<td></td>
<td>.00 (.0%)</td>
<td>.64 (1.9%)</td>
<td>3.77 (11.3%)</td>
<td>3.18 (9.5%)</td>
<td>7.60 (22.8%)</td>
<td></td>
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<tr>
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<td>.00 (.0%)</td>
<td>.14 (1.7%)</td>
<td>.57 (1.7%)</td>
<td>1.88 (5.6%)</td>
<td>2.59 (7.8%)</td>
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<tr>
<td></td>
<td>.00 (0%)</td>
<td>9.30 (27.9%)</td>
<td>15.57 (46.7%)</td>
<td>8.47 (25.4%)</td>
<td>33.34 (100.0%)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
time variation in the kinetic energy of the stationary eddies, the total contribution of the linear and nonlinear terms on the right-hand side of (2-5.1) and (2-5.2) should be zero. In the corresponding equations in wavenumber frequency space, which are (2-5.3) and (2-5.4) multiplied by \( n \), the left-hand sides of the equations are zero for stationary eddies \( (n = 0) \). Therefore, no analysis of the total contribution of linear and nonlinear interactions to the stationary eddies is made.
2-5.4 Interaction Computations

The terms of the energy equation are computed for the separate wave numbers and frequencies of each category. The interaction terms are each computed for their 406 interaction combinations in order that the important combinations will be apparent. The results for each wave number and frequency are then summed over each category and presented in block diagram form shown in Figures 2.25 through 2.30. In these diagrams the total contributions of each of the terms in (2-5.3) and (2-5.4) are shown in the small blocks in the center of the diagram and labeled with the symbols for the terms they represent.

The terms represented by $U_l$ and $V_l$ stem from the transformed zonal advection of the zonal and meridional components of the kinetic energy. Those represented by $U_2$ and $V_2$ arise from the transformed meridional advection of the zonal and meridional components of the kinetic energy. These terms represent contributions to the energy through an inertial transfer process. The symbols $U_3$ and $V_3$ denote the terms in the transformed energy equations which are due to the sphericity of the earth. The symbols $U_4$ and $V_4$ represent the terms which contribute to the spectral energy through the ageostrophic motion. These last two terms do not involve nonlinear interactions. The ageostrophic terms represent contributions to the energy through the rate of
work done by the pressure and Coriolis forces. All the above terms are computed directly from the available pressure and wind data.

The terms \( U_5 \) and \( V_5 \), however, cannot be computed directly because no data concerning the eddy frictional force is available. The sums of the quantities \( U_1 \) through \( U_4 \) and \( V_1 \) through \( V_4 \) are displayed below the appropriate blocks. To the left of these sums are shown the sums of the spectral energy over the category. The differences between the sums of the terms of the equations and the spectral energy for the category are displayed to the right of the sums of the terms and labeled \( U_5 + E_1 \) and \( V_5 + E_2 \). These quantities contain, in addition to the eddy friction terms, the results of error in the data, and computational error.

The larger blocks, connected with arrows to the small blocks labeled \( U_1 \), \( U_2 \), \( V_1 \) and \( V_2 \), contain positive and negative interaction combinations which represent the major contributions to the totals in the small blocks. The partitioned interaction combinations for \( U_3 \) and \( V_3 \) were not computed because test computations indicated that the resultant contribution through these terms is generally quite small. The figures at the bottom of the large blocks represent the total positive and negative contributions from all the interaction combinations. Since only the interaction combinations with significant magnitude are listed, the sums of those listed will only approximate the totals at the bottoms of the blocks.

Since the spectral energy, \( E_{uu}(k,n) \) and \( E_{vv}(k,n) \) is inherently a positive quantity, a negative value for an interaction combination or the sum over all interaction combinations, for a given \( k \) and \( n \),
is interpreted as a detraction from the spectral energy. Positive values are interpreted as contributions to the spectral energy.

Before discussing the results for the selected categories, there are some features characteristic to the interaction results which may be pointed out for all six of the selected categories. The most persistent features are the dominance of the terms $U_l$ and $V_l$ over the other interaction contributions, and the dominance within these terms of the interaction combination involving the mean zonal flow and the category under consideration. The second feature mentioned would indicate that, at least for the wave numbers and frequencies which provide the major contributions in the category, the $(M,0)^*(k,n)$ interaction combination is predominant by about an order of magnitude for $U_l$ and $V_l$. Examination of the results for individual values of $k$ and $n$ reveals that the above is true for the relatively large values of $E_{uu}(k,n)$ and $L_{vv}(k,n)$. 
2-5.5 *Low Frequency Planetary Waves: (P,+L) and (P,-L)*

For comparison purposes these categories, shown in Figures 2.25 and 2.26 will be discussed together because they differ only in the direction of the motion of the waves. One of the major similarities between these two categories is that the interaction terms $U_l$ and $V_l$, summed over $k$ and $n$ for each category, are by far the largest of the interaction terms. Since $(M,0)*(P,+L)$ are the major interaction combinations of $U_l$ and $V_l$ and the signs of their contributions depend on the sign of the frequency, the direction of the motion of the waves determines the signs of the contributions from $U_l$ and $U_2$.

If we turn attention to the secondary interactions it may be noted that for $U_l$ in $(P,+L)$ the predominant secondary interactions involve the zonally-averaged flow $(M)$, the planetary scale $(P)$, or cyclone scale waves in either a stationary condition or moving with low frequency. The reason for this lies in the fact that $U_l$ contains only interactions involving the zonal component of the wind and from Figures 2.2 and 2.4 or Table 3 we see that the spectral energy $E_{uu}(k,n)$ is concentrated primarily in the categories $(P,+L)$ and $(P,-L)$. It is interesting to note that the interaction combination $(P,+L)*(P,-L)$ is positive and about the same magnitude for both categories discussed here, because the primary interactions generally change signs with category. The secondary interaction combinations for term $V_l$, which includes the meridional component of the wind, involve all scales of motion and the full range of frequencies.
For the categories (P,+L) and (P,-L), the terms U2 and V2 contain a number of interaction combinations of the same order of magnitude. In the category (P,+L) the (M,0)*(P,+L) interaction combination for U2 ranks sixth in magnitude and is about one-seventh the size of the resultant interaction term. For (P,-L) the (M,0)*(P,-L) interaction ranks second and is more than one-third of the summed interaction term. For the terms U2 and V2 we also find that the significant interaction combinations range over all scales and frequencies. A fairly persistent feature of the significant interaction combinations in both (P,+L) and (P,-L) categories is that, if the interactions do not involve stationary eddies, they generally involve waves moving in both directions.

