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COO-30-97

A TECHNIQUE FOR STUDYING THE PHYSICAL SCATTERING MATRIX*

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This paper was submitted for publication in the open literature at least 9 months prior to the issuance date of this Microcard. Since the U.S.A.E.C. has no evidence that it has been published, the paper is being distributed in Microcard form as a preprint.

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ABSTRACT

The Fredholm theory of integral equations is used in a Hilbert space of functions which are square integrable on a contour in order to discuss the Lippmann-Schwinger equation. By exploiting the analyticity of the potential matrix in its indices a simple proof of the existence of the physical scattering matrix is given. The high energy behavior of the physical scattering matrix is also discussed.

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I. INTRODUCTION

The Fredholm theory of integral equations has been used many times to study the Lippmann-Schwinger equation. In this discussion a slight extension of the usual ideas of Fredholm theory is employed to re-examine the scattering of a particle from a static potential in momentum space. Most of the results obtained are known and the only excuse for presenting this discussion is to illustrate the techniques involved.

In the next section, the basic ideas for discussing Fredholm theory on contours are introduced and the existence of the physical partial wave scattering matrix is proved. In Section III, unitarity is discussed and used to analytically continue the scattering matrix in the energy. The high energy behavior of the physical scattering matrix is then discussed. In Section V, the partial wave dispersion relation is proved. Finally, the three dimensional scattering matrix is briefly discussed.

II. FREDHOLM THEORY ON CONTOURS

The simplest type of Lippmann-Schwinger equation will be used to introduce the ideas which are involved in extending Fredholm theory to contours in a manner which is useful for scattering theory.

A scattering matrix, $T(s, g)$, is defined by the equation

$$T(s, g) = gV + gK(s) T(s, g) \quad (1)$$

where the kernel, $K(s)$, has the specific form

$$\langle p' | K(s) | p \rangle = \langle p' | V | p \rangle \frac{1}{s - p^2} \quad (2)$$

Equation (1) is understood as an operator equation on the Hilbert space of square integrable functions $L_2(0, \infty)$. On this space, it is assumed that $K(s)$ is a family of analytic Schmidt operators for all $s \in [0, \infty]$. This means that for any $s \in [0, \infty]$:

$$\|K(s)\|^2 = \int_0^\infty dp \int_0^\infty dp' |\langle p' | K(s) | p \rangle|^2 \quad (\text{Schmidt Condition})$$

and, for any ψ and $\phi \in L_2(0, \infty)$, $(\psi, K(s)\phi)$ is analytic in s .

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For any $K(s)$ which is a Schmidt operator, the resolvent, $R(s, g)$, of $K(s)$ exists, where

$$R(s, g) \equiv [1 - gK(s)]^{-1}$$

$R(s, g)$ is given explicitly by the Fredholm expansion.¹

Thus, for $s \notin [0, \infty]$,

$$T(s, g) = gR(s, g) = gV + g^2 K(s) R(s, g) V. \quad (3)$$

The first question to arise is whether the physical scattering matrix,

$$t_+(k^2, g) \equiv \lim_{\epsilon \rightarrow 0} \langle k | T(k^2 + i\epsilon, g) | k \rangle \quad k > 0, \quad (4)$$

exists. This question has been answered affirmatively many times.

A sufficient condition which ensures that the limit in Eq.(4) exists is that $\langle p' | V | p \rangle$ be analytic in its indices around the real, positive p, p' axes. To see this, consider the resolvent equation

$$R_c(s, g) = 1 + gK_c(s) \cdot R_c(s, g). \quad (5)$$

This is to be understood as an equation in the Hilbert space of functions square integrable on a contour c , $L_2(c)$, in the sense that

$$(\psi, \phi)_c \equiv \int_c |dp| \overline{\psi(p)} \phi(p).$$

The contour c runs from 0 to ∞ .

Matrix multiplication in $L_2(c)$ is defined in terms of the bilinear form

$$\phi \cdot \psi \equiv \int_c \phi(p) dp \psi(p).$$

Note that $|\phi \cdot \psi| \leq \|\phi\|_c \|\psi\|_c$, where $\|\phi\|_c^2 \equiv (\phi, \phi)_c$.