The ageostrophic terms, U4 and V4, are fairly consistently positive over the category (P,+L) and fairly consistently negative for the category (P,-L). These terms serve to counteract the large positive or negative contributions through U1 and V1. The contributions due to the friction and error terms summed over k and n in (P,-L) seem quite large. However, examination of these terms for each value of k and n reveals that the largest relative values occur for the first 3 to 4 frequencies and the relative size of these terms diminishes sharply as n increases.
### Contributions to the Spectral Kinetic Energy of the Category (P,+L)

#### Negative Contributions

<table>
<thead>
<tr>
<th>Term</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>((P,+L)*(M,0))</td>
<td>-68.15</td>
</tr>
<tr>
<td>((P,+L)*(C,-H))</td>
<td>-0.33</td>
</tr>
<tr>
<td>((P,+L)*(M,-L))</td>
<td>-1.85</td>
</tr>
<tr>
<td>((C,+L)*(P,-L))</td>
<td>-1.49</td>
</tr>
<tr>
<td>((P,+I)*(C,-I))</td>
<td>-1.36</td>
</tr>
<tr>
<td>((S,+I)*(S,-I))</td>
<td>-0.80</td>
</tr>
</tbody>
</table>

**Total**

-69.70

#### Positive Contributions

<table>
<thead>
<tr>
<th>Term</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>((P,+L)*(P,0))</td>
<td>2.21</td>
</tr>
<tr>
<td>((P,+L)*(P,-L))</td>
<td>1.72</td>
</tr>
<tr>
<td>((P,+L)*(C,-L))</td>
<td>1.31</td>
</tr>
<tr>
<td>((M,+L)*(P,0))</td>
<td>1.15</td>
</tr>
<tr>
<td>((M,+L)*(P,-L))</td>
<td>1.15</td>
</tr>
</tbody>
</table>

**Total**

9.85

---

#### Contributions from Individual Terms

**Term U1**

-59.85

**Term U2**

15.79

**Term U3**

1.09

**Term U4**

44.94

**Total**

-17.72 + 13.39 = 1.97 + 11.42 = 33.51

---

**Term V1**

-49.92

**Term V2**

11.61

**Term V3**

1.11

**Term V4**

51.80

**Total**

-2.96 + 7.85 = 14.60 + -6.75 = 14.57

---

**Fig. 2.25** Contributions to the spectral kinetic energy of the category (P,+L) from the individual terms of the transformed energy equations. The total contributions from each term are shown in the small blocks. The contributions from the major interaction combinations for each term are shown in the large blocks.
Fig. 2.26 Same as figure 1.25 for the category (C, -L).
2-5.6. **Low Frequency Cyclone Waves: (C,+L) and (C,-L)**

These two categories, shown in Figures 10 and 11, will also be discussed together for comparison purposes. In contrast with the low frequency planetary scale categories the Ul interaction terms are far less predominant in both the positive and negative frequency categories. The reason for this may be seen from the spectral energy distribution of $E_{uu}(k,n)$ over the (C,+L) and (C,-L) categories in Figure 1. With the comparatively smaller amount of energy occurring in these categories, especially in (C,+L), the $(M,0)\cdot(k,n)$ interaction combinations are somewhat smaller than for (P,+L) or (P,-L). For the V1 interaction term, the contribution for the category (C,+L) is about one-fourth the size of the same term in (P,+L), but the term (V1) for category (C,-L) is slightly larger than for the same term (P,-L). These results reflect the relative amounts of energy contained in the respective categories.

The pattern of interaction combinations for Ul and V1 are similar to those discussed in (P,+L) and (P,-L). For the term U2, in both (C,+L) and (C,-L), it may be noted that (P,+L)*'(P,0) and (P,+L)*'(P,-L) are the major contributions. However, for (C,+L) these contributions are positive and for (C,-L) their contributions are negative. The term V2, when computed for each wave number and frequency, is found to have enough variation in sign that the resulting sum over the category is very small. This variation in sign was also the case for the term U4 over the categories (C,+L) and (C,-L) and caused the ageostrophic term to lose a lot of its relative magnitude in the category sum through cancellation.

The friction-plus-error term is smaller in magnitude than for the categories (P,+L) and (P,-L) but is still as large as the spectral energy energy summed over k and n in the categories (C,+L) and (C,-L), except
Fig. 2.27 Same as figure 1.25 for the category (C, +L).
Fig. 2.28 Same as figure 1.25 for the category (C, -L).
for the case involving the $E_{vv}(k,n)$ in $(C,-L)$. As in the cases of $(P,+L)$ and $(P,-L)$, the friction-plus-error term diminishes in size as $n$ increases. Also as in the categories $(P,+L)$ and $(P,-L)$ we note the phenomenon of a large number of interactions between waves moving in opposite directions.

2-5.7 Intermediate Frequency Planetary Waves: $(P,-I)$.

The interactions for the category $(P,+I)$ were not computed because, as may be seen in Fig.2.1 or Table 3, it contains a very small amount of energy for either the zonal or meridional components. The primary interactions for the terms $U_l$ and $V_l$ in the category $(P,-I)$ have similar characteristics to those of the other selected categories. The secondary interactions for $U_l$ involve waves in this category interacting with planetary scale waves both stationary and retrogressing with low frequency. It is noted for the term $V_l$ that, in this category, $(M,0)^*(P,-I)$ is the largest interaction combination, but it is not as large in comparison to the next largest interaction, $(P,0)^*(C,-I)$, as has been the case in the categories discussed previously. This is probably due to the fact that, as may be observed from Figure 1, the spectral energy, $E_{vv}(k,n)$, is relatively large over the small portion of the category around $k = 4$ and $5$ and $n = -20$ to $-10$. The interaction combination, $(M,0)^*(k,n)$, is dominant only over this portion of the spectrum. The term $U_2$ summed over the category $(P,-I)$ has a significant magnitude when compared to $E_{uu}$. The interaction components which serve as contributors to $U_2$ involve primarily the planetary scale waves in all but the high frequency category. The main detractors from the spectral energy are the interaction combinations involving the intermediate
frequency cyclone waves with both categories of low frequency planetary waves. For the term V2, the interaction combinations are smaller and with enough variation in sign that, summed over the category, this term has practically an insignificant magnitude. This is also true of the terms U3 and V3. The terms U4 and V4, however, are predominantly negative over this category and serve to counteract the contributions of the other terms.

It may be noted that the friction-plus-error terms are considerably smaller with respect to the spectral energies than in the low frequency categories. As in the other categories this term decreases in relative size over the category as n increases in magnitude. Also this category has more interaction combinations, which involve waves moving in the same direction (from west to east) than the low frequency categories. However, interactions between waves moving in opposite directions is still a definite characteristic.

2-5.8 Intermediate Frequency Cyclone Waves: (C,-I)

The interactions for the cyclone scale waves retrogressing with intermediate frequency were not computed because of the very small amount of energy contained by waves in the category. It may be noted from Figure 1 that the transient energy in this category comes almost entirely from the energy of the meridional component of the wind. This is why the interaction term V1 is by far the biggest contribution of the interaction terms of both components. As would be expected, the primary interaction combination is \((M,0)*(C,-I)\) and V1 for each wave number and frequency has a strong positive correlation with \(E_{vv}(k,n)\) over the category. The secondary interactions indicate a significant
Fig. 2.29 Same as figure 1.25 for the category (P, -I).
Fig. 2.30 Same as figure .25 for the category (C, -I).
contribution through U1 from the stationary planetary scale waves. The small magnitude of V2 is a true reflection of the situation for the individual wave numbers and frequencies over the category. The interaction combinations cover all scales, except the zonal average, M, and all the frequency categories. The zonal average category, M, does not appear because the term V2 only involves the meridional component of the wind, and as was mentioned earlier, this analysis involves the streamfunction-derived winds. Both the terms U3 and V3 were small over the category, but they also had considerable variation in sign and therefore appear as insignificant contributions. The ageostrophic terms, U4 and V4, were consistently negative and acted in opposition to U1 and V1 for the individual wave numbers and frequencies of the categories.

The relative size of the terms U5 + E and V5 + E are smaller for this category than for any of the selected categories. They also had the characteristic of decreasing as the magnitude n increased. The interaction of waves moving in opposite directions is also a notable feature of this category.
2-6. The Spectra of the Potential Temperature and its Similarity to those of the Velocity

Studies of the statistical characteristics of turbulent-scalar quantities have mostly been confined to the field of isotropic, homogeneous turbulence (Obukhoff, 1949; Yaglom, 1949; Corrsin, 1951a, 1951b; Kistler, O'Brien, Corrsin, 1954). It was found that in such a field of turbulence similarity between the spectra of the scalar quantities and those of the velocity exists.

The large-scale motion in the atmosphere is generally anisotropic. Similarity between the spectra of scalar quantities and those of the velocity cannot generally be expected. However, if certain degree of similarity between the two can be established, predicability of the dispersion of particles, such as volcanic dust and radioactive debris, by the large-scale atmospheric motion would greatly be increased.

In view of the quasi-conservative characteristic of the potential temperature in the atmosphere, particularly along the same latitudinal circle, the wavenumber-frequency spectra of the potential temperature at 20°, 40°, 60°, and 80°N at 100-, 200-, and 500-mb levels for the summer and winter seasons of 1964 have been computed with the use of (1-1) and (1-9). Because of the similarity in the spectral distribution at 100-, 200-, and 500-mb levels, and of the fact that more wave activities occur at 200-mb, only the wavenumber frequency spectra of the potential temperature at 200-mb level are presented in Figures 2.31 and 2.32. In these two figures, only the spectra at 40°, 60° and 80°N are presented, since at 20°N there was comparatively little wave activities appeared in the spectra of the potential temperature in both the summer and winter seasons. It may be noted in Figs. 2.31 and 2.32 that the wavenumber-frequency spectra of
Fig. 2.31 Wave-number frequency spectra of the sensible heat at 200 mb, Summer 1964.
Fig. 2.32 Wave-number frequency spectra of the sensible heat at 200 mb, Winter 1964.
the potential temperature are very much similar to those of the horizontal velocity rather than the spectra of either the zonal or the meridional components of the velocity.

The frequency spectra of the potential temperature at 200-mb level for the summer and winter seasons of 1964 have been computed with the use of the (1-2.13), and are shown in Figures 2.33 and 2.34. In these figures, the solid curves are spectra of the potential temperature contributed by the waves moving from west to east, and the dashed curves are those contributed by the waves moving from east to west. It is seen from Figures 2.33 and 2.34 that the frequency spectra of the potential temperature generally decreases with increasing frequency. It may be noted that spectra contributed by the waves moving west to east is generally greater than those contributed by the waves moving from east to west, except at 20°N in the summer the easterly waves rather than the westerly waves contributed more to the spectral value of the potential temperature.

In the high frequency portion, the spectra of the potential temperature are approximately proportional to the minus three-half power of the frequency. The similarity between the spectra of the potential temperature and those of the horizontal velocity indicates that the spectra of the potential temperature are closely associated with the planetary waves in the atmosphere as shown in Figs. 2.1 through 2.4 and Figs. 2.31 and 2.32.
Fig. 2.33 Frequency spectra of the sensible heat at 200 mb, Summer 1964. The solid and dashed curves represent the spectra contributed by the waves moving from west to east and from east to west, respectively.
Fig. 2.34 Frequency spectra of the sensible heat at 200 mb, Winter 1964. The solid and dashed curves represent the spectra contributed by the waves moving from west to east and from east to west, respectively.
Fig. 2.35 Wavenumber spectra of the sensible heat at 200 mb, Summer 1964. The solid curves represent the spectra contributed by the stationary and moving waves, and the dashed curves are those contributed by the stationary waves only.
Fig. 2.36 Wavenumber spectra of the sensible heat at 200 mb, Winter 1964. The solid curves represent the spectra contributed by the stationary and moving waves, and the dashed curves are those contributed by the stationary waves only.
2-7. The Mean Sensible Heat Associated with the Motion in the Atmosphere

To examine the characteristics of the distribution of the mean sensible heat associated with the motion in the atmosphere, it is convenient to express the mean sensible heat as

$$\sum_{k} \sum_{n} E_{TT}(k,n) = E_{TT}(0,0) + \sum_{n \neq 0} E_{TT}(0,n) + \sum_{k \neq 0} E_{TT}(k,0) + \sum_{k \neq 0} \sum_{n \neq 0} E_{TT}(k,n)$$

(2-7.1)

The term on the left-hand side of the above equation represents the power of mean sensible heat, divided by $(\rho c_p^2)$, associated with the total motion (stationary and nonstationary) in the atmosphere. On the right-hand side of (2-5.6), the first term represents the mean sensible heat associated with the stationary zonal mean motion, the second term represents that associated with the nonstationary zonal mean motion, the third term represents that associated with the stationary waves, and the last term represents that associated with the transient waves. The distribution of the values of these terms is shown in Figures 2.37 to 2.41.

It is seen from Fig. 2.37 that at all levels the mean sensible heat associated with the total motion in the summer is greater than that in the winter, except at 20°N, 100-mb the potential temperature in the winter is greater than that in the summer. Comparing Figs. 2.38 to 2.41 one finds that the mean sensible heat associated with the stationary zonal mean motion (Fig. 2.38) is about two to three order of magnitude greater than that nonstationary associated with the nonstationary wave and/zonal mean motions. Fig. 2.38
shows also the square of the mean potential temperature distribution in the atmosphere for the summer and winter of 1964.

It may be noted that most of the mean sensible heat associated with the nonstationary motion is related to the transient waves (Fig. 2.41) rather than to the nonstationary zonal mean motion (Fig. 2.39). The maximum of the former in both the troposphere and the stratosphere occurs near 60°N, except that in the summer at 100-mb it occurs at 40°N and that in the winter at 500-mb it occurs near 80°N.