Thus if $\phi(p)$ is analytic in p in a region including c and the real, positive p -axis, then

$$\phi \cdot \phi = \int_c \phi^2(p) dp = \int_0^\infty \phi^2(p) dp.$$

In general, however,

$$\|\phi\| \neq \|\phi\|_c$$

so that the mapping $\phi \in L_2(0, \infty) \rightarrow \phi \in L_2(c)$ is not isometric.

If $\|K_c(s)\|_c < \infty$, then equation (5) can be solved by Fredholm theory and $R_c(s, g)$ exists. Furthermore, it has the same initial analyticity as $K(s)$.

Thus if $\langle p | V | p' \rangle$ is analytic in a region of the real positive p and p' -axes, then

$$\langle k | T(k^2 + i\epsilon, g) | gV | k \rangle = g^2 \int_0^\infty \int_0^\infty \langle k | K(k^2 + i\epsilon) | p \rangle dp \times$$

$$\langle p | R(k^2 + i\epsilon, g) | p' \rangle dp' \langle p' | V | k \rangle = g^2 \int_c^\infty \int_c^\infty \langle k | K(k^2 + i\epsilon) | p \rangle dp \times$$

$$\langle p | R_c(k^2 + i\epsilon, g) | p' \rangle dp' \langle p' | V | k \rangle.$$

A possible contour is shown in Figure 1. It is assumed that $\langle p | V | k \rangle$ and $\langle k | V | p \rangle$ are analytic in p for all $k \in [0, \infty]$ in a region of the p -plane including $[0, \infty]$ and c .

Finally, in the limit $\epsilon \rightarrow 0$, the last term in Equation (6) exists. Thus, $t_+(k^2, g)$ exists. Similarly,

$$t_-(k^2, g) \equiv \lim_{\epsilon \rightarrow 0} \langle k | T(k^2 - i\epsilon, g) | k \rangle$$

exists.

The essential point is that $\|K_c(k^2 + i\epsilon)\|_c \neq \|K(k^2 + i\epsilon)\|$.

Indeed, $\|K(k^2 + i\epsilon)\| \xrightarrow{\epsilon \rightarrow 0} \infty$.

III. UNITARITY

It is a simple matter to show that

$$t_+(k^2, g) - t_-(k^2, g) = \frac{\pi i}{k} t_+(k^2, g) t_-(k^2, g). \quad (7)$$

First, define

$$\langle k | \Delta T(k^2, g) | k \rangle = \langle k | T(k^2 + i\epsilon, g) - T(k^2 - i\epsilon, g) | k \rangle.$$

From eq. (1), it follows that

$$\langle k | \Delta T(k^2, g) | k \rangle = \langle k | g \Delta K(k^2) R(k^2 - i\epsilon, g) V + gK(k^2 + i\epsilon) \Delta R(k^2, g) V | k \rangle \quad (8)$$

Similarly,

$$\Delta R(k^2, g) = g \Delta K(k^2) R(k^2 - i\epsilon, g) + gK(k^2 + i\epsilon) \Delta R(k^2, g).$$

Since $R(k^2 + i\epsilon, g)$ exists this last equation can be solved for ΔR ,

$$\Delta R(k^2, g) = gR(k^2 + i\epsilon, g) \Delta K(k^2) R(k^2 - i\epsilon, g). \quad (9)$$

Equation (9) is the general unitarity condition for the resolvent.

Combining Eq. (8) and (9) gives

$$\langle k | \Delta T(k^2, g) | k \rangle = \langle k | R(k^2 + i\epsilon, g) \Delta K(k^2) T(k^2 - i\epsilon, g) | k \rangle. \quad (10)$$

Finally using the fact that

$$\Delta K(k^2) = V | k \rangle \frac{i\pi}{K} \langle k | + O(\epsilon),$$

Equation (10) may be reduced to Equation (7) in the limit $\epsilon \rightarrow 0$.

Equation (9) may be used to continue $R(s, g)$ to the second sheet.³

If $\Delta K(k^2)R(k^2 - i\epsilon, g)$ is a Schmidt kernel, then

$$R(k^2 + i\epsilon, g) = R(k^2 - i\epsilon, g) [1 - g \Delta K(k^2) R(k^2 - i\epsilon, g)]^{-1} \\ \equiv R_{II}(k^2 - i\epsilon, g).$$

IV. HIGH ENERGY BEHAVIOR

In order to discuss some further applications of the techniques developed in Section II, it is convenient to consider a special example.