The mean sensible heat associated with the stationary waves (Fig. 2.40) is of the same order of magnitude as that associated with the transient waves (Fig. 2.41). The maximum of the former in the winter occurs near 60°N in both the troposphere and stratosphere, but shifts to the low latitudes in the summer.
Fig. 2.37 Sensible heat associated with the stationary and non-stationary motions.

Fig. 2.38 Mean sensible heat associated with the stationary zonal mean motion.
Fig. 2.39 Mean sensible heat associated with the nonstationary mean motion.

Fig. 2.40 Mean sensible heat associated with the stationary waves.

Fig. 2.41 Mean sensible heat associated with the transient waves.
Chapter 3

Spectral Characteristics of the Meridional Transports of Sensible Heat and Angular Momentum

3-1.1 Wavenumber-Frequency Spectra of the Meridional Transport of Sensible Heat

3-1.2 Spectra of the meridional Transport of Sensible Heat in Wavenumber and Frequency Domains

3-1.3 Mean Rate of the Meridional Transport of Sensible Heat in the Atmosphere

3-1.4 Meridional Transport of Sensible Heat in Relation to the latitudinal Mean Temperature Distribution

3-2.1 Wavenumber-Frequency Spectra of the meridional transport of Angular Momentum

3-2.2 Wavenumber and Frequency Spectra of the meridional Transport of the angular Momentum

3-2.3 Mean Rate of the Mean meridional Transport of angular Momentum in the Atmosphere.

3-3.1 Linear and Nonlinear Contributions to the meridional Transport of angular Momentum in Wavenumber-Frequency Space

3-3.2 Interaction Computations and Analyses.
3-1.1 The Wave-Number Frequency Co-Spectra of the Meridional Transport of Sensible Heat

The mean meridional flux of sensible heat over the Northern Hemisphere has been computed by White (1951, 1954). To investigate the effect of waves of various wave-lengths and frequencies on the meridional transport of sensible heat, the wave-number frequency co-spectra of the meridional heat flux will be analyzed. To do so, the longitude-time co-spectra of the meridional transport of sensible heat across 20°, 40°, 60°, and 80°N, at the 100-, 200-, 500- and 850-mb for the summer and winter seasons, 1964, have been computed with the use of Eqs. (1.2.1) and (1-2.9) (Kao and Sagendorf, 1969). Because of limited space, only the co-spectra of the heat transport at the 200-mb level for the summer and winter seasons are shown in Figs. 3.1 and 3.2. In these figures, the horizontal axis represents the frequency in unit cycles per 90 days and the vertical axis represents the wavenumber on the latitude circle. It should be noted that the scales in Fig. 3.1 and Fig. 3.2 are not the same.

A striking feature in the co-spectra of the meridional sensible heat transport is that there exists a preferred spectral band at various latitudes, which indicates the wavenumber frequency domain of interaction between the meridional velocity and temperature. In the middle latitudes, (40° and 60°N), the spectral band is oriented in a domain extending from a region of low wavenumbers and frequencies to a region of high wavenumbers and negative frequencies. In the high latitudes (80°), the heat transport is confined to a domain of very low wavenumbers (k=1 and 2) and frequencies. In the lower (20°N) latitudes, however, the co-spectra is confined in a very narrow band
centered near zero frequency. The change of the orientation of the spectral band with latitude may partially be explained by the increase in the Coriolis effect with increasing latitude on the wave motion in the atmosphere.

In both the troposphere (500 mb) and near the tropopause (200 mb) most of the heat transport occurs near 40°N, whereas in the stratosphere (100 mb) it shifts to near 60°N. In general, the transport in the winter is greater than that in the summer.

In the middle latitudes, most of the waves contribute to the poleward transport of sensible heat, except for a few slowly moving medium waves contributing to the equatorward transport of heat. In the low and high latitudes, the stationary and slowly moving waves contribute to most of the equatorward transport of heat in the atmosphere.
Fig. 3.1 Wave-number frequency spectra of the meridional sensible heat flux at 200-mb, Winter, 1964.
Fig. 3.2 Wave-number frequency spectra of the meridional sensible heat flux at 200-mb, Summer, 1964.
3-1.2 The Spectrums of the Meridional Transport of Sensible Heat in the Frequency and Wavenumber Domains

The distribution of the frequency spectrums of the rate of the meridional transport of sensible heat across 20°, 40°, 60°, and 80°N, at the 100-, 200-, 500-, and 850-mb for the winter and summer seasons of 1964 are computed with the use of Eqs. (1-2.9b). Because of limited space, only the spectrums at 100- and 850-mb are shown in Figs. 3.3 and 3.4. In these figures, the solid curves are the frequency spectrums of the transport contributed by waves moving from west to east, and the dashed curves are those contributed by waves moving from east to west. It is seen from these figures that most of the meridional transport of sensible heat is accomplished by the medium frequency eastward moving waves and the low frequency waves moving both east and west.

It may be noted that most of the transport is directed toward the North Pole and is accomplished by the medium frequency waves. The rate of transport across the middle latitudes (40° and 60°N) in both the troposphere and stratosphere is greater than that across the low (20°N) and high (80°N) latitudes. In general, the rate of transport in the winter is greater than that in the summer, and the transport in the lower levels is greater than that in the upper levels. In the lower level near 850-mb most of the transport is accomplished by the medium frequency and stationary waves, whereas in the upper levels near 100-mb most of the transport is accomplished by the low frequency waves.
The distribution of the wavenumber spectrums of the rate of the meridional transport of sensible heat at 100-, 200-, 500-, and 850-mb for the winter and summer seasons of 1964 are computed with the use of Eqs. (1-2.9a), and those at 100- and 850-mb are shown in Figs. 3.5 to 3.6. In these figures, as a visual aid the points of line spectra are connected by solid and dashed lines. The solid lines represent the contribution to the transport due to both the stationary and moving waves, whereas the dashed lines represent that due to the stationary waves only. The space between the solid and dashed lines in these figures represents, therefore, the transport due to the contribution of the moving waves.

It is seen from the wavenumber spectrums that the poleward transport of sensible heat is greatest across the middle latitudes near 850-mb. In the winter, most of the poleward transport is accomplished by the medium and long stationary waves near 850-mb, whereas the equatorward transport is accomplished by the long waves near 100-mb. In the summer, most of the poleward transport is contributed by the medium waves, and the equatorward transport is contributed by the long waves near the 850-mb; in the stratosphere near 100-mb, however, both the medium and long waves contribute to the poleward transport of sensible heat. In general, the transport across the middle latitudes is greater than that across the low and high latitudes, and the transport in the winter is greater than that in the summer.

The meridional convergence and divergence of the heat transport across 20°, 40°, 60°, and 80°N have been computed and found to be primarily the contribution of the long stationary waves and the medium transient waves in the atmosphere.
Fig. 3.3a. Frequency spectra of the meridional heat flux at 100 mb, Summer, 1964.
Fig. 3.3b. Frequency spectra of the meridional heat flux at 100 mb, Winter, 1964.
Fig. 3.4a. Frequency spectra of the meridional flux of sensible heat at 850 mb, Summer, 1964.
Fig. 3.4b  Frequency spectra of the meridional flux of sensible heat at 850 mb, Winter, 1964.
Fig. 3.5a. Wave-number spectra of the meridional heat flux at 100 mb, Summer, 1964.
Fig. 3.5b. Wave-number spectra of the meridional heat flux at 100 mb, Winter, 1964.
Fig. 3.6a. Wavenumber spectra of the meridional flux of sensible heat at 850 mb, Summer, 1964.
Fig. 3.6b. Wavenumber spectra of the meridional flux of sensible heat at 850 mb, Winter, 1964.
3-1.3 The Mean Rate of Meridional Transport of Sensible Heat in the Northern Hemisphere

To obtain a general picture of the maintenance of the budget of the sensible heat in the atmosphere, the rates of the total transport of sensible heat across 20°, 40°, 60°, and 80°N latitudes in the 100-, 200-, 500-, and 850-mb levels for the winter and summer seasons of 1964 are computed and shown in Fig. 3.7. It is seen from this figure that the total meridional transport of sensible heat at all four levels is mostly directed toward the North Pole, except at 20°N there is a weak equatorward transport of sensible heat at the 100- and 200-mb levels. It may be noted that the maximum poleward transport of sensible heat at the 100-mb level occurs near 60°N, whereas at the 200-, 500-, and 850-mb levels occurs near 40°N. Therefore, there is a divergence of sensible heat transport between 20° and 60°N and a convergence between 60° and 80°N at the 100-mb level, whereas at the 200-, 500-, and 850-mb there is a divergence of the sensible heat transport between 20° and 40°N and a convergence of it between 40° and 80°N. The larger latitudinal region of the meridional convergence of sensible heat in the middle and high latitudes in the troposphere than in the stratosphere indicates that in the middle and high latitudes in the troposphere there is a need for more import of sensible heat to compensate for the vertical heat transport and heat loss due to radiation than in the stratosphere. The rate of the total meridional transport of sensible heat across 20°, 40°, 60° and 80°N at the 100-, 200-, 500- and 850-mb levels in the summer season show a similar distribution as in the winter, except that the magnitude of the transport rate for the
summer is less than 50% of that for the winter, and the maximum poleward transport of sensible heat at 850 mb shifts from 40°CN in winter to 60°CN in the summer. This indicates the effect of the northward migration of the sun with respect to the earth in the summer season on the heat flux in the troposphere. In the stratosphere, however, the maximum poleward transport of sensible heat shifts from 60°CN in the winter to 50°CN in the summer. A similar result was found by Peixoto(1960).

To examine the mechanism for the meridional heat transport, it is convenient to express the rate of the total meridional heat transport as the sum of the transports due to the stationary and nonstationary waves, that is

\[
\sum \sum E_{VT}(k,n) = \sum E_{VT}(k,0) + \sum \sum E_{VT}(k,n),
\]

since \(E_{VT}(0,0) = E_{VT}(0,n) = 0\). The values of the terms in the above equation at 100-, 200-, 500- and 850-mb for summer and winter seasons are plotted in Figs. 11a,b,c and are shown in the Table of the Appendix.

The distributions of the rate of the meridional transport of sensible heat across 20°, 40°, 60° and 80°N latitudinal circles at the 100-, 200-, 500- and 850-mb due to the stationary waves in the winter and summer seasons of 1964 are shown in Fig. 3.8. It is seen from this figure that the stationary waves provide a poleward transport of sensible heat at nearly all four levels and the four latitudinal circles. The exceptions are at 80°N in the winter and at the 500- and 850-mb level in the summer where there is a weak transport toward the equator. Again, the rate of the meridional transport in the winter is generally
Fig. 3.7. Meridional flux of sensible heat due to stationary and nonstationary motion.

Fig. 3.8. Meridional flux of sensible heat due to stationary motion.

Fig. 3.9. Meridional flux of sensible heat due to nonstationary motion.
greater than that in the summer. However, the maximum poleward transport of sensible heat due to the stationary waves occurs near 60°N at the 100-, 500- and 850-mb, whereas at the 200-mb level it occurs near 50°N. These indicate that the stationary waves provide a meridional divergence of sensible heat transport between 20° and 60°N and a convergence between 60° and 80°N at the 100-, 500-, and 850-mb, whereas at the 200-mb level the stationary waves contribute to a divergence of sensible heat transport between 20° and 50°N, but a convergence between 50° and 80°N.

The distribution of the rate of the meridional transport of sensible heat across 20°, 40°, 60° and 80° latitudinal circles at the 100-, 200-, 500-, and 850-mb due to transient waves in the winter and summer seasons of 1964 are shown in Fig. 1.50. It is seen from this figure that transient waves contribute most of the poleward transport of sensible heat in the middle latitudes of the troposphere, and produce a maximum transport near 40°N at 200, 500 and 850 mb, and near 60°N at the 100-mb level. In the summer, the maximum transport at the 850-mb level shifts to 50°N, and in general the rate of the transport due to transient waves in summer is less than that in winter.

It may be noted from Figs. 3.8 and 3.9 that most of the convergence and divergence of the meridional transport of sensible heat in the troposphere is primarily the contribution of the transient waves, whereas in the stratosphere it is mainly the result of the stationary waves.
3-1.4 Meridional Transport of Sensible Heat in Relation to the Latitudinal Mean Temperature Distribution

We have found that the meridional sensible heat transport in both the troposphere and stratosphere is directed toward the North Pole, except in the low latitudes in the stratosphere where there is a weak transport of sensible heat toward the equator. To examine the relation between the meridional heat transport and the meridional gradient of temperature, the squares of the latitudinal mean temperature at the 850-, 100-, 200-, and 500-mb levels for the winter and summer seasons are respectively shown in Fig. 2.37. It may be noted that in both winter and summer, the meridional gradient of the temperature in the troposphere is directed toward the North Pole, indicating that the sensible heat transport in the troposphere is along the temperature gradient. In the stratosphere, however, the meridional gradient of temperature is directed toward the equator except north of 60°N in the winter where it is directed toward the North Pole. Therefore, the sensible heat transport in the stratosphere is mostly directed against the temperature gradient.

It is likely that there are two processes which result in this transport against the gradient of sensible heat in the stratosphere. The most obvious is the horizontal transport due to turbulent motion which may be along or against the temperature gradient. The second process that should be considered is the vertical transport of heat. It is likely that there is a downward transport from the stratosphere to the troposphere at high latitudes to compensate the heat lost by radiation in the lower levels.
3.2.1 The Wavenumber-Frequency Spectra of the Meridional Transport of Angular Momentum

The importance of the meridional transport of angular momentum in relation to the general circulation in the atmosphere was first pointed out by Jeffreys (1926) and was later reemphasized by Bjerknes (1948), Starr (1948), and others. Quantitative analyses of the mean flux of angular momentum were made by Widger (1949), Mintz (1951), Kubota and Iida (1954), Obasi (1963), and Tewelles (1963). Investigations of the wavenumber spectra of the meridional flux of angular momentum were made by Kao (1954), Benton and Kahn (1958), and Saltzman (1958).