It will come as no surprise if the Yukawa potential is used.

For the l th partial wave,

$$\langle p' | V | p \rangle = \frac{1}{\pi} Q_\ell \left(\frac{p^2 + p'^2 + \mu^2}{2pp'} \right) \quad (11)$$

where an extraneous pp' has been removed.

It is convenient to study a symmetric kernel which is simply related to $K(s)$ rather than $K(s)$ itself.

$$K'(s) \equiv G_0^{1/2}(s) V G_0^{1/2}(s)$$

$$\langle p' | G_0(s) | p \rangle \equiv \delta(p' - p) [s - p^2]^{-1/2}.$$

Then

$$K'(s) G_0^{1/2}(s) = G_0^{1/2}(s) K(s)$$

and the resolvent of K' , R' , is related to R by the equation

$$R'(s, g) G_0^{1/2}(s) = G_0^{1/2}(s) R(s, g).$$

Therefore,

$$T(s, g) = gV + g^2 V G_0^{1/2}(s) R'(s, g) G_0^{1/2}(s) V.$$

It will now be shown that in the high energy limit, $t_+(E, g)$ is given by the Born-Neumann series. First, recall that if $g \| K'_c(E) \|_c < 1$, $R'_c(E, g)$ is given by the B.N. series.

$$\| K'_c(E) \|_c^2 = \frac{1}{\pi^2 E} \int_c |dq| \int_c |dq'| |1 - q^2|^{-1} |Q_\ell \left(\frac{q^2 + q'^2 + \frac{\mu^2}{E}}{2qq'} \right)| |1 - q'^2|^{-1} \quad (12)$$

where the substitution $p = \sqrt{E} q$ has been made. The integral converges to a finite limit as $E \rightarrow \infty$, so $g \| K'_c(E) \|_c < 1$ for all E greater than some $E(g)^4$.

It is easy to show that

$$\| K'_c(E) \|_c \xrightarrow{E \rightarrow \infty} O\left(\frac{\log E}{\sqrt{E}}\right).$$

Since $R'_c(E, g)$ is given by the B.N. series for large E , $t_+(E, g)$ will also be given by the B.N. series provided $\langle p | V | k \rangle$, the "mixed"

elements satisfy the condition assumed following Equation (6). This restricts c .

V. THE DISPERSION RELATION FOR THE SCATTERING MATRIX

For the V defined by Equation (11), the well-known dispersion relation

$$\begin{aligned} \langle \sqrt{s} | T(s, g) | \sqrt{s} \rangle \equiv t(s, g) = & \int_0^{\infty} \frac{dE}{2\sqrt{E}} \frac{t_+(E, g) t_-(E, g)}{s - E} \\ & + \int_{-\infty}^{-\frac{\mu^2}{4}} \frac{dE}{2\pi i} \frac{\Delta t(E, g)}{s - E} \end{aligned}$$

holds. $t(E, g)$ is the discontinuity of t across the left-hand cut.

First note that $\langle \sqrt{s} | V | \sqrt{s} \rangle$ is analytic in the s -plane cut from $-\infty$ to $-\frac{\mu^2}{4}$.

Secondly, note that $R_c(s, g)$ is analytic in the s -plane cut from 0 to ∞ along c^2 where c^2 is a contour defined by the mapping $x \in c \rightarrow x^2 \in c^2$.

Finally, it remains to discuss the "mixed" matrix elements $\langle p | V | \sqrt{s} \rangle$, $p \in c$. It is simplest to define c by $p = qe^{-i\phi}$ where $q \in [0, \infty]$ and $\phi > 0$. If $\psi = \arg \sqrt{s}$, then ϕ must be restricted by the relation $\min(\psi, \frac{\pi}{2}) > -\phi > \max(\pi + \psi - \frac{\pi}{2})$ so that $\langle p | K_c(s) | p' \rangle$ is a Schmidt kernel.