The meridional transport of angular momentum across a latitude \( \phi \) per unit pressure difference per unit time may be written as

\[
\frac{2 \pi a^2 \cos \phi}{g} \int_{0}^{2\pi} \int_{-\infty}^{\infty} u(\lambda, t) v(\lambda, t) d\lambda dt
\]

With the use of (1-2.10) the above expression may be written as

\[
\sum_{k=0}^{\infty} \int_{-\infty}^{\infty} \frac{2 \pi a^2 \cos \phi}{g} E_{uv}(k,n) dn = \sum_{k=0}^{\infty} EUV(k,n) dn
\]

where \( EUV(k,n) \) is the wavenumber frequency spectrum for the meridional transport of angular momentum. The values of \( EUV(k,n) \) at 100-, 200-, and 500-mb, at 20°, 40°, 60°, and 80°N for the Summer and Winter of 1964 have been computed (Kao, Tsay and Wendell, 1969). Because of limited space, only those at 200-mb level are shown in Figs. 3.10 and 3.11.

Similar to the wavenumber-frequency spectra of the meridional transport of the sensible heat, the spectra of the meridional transport of angular momentum show a preferred spectral band at various latitudes, which indicates the wavenumber frequency domain of interaction between the zonal and meridional components of the velocity. However, there
are basic differences in the spectral and mean distributions of the meridional transport of angular momentum and sensible heat. For example, at 60°N, 200-mb level, the mean meridional transport of angular momentum is directed toward the equator, whereas that of the sensible heat is directed toward the North Pole as shown in Figs. 3.7 and 3.16. Differences can also be found in the detail distribution of the wavenumber-frequency spectra of the meridional transport of angular momentum and sensible heat. The reason for these differences lies in the fact that the rate of the meridional transport of angular momentum is affected by the surface friction and mountain-pressure, whereas that of the sensible heat is affected by the radiative heating and cooling in the atmosphere.

It is seen from Figs. 3.10 and 3.11 that most of the poleward transport of angular momentum is accomplished by the medium-size waves moving from west to east, and that of the equatorward transport is contributed by the medium-size waves moving from east to west. This indirectly shows that most of the eastward moving waves tend to orient the principal axis of dispersion along the direction of WSW-ENE, whereas most of the westward moving waves tend to orient the principal axis of dispersion in the direction of WNW-ESE.

However, some of the eastward moving waves provide an equatorward transport of angular momentum, particularly those slowly moving waves of wavenumbers 4 and 5 at the low latitudes (20°N) in the summer, and waves of wavenumber 6 in the frequency range, -3 and -6, in the middle latitudes (40°N) in the winter.
Fig. 3.10. Wavenumber-frequency spectra of the meridional transport of angular momentum at 200 mb, Summer, 1964.
Fig. 3.11. Wavenumber-frequency spectra of the meridional transport of angular momentum at 200 mb, Winter, 1964.
3-2.2 The Frequency and Wavenumber Spectra of the Meridional Transport of Angular Momentum

The frequency spectra of the meridional transport of angular momentum at 200-mb level at 20°, 40°, 60°, and 80°N for the summer and winter of 1964 are shown in Figs. 3.12 and 3.13. In these two figures, the solid curves are the spectra contributed by the waves moving from west to east, and the dashed curves are those contributed by the waves moving from east to west. Figure 3.12 shows that in the summer most of the poleward transport of angular momentum occurs near 40°N and is accomplished by the eastward moving waves, and that in the winter both the westward and eastward moving waves contribute to the poleward transport of angular momentum. In the summer, most of the equatorward transport of angular momentum is accomplished by the westward moving waves of low frequencies (Fig. 3.12). In the winter, however, only the westward moving waves at the low latitudes (20°N) contribute to the equatorward transport (Fig. 3.13).

The wavenumber spectra of the meridional transport of angular momentum at 200-mb level are shown in Figs. 3.14 and 3.15. In these two figures, the solid curves are the spectra contributed by both the stationary and nonstationary waves, and the dashed curves are those contributed by the stationary waves only. It may be noted that in both the summer and winter seasons waves of practically all wavelengths in the low and middle latitudes contribute to the poleward transport of angular momentum. In the high latitudes (60°N), however, only the very long waves, particularly of \( k = 1 \), contribute to the equatorward transport of angular momentum.
In the winter, most of the meridional transport of angular momentum is accomplished by the stationary waves (Fig. 3.15), whereas in the summer most of the transport is accomplished by the moving waves, except in the low latitudes (20°N) the stationary waves contribute most of the transport (Fig. 3.14).
Fig. 3.12. Frequency spectra of the meridional transport of angular momentum at 200 mb, Summer, 1964. The solid and dashed curves represent respectively the spectra contributed by waves moving from west to east and from east to west.
Fig. 3.13. Frequency spectra of the meridional transport of angular momentum at 200 mb. Winter, 1964. The solid and dashed curves represent respectively the spectra due the waves moving from west to east and from east to west.
Fig. 3.14. Wavenumber spectra of the meridional transport of angular momentum at 200 mb, Summer, 1964. The solid curves represent the spectra due to the moving and stationary waves, the dashed curves represent those due to the stationary waves only.
Fig. 3.15. Wavenumber spectra of the meridional transport of angular momentum at 200 mb, Winter, 1964. The solid curves represent the spectra due to the moving and stationary waves, the dashed curves represent those due the stationary waves only.
3-2.3  The Mean Rate of the Mean Meridional Transport of Angular Momentum in the Northern Hemisphere

To obtain a general picture of the distribution of the meridional transport of angular momentum in the troposphere and stratosphere, the rate of the mean meridional transport of angular momentum by both the stationary and nonstationary motions at 100-, 200-, and 500-mb levels, at 20°, 40°, 60°, and 80°N for the summer and winter of 1964 are shown in Fig. 3.16. This figure shows that in the winter the maximum poleward transport of angular momentum occurs near 40°N at all three levels, whereas the maximum equatorward transport occurs near 60°N. Of the three levels, the maximum poleward transport of angular momentum occurs near the tropopause (200-mb), and the maximum of the equatorward transport occurs near 500-mb. In the summer, the maximum poleward transport occurs near 40°N in the troposphere (500-mb), and it occurs near 20°N in the stratosphere (100-mb). There is no noticeable equatorward transport of angular momentum at all three levels in the summer.

Figure 3.16 indirectly shows that in the winter the field of motion tends to orient the principal axis of dispersion in the direction of WSW-ENE between 20° and 60°N, but in the direction of WNW-ESE between 60° and 80°N; in the summer the field of motion tends to orient the principal axis of dispersion in the direction of WSW-ENE at all latitudes.