$\langle p | V | \sqrt{s} \rangle$ is analytic unless the relation

$$-1 \leq \frac{p^2 + s + \mu^2}{2p \sqrt{s}} \leq 1 \quad (13)$$

is satisfied. By studying Equation (13) in detail, it is found that the last term in Equation (6) may be analytically continued to all s excluding the line $[-\infty, -\mu^2]$. To see that there is actually a cut along this line, it is necessary to calculate the discontinuity and show that it is non-vanishing.

The results of Section IV ensure that no subtraction are necessary.

VI. THE FULL SCATTERING MATRIX

So far the discussion has been confined to one dimensional problem. It is interesting to consider the three-dimensional scattering matrix. For this case, Equation (2) is replaced by

$$\langle \vec{p}' | K(s) | \vec{p} \rangle = \langle \vec{p}' | V | \vec{p} \rangle \frac{1}{s - p^2}.$$

It is convenient to define $\vec{p} = p \hat{e}$ and $\vec{p}' = p' \hat{e}'$ and to continue $K(s)$ to $K_c(s)$ by making p and p' complex.

For concreteness, take

$$\langle \vec{p}' | V | \vec{p} \rangle = \frac{1}{(\vec{p} - \vec{p}')^2 + \mu^2}.$$

Then

$$\|K(s)\|^2 = (4\pi)^2 \int_0^\infty dq q^2 \frac{1}{(q^2 + \mu^2)^2} \int_0^\infty dx x^2 \frac{1}{|s - x^2|^2}$$

for $s \notin [0, \infty]$.

As before, introduce $K_c(s)$ and $R_c(s, g)$ where c is chosen as shown in Figure 1. Then $\|K_c(s)\|_c < \infty$ for all $s \notin c^2$ provided the radius of the small semi-circle, a , is less than μ .

If $a < \mu$ then the "mixed" matrix elements, $\langle \vec{p} | V | \vec{k} \rangle$, exist and hence

$$t_+(k, \hat{e}' \cdot \hat{e}) \equiv \lim_{\epsilon \rightarrow 0} \langle k \hat{e}' | T(k^2 + i\epsilon) | k \hat{e} \rangle$$

exists.

To study the high energy behavior of t_+ , it is convenient to introduce a new kernel, as was done before.

$$K'(s) = G_0^{1/2}(s) V G_0^{1/2}(s)$$

$$\langle \vec{p}' | G_0^{1/2}(s) | \vec{p} \rangle \equiv \delta(\vec{p}' - \vec{p}) (s - p^2)^{-1/2}.$$

Then if $a < \mu$ it is shown in the Appendix, that

$$\|K_c(k^2)\|_c \xrightarrow{k \rightarrow \infty} O\left(\sqrt{\frac{\mu}{k}}\right).$$

Thus, $t_+(k^2, \hat{e} \cdot \hat{e})$ is given by the B.N. series at high energies.

Finally, a few remarks on the restriction $\mu > a$ are in order. To understand what it means, it is easiest to choose c to be a ray from 0 to $\infty e^{-i\phi}$, $\phi > 0$. Then it is easy to show that $R_c(k^2, g)$ exists. However, when the "mixed" matrix elements are examined, it is found that $\langle \vec{k} | V | \vec{p} \rangle$ becomes infinite if

$$p^2 = k^2 + \mu^2 \quad 0 \leq \cos \phi \leq \frac{k}{\sqrt{k^2 + \mu^2}}.$$

These relations can be rewritten as $p \sin \phi < \mu$. This means that ϕ must be restricted to ensure that the $e^{-\mu r}$ damps out the large r behavior in the integral $\langle \vec{k} | V | \vec{p} \rangle =$
 $\int e^{+i[\vec{p} - \vec{k}] \cdot \vec{r}} e^{-\mu r} e^{+i\vec{p} \cdot \vec{r}} \sin \phi \, r dr d\Omega.$