To examine the mechanism of the meridional transport of angular momentum in the atmosphere, it is convenient to separate the transport into two terms: one representing the transport contributed by the stationary waves, and the other by the nonstationary waves as the follows:

\[ \sum_{k} \sum_{n} E_{vu}(k,n) = \sum_{k \neq 0} E_{vu}(k,0) + \sum_{k \neq 0} \sum_{n \neq 0} E_{vu}(k,n) \]  

(3-2.1)
since $E_{vu}(0,0) = E_{vu}(0,n) = 0$. The values of the first and second terms on the right-hand side of the above equation, representing respectively the meridional transport of angular momentum contributed by the stationary and transient waves, at 100-, 200-, and 300-mb levels for the summer and winter are plotted in Figs. 3.17 and 3.18. It is seen from these two figures that in the winter most of the meridional transport of angular momentum is accomplished by the stationary waves, and that the transient waves contribute about one third of the total transport in the low ($20^\circ$N) and middle ($40^\circ$N) latitudes. In the summer, however, most of the transport in the middle latitudes is accomplished by the moving waves in the middle latitudes, and that in the low latitudes, particularly in the stratosphere, is accomplished by the stationary waves.

It may be noted from Figs. 2.15 and 3.18 that in the winter the meridional transport of westerly momentum at all three levels is against the gradient of the zonal mean momentum in the regions between $20^\circ$ and $40^\circ$N.

In the summer, the transport is against the gradient of the mean momentum in the regions between $20^\circ$ and $40^\circ$N at 200- and 500-mb levels.
Fig. 3.16. Meridional transport of angular momentum by stationary and nonstationary motion.

Fig. 3.17. Meridional transport of angular momentum by the stationary waves.

Fig. 3.18. Meridional transport of angular momentum by the transient waves.
3-3.1 The Linear and Nonlinear Contributions to the Meridional Transport of Angular Momentum in Wavenumber-Frequency Space

To evaluate the linear and nonlinear contributions to the meridional transport of angular momentum, it is convenient to express (1-5.8) in the following form,

\[
\frac{2-\tau^2}{g} \mathcal{E}_{uv}(k,n) = \frac{aT \cos^2 \phi}{g n} \left\{ \sum_j \sum_m j U(k-j,n-m) \left[ U(j,m)V(-k,-n) + V(j,m)U(-k,-n) \right] \right. \\
+ \left. V(j,m)U(-k,-n) \right] + i \sum_j \sum_m V(k-j,n-m) \left[ U(\phi,j,m)V(-k,-n) + V(\phi,j,m)U(-k,-n) \right] \\
+ \left. i \tan \phi \sum_j \sum_m U(j,m) \left[ U(-k,-n)U(k-j,n-m) - V(-k,-n)V(k-j,n-m) \right] \right\} \\
+ i \frac{T \alpha^2 \cos^2 \theta}{g n} \left\{ \left[ U(-k,-n) \left[ U(k,n) - U_g(k,n) \right] - V(-k,-n) \left[ V(k,n) - V_g(k,n) \right] \right] \right. \\
+ \left. \left[ U(-k,-n) G_2(k,n) - V(-k,-n) G_1(k,n) \right] \right\} \\
+ \left. \left[ \sum_k \sum_m W(k-j,n-m) \left[ U(\phi,j,m)U(-k,-n) + V(\phi,j,m)V(-k,-n) \right] \right] \right\} \\
(EUV) \hspace{1cm} (UV1) \hspace{1cm} (UV2) \hspace{1cm} (UV3) \hspace{1cm} (UV4) \hspace{1cm} (UV5) \hspace{1cm} (UV6) \hspace{1cm} (3-3.1)
\]

where \( U_g \) and \( V_g \) are the transforms of the zonal and meridional components of the geostrophic velocity. The symbols below the terms on the right-hand side of the above equation are for future reference. The summations over the index \( j \) are from \(-j_s\) to \(+j_s\) where \( j_s \) depends on
the number of discrete points on the latitude circle. The summations
over \( m \) are from \( -m_s \) to \( +m_s \) depending on whether frequency \( n \) is posi-
tive or negative, where \( m_s \) depends on the number of discrete time
points used. Summation is used here in place of integration with
respect to \( m \), as in (1-5.8), because the transforms are computed for
integer values of frequency.

In order to compute the interaction terms involving derivatives
with respect to latitude, velocity values were interpolated onto lati-
tude circles \( 21^{\circ} \) north and south of the latitude of concern in order
to provide a centered difference of \( 5^{\circ} \). The quantities \( U_\phi(k,n) \) and
\( V_\phi(k,n) \) are the transforms of the centered difference approximations
to \( \partial u/\partial \phi \) and \( \partial v/\partial \phi \). A factor of \( T/2\pi \) was multiplied to the right-
hand side of (3-3.1) because of the time transformation, \( t = 2\pi t'/T \),
which results in \( \partial/\partial t' = 2\pi T^{-1} \partial/\partial t \).

The terms represented by (UV1), (UV2) and (UV6) stem respectively
from the transformed longitudinal, latitudinal and vertical convergence
of the meridional flux of angular momentum. The terms denoted by (UV3)
arise from the sphericity of the earth. The terms denoted by (UV4)
represent the contribution to the meridional flux of angular momentum
due to the ageostrophic motion, and those denoted by (UV5) represent
the contribution due to the eddy and molecular stress forces.
3-3.2 Interaction Computations and Analyses

The terms of the angular momentum transport equation (1-11.1) are computed for the separate wavenumbers and frequencies of each category. The interaction terms are each computed for 406 interaction combinations in order that the important combinations will be apparent. The results for each wavenumber and frequency are then summed over each category and presented in block diagram form shown in Figs. 1.60 through 1.65. In these diagrams the total contributions of each of the terms in (3-3.1) are shown in the small blocks in the center of the diagram and labeled with the symbols for the terms they represent. All the terms UV1 through UV4 are computed directly from the available pressure and wind data. The term UV5, however, cannot be computed directly because no data concerning the eddy frictional force is available. The value of the term UV6, which involves the vertical motion, is generally small and may be neglected.

The sums of the quantities UV1 through UV4 are displayed below the appropriate blocks. To the left of these sums are shown the sums of the spectral value of the meridional flux of angular momentum over the category. The differences between the sums of the terms and the spectral value for the category are displayed to the right of the sums of the terms and labeled UV5 + E which contains, in additional to the eddy frictional effect, the contribution due to the vertical motion and the results of error in the data.

The larger blocks, connected with arrows to the small blocks labeled UV1 and UV2, contain positive and negative interaction combinations which represent the major contributions to the totals in the small blocks.
The partitioned interaction combinations for UV3 were not computed because test computations indicated that the resultant contribution through these terms is generally small. The figures at the bottom of the large blocks represent the total positive and negative contributions from all the interaction combinations. Since only the interaction combinations with significant magnitude are listed, the sums of those listed will only approximate the totals at the bottoms of the blocks. A positive value for an interaction combination or the sum over all interaction combinations, for a given k and n, is interpreted as a contribution to the poleward flux of the angular momentum, whereas a negative value is interpreted as a contribution to the equatorward flux of the angular momentum.

It may be noted from Figs 3.19 through 3.24 that \((M,0)*(k,n)\) and \((P,0)*(k,n)\) always are the primary interaction combinations which provide the major contribution to the wavenumber frequency spectra of the meridional flux of the angular momentum, and that the secondary interaction combinations for UV1 and UV2 generally involve all scales of motion and the full range of frequencies. However, comparatively small contribution involves in the high frequency short waves. It may also be noted that the positive contribution to the meridional flux of the angular momentum comes primarily from the interactions between the low frequency eastward moving planetary waves and the stationary mean zonal motion, whereas the negative contribution comes primarily from the interactions between the low frequency cyclone waves and the stationary planetary waves and the mean zonal motion.

A summary of the primary and secondary nonlinear interaction contributions to UV1 and UV2 is shown in Fig. 3.25. In this figure, only those interaction combinations whose values are greater than 2 are listed, and the primary interactions are underlined. It may be noted that the positive contribution to UV1 involves primarily the interaction combinations \((M,0)*(P,-L)\) and \((P,+L)*(P,-L)\), whereas the negative contribution involves \((M,0)*(C,+L)\), \((P,0)*(C,-L)\) and \((P,0)*(P,-L)\). The positive contribution to UV2 involves \((P,0)*(P,+L)\), whereas the negative contribution involves \((P,0)*(P,-L)\) and \((P,+L)*(C,-L)\).
Fig. 3.19. Linear and nonlinear contributions to the spectra of the meridional flux of the angular momentum in the domain (P,-L), at 500 mb, 40°N, Winter, 1964.
Fig. 3.20. Linear and nonlinear contributions to the spectra of the meridional flux of the angular momentum in the domain \((P,+L)\), at 500 mb, 40°N, Winter, 1964.
Fig. 3.21. Linear and nonlinear contributions to the spectra of the meridional flux of the angular momentum in the domain (P,-I), at 500 mb, 40° N, Winter, 1964.
Fig. 3.22. Linear and nonlinear contributions to the spectra of the meridional flux of the angular momentum in the domain (C, -L), at 500 mb, 40°N, Winter, 1964.
Fig. 3.23. Linear and nonlinear contributions to the spectra of the meridional flux of the angular momentum in the domain (C,+L), at 500 mb, 40°N, Winter, 1964.
Fig. 3.24. Linear and nonlinear contributions to the spectra of the meridional flux of the angular momentum in the domain \((C,-I)\), at 500 mb, 40°N, Winter, 1964.
Fig. 3.25. Positive and negative nonlinear contributions to UV1 and UV2.
Table 3.1
Linear and Nonlinear Contributions to the Meridional Transport of the Angular Momentum in Various Wavenumber-Frequency Domains

<table>
<thead>
<tr>
<th></th>
<th>EUV</th>
<th>UV1</th>
<th>UV2</th>
<th>UV3</th>
<th>UV4</th>
<th>UV5+E</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P,-L)</td>
<td>7.65</td>
<td>29.12</td>
<td>-27.95</td>
<td>2.34</td>
<td>-22.81</td>
<td>26.95</td>
</tr>
<tr>
<td>(P,+L)</td>
<td>3.67</td>
<td>-6.03</td>
<td>17.51</td>
<td>-9.02</td>
<td>40.75</td>
<td>-39.54</td>
</tr>
<tr>
<td>(P,-I)</td>
<td>3.12</td>
<td>5.74</td>
<td>0.68</td>
<td>0.27</td>
<td>-4.39</td>
<td>0.82</td>
</tr>
<tr>
<td>(C,-L)</td>
<td>-5.13</td>
<td>-20.60</td>
<td>2.10</td>
<td>-0.09</td>
<td>14.05</td>
<td>-0.59</td>
</tr>
<tr>
<td>(C,+L)</td>
<td>0.39</td>
<td>-8.34</td>
<td>1.40</td>
<td>0.57</td>
<td>0.75</td>
<td>6.01</td>
</tr>
<tr>
<td>(C,-I)</td>
<td>5.18</td>
<td>12.53</td>
<td>-2.84</td>
<td>0.01</td>
<td>-6.24</td>
<td>1.72</td>
</tr>
</tbody>
</table>

To gain an insight into the linear and nonlinear contributions to the meridional flux of the angular momentum in various wavenumber-frequency domains, the resultant values of EUV, UV1, UV2, UV3, UV4 and (UV5 + E) in various domains are listed in Table 3.1. It may be noted from this Table that of the six wavenumber-frequency domains the domain of the low frequency eastward moving cyclone waves, (C,-L), is the only one contributes to an equatorward flux of the angular momentum. Because of its distinct characteristic, we shall discuss it separately from the other five domains.

(A) In the domain of the low frequency eastward moving cyclone waves, (C,-L), (1) the meridional flux of the angular momentum (EUV) is directed toward the equator, (2) the resultant of the nonlinear interactions due to the longitudinal convergence of the meridional flux of angular momentum (UV1) provides an equatorward transport of angular momentum, (3) the resultant of the nonlinear interactions due to the latitudinal convergence of the meridional flux of the
angular momentum (UV2) provides a poleward transport of the angular momentum, (4) the ageostrophic effect (UV4) contributes a poleward flux of the angular momentum, which tends to counteract the nonlinear interactions (UV1) and (UV2), (5) the effect of the sphericity of the earth is small, (6) the frictional force and vertical motion contribute a slight, equatorward flux of the angular momentum.

(B) In the domains (P,-L), (P,+L), (P,-I), (C,+L) and (C,-I), the following characteristics are in common: (1) the meridional flux of the angular momentum is directed toward the north pole, (2) the resultant of the nonlinear interactions due to the longitudinal convergence of the meridional flux of the angular momentum (UV1) provides a poleward flux of the angular momentum in the domains of eastward moving waves, but provides an equatorward transport in the domains of westward moving waves, (3) the resultant of the nonlinear interactions due to the latitudinal convergence of the meridional flux of the angular momentum (UV2) generally contributes to a poleward flux of the angular momentum in the domains of westward moving waves, but to an equatorward flux in the domains of eastward moving waves, (4) the ageostrophic effect (UV4) provides a poleward transport of the angular momentum in the domains of westward moving waves, but an equatorward flux in the domains of eastward moving waves, (5) the effects of the frictional force and vertical motion generally work against the ageostrophic effect, i.e., have the opposite sign to UV4, (6) the effect of the frictional force generally decreases with decreasing wavelength and increasing frequency of the waves, (7) the effect of the sphericity of the earth is generally small as compared with other terms in the equation.
REFERENCES


Eliasen, E. and B. Mannenhauer, 1969: On the observed large-scale atmospheric wave motion. Tellus, 21, 149-165.