APPENDIX

$$|K_c'(k^2)|_c^2 = \int d\Omega \int d\Omega' \int_c |dp|^2 \left| \frac{1}{k^2 - p^2} \right| \int |dp' p'^2| \left| \frac{1}{k^2 - p'^2} \right| \times$$

$$\left| \frac{1}{(\vec{p} - \vec{p}')^2 + \mu^2} \right|^2$$

where c is chosen as in Figure 2. Let $\vec{p} = k\vec{q}$, $\vec{p}' = k\vec{q}'$ and $e = \frac{\mu}{k}$, then

$$|K_c'(k^2)|_c^2 = \left(\frac{4\pi}{\mu}\right)^2 e^2 \int |dq| |q|^2 \int |dq'| |q'|^2 \frac{1}{|1 - q^2|} \frac{1}{|1 - q'^2|} \times$$

$$\frac{1}{-1} \int \frac{dz}{2} \frac{1}{|q^2 + q'^2 - 2qq'z + e^2|^2}$$

$$= I_{11} + I_{22} + I_{33} + 2(I_{12} + I_{13} + I_{23}).$$

The subscripts refer to the segment of c that q and q' are restricted to.

$$I_1 = \left(\frac{4\pi}{\mu}\right)^2 e^2 \int_0^{1-a} dq \frac{q^2}{1-q^2} \int_0^{1-a} dq' \frac{q'^2}{1-q'^2} \frac{1}{4qq'}$$

$$\left[\frac{1}{(q-q')^2 + e^2} - \frac{1}{(q+q')^2 + e^2} \right] < \left(\frac{2\pi}{\mu}\right)^2 e^2 \left[\frac{1-a}{a(2+a)} \right]^2 x$$

$$\int_0^{1-a} dq \int_0^{1-a} dq' \left[\frac{1}{(q'-q)^2 + e^2} - \frac{1}{(q+q')^2 + e^2} \right]$$

$$\xrightarrow{e \rightarrow 0} \left(\frac{2\pi}{\mu}\right)^2 \left[\frac{1-a}{a(2+a)} \right]^2 e(1-a) \frac{\pi}{2} = 0 \left(\frac{\mu}{k}\right)$$

$$I_{12} = \left(\frac{4\pi}{\mu}\right)^2 e^2 \int_0^{1-a} dq \frac{q^2}{1-q^2} \int_{-\pi}^{2\pi} d\psi \frac{|1+ae^{i\psi}|^2}{|1-a^2 e^{2i\psi}|^2} \int_{-1}^1 \frac{dz}{2} x$$

$$\left| \frac{1}{q^2 + (1 + ae^{i\psi})^2 - 2a(1 + ae^{i\psi})^2 Z + \mu^2} \right|^2$$

$$< \left(\frac{4\pi}{\mu}\right)^2 e^2 \frac{1-a}{a(2+a)} \frac{(1+a)^2}{a(2-a)} \int_0^{1-a} dq \int_{-\pi}^{2\pi} d\psi \int_{-1}^1 \frac{dz}{2}$$

$$[f(q, \psi, Z) - f(q, \pi, Z)] + f(q, \pi, Z)$$

The term in square brackets has no singularities in the region of integration if $e \rightarrow 0$ under the integral, hence the first term in I_{12} is $O(e^2)$. The second term is equal to

$$\left(\frac{4\pi}{\mu} \right)^2 e^2 \frac{(1-a)}{a(2+a)} \frac{(1+a)^2}{a(2-a)} \frac{\pi}{2} \frac{1}{(1-a)} \int_0^{1-a} dq$$

$$\left[\frac{1}{q^2 - (1-a)^2 + e^2} - \frac{1}{q^2 + (1-a)^2 + e^2} \right] \xrightarrow{e \rightarrow 0} O(e) = O\left(\frac{1}{k}\right).$$

The other four terms are evaluated in a similar manner

$$I_{22} \rightarrow O\left(\frac{1}{k}\right), \quad I_{33} \rightarrow O\left(\frac{1}{k}\right), \quad I_{23} \rightarrow O\left(\frac{1}{k}\right) \quad \text{and} \quad I_{13} \rightarrow O\left(\frac{1}{k^2}\right).$$

The singularity in the integrand comes when $q = q'$ if $e = 0$.

ACKNOWLEDGEMENT

The author wishes to express his gratitude to Professor R. Blankenbecler and Dr. G. Tiktopoulos for many useful discussions.

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*This work is part of a Ph.D. thesis submitted to Princeton University.

+Supported in part by the U. S. Atomic Energy Commission under Contract No. AT(11-1)-30 and by funds granted by the University Alumni Research Foundation.

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Figure 1. A contour which may be used to prove the existence of t_+ .

Figure 2. The contour used to study $\|K'_c(k^2)\|_c$
in the 3-dimensional case.