MASTER

STEADY STATE HEAT CONDUCTION WITH
IRREGULAR SHAPED BOUNDARIES

W. A. Scheffler

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IN NUCLEAR SCIENCE ABSTRACTS

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NATIONAL REACTOR TESTING STATION
IDAHO FALLS, IDAHO

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Abstract

The effect of irregular shaped boundaries on the temperature distribution in a solid is calculated. The method used to calculate the effect of the boundary perturbation is straightforward and of a general nature. The application of such a method to problems in heat conduction will provide temperature corrections to thermocouples due to solid boundary irregularities.

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STEADY STATE HEAT CONDUCTION WITH IRREGULAR SHAPED BOUNDARIES

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I. Introduction

Most engineering problems of practical interest are only simulated by the reference geometries in which the heat conduction equation can be readily solved. When the boundaries of the solid deviate from the geometry in which Laplace's Equation is separable one has the alternative of using complex variables for two dimensional problems or numerical methods for two or three dimensional problems. Therefore, it is of some interest to have an accurate order of magnitude calculation method to predict the effect of an irregular shaped boundary on the temperature distribution. The purpose of this paper is to describe such a method.

The method incorporated here is quite general and flexible and can be easily applied to problems in two or three dimensions. The real advantage of the method is that the effect of an irregular shaped boundary can be found by solving problems with regular shaped boundaries, and since these solutions are readily available⁽¹⁾, the effect of the boundary irregularity can be easily calculated.

The method discussed here is regular perturbation theory. Since the regular perturbation method is of such a general nature, it is difficult to describe a step by step procedure to obtain a solution to a specific problem. However, only a slight variance of the illustration below should serve to solve most problems. The method is almost classical and has been used with much success in problems of fluid flow.

It was found expedient to make an adjustment of the usual regular perturbation method to facilitate the calculation in heat conduction problems. The description below only includes first order effects due to the non-symmetry of the boundary. However, if it is desired, second order and higher corrections can be readily calculated by carrying terms of higher order. Calculation of higher order corrections due to non-symmetry do not include any new features in the analysis procedure.

II. Analysis

For the purpose of illustration, consider two annular cylindrical surfaces. The inner surface being of irregular shape and at a constant temperature, T_i . To first order this surface can be represented by

$$r_p = a[1 + \epsilon f(\theta) + O(\epsilon^2)] \quad (1)$$

where ϵ is the eccentricity of the perturbation in the boundary of the annular surface. Let the outer surface be regular ($r_o = \text{const.}$), and have a constant surface temperature T_o . The temperature is now represented at any point (r, θ) by a power series in ϵ

$$T(r, \theta, \epsilon) = T_o(r, \theta) + \epsilon T_1(r, \theta) + O(\epsilon^2) \quad (2)$$

where

$$T_o(r, \theta) = A \ln r + B \quad (3)$$

is the solution for the general unperturbed problem. For the time being the constants A and B are left arbitrary and will be determined in the subsequent analysis.

The temperature at the non-symmetrical boundary is

$$T(r_p, \theta) = T_i = T_o(r_p, \theta) + \epsilon T_1(r_p, \theta) \quad (4)$$

where terms of order ϵ^2 have been dropped.

We now expand the terms on the right hand side of equation (4) in a Taylor series about a ,

$$T_o(r_p, \theta) = T_o(a, \theta) + \left. \frac{\partial T_o}{\partial r_p} \right|_{r_p = a} (r_p - a) \quad (5)$$

$$T_1(r_p, \theta) = T_1(a, \theta) + \left. \frac{\partial T_1}{\partial r_p} \right|_{r_p = a} (r_p - a) \quad (6)$$

From equation (2) we may write

$$r_p - a = a\epsilon f(\theta) \quad (7)$$

Equations (5), (6), and (7) are now combined and terms of order ϵ^2 dropped to obtain

$$T_i = T_o(a, \theta) + \epsilon[a T_o'(a, \theta)f(\theta) + T_1(a, \theta)] \quad (8)$$

Examining equation (8) and matching coefficients of ϵ

$$T_i - T_o(a, \theta) = 0 \quad (9)$$

and

$$a T_o'(a, \theta)f(\theta) + T_1(a, \theta) = 0 \quad (10)$$

Now equation (3) is used to obtain

$$T_i - A \ln a - B = 0 \quad (11)$$

and

$$T_i(a, \theta) = \frac{A}{a} f(\theta) \quad (12)$$

We now proceed in a similar manner to evaluate the temperature $T(r, \theta)$ at the outer surface, r_o ,

$$T(r_o, \theta) = T_b = T_o(r_o, \theta) + \epsilon T_1(r_o, \theta) \quad (13)$$

Again, matching coefficients of ϵ

$$T_b - A \ln r_o - B = 0 \quad (14)$$

and

$$T_1(r_o, \theta) = 0 \quad (15)$$

Equation (15) is physically correct since at the outer annular surface the perturbation in the temperature profile due the non-symmetry of the inner surface must vanish. Equations (11) and (14) are used to determine the constants A and B while equations (12) and (15) will be used as boundary conditions to determined $T_1(r, \theta)$.

To determine $T_1(r, \theta)$ we note that the temperature profile $T(r, \theta)$ must satisfy the heat conduction equation. The unperturbated temperature solution $T_o(r, \theta)$ likewise satisfies the heat conduction equation. Hence due to the linearity of this equation $T_1(r, \theta)$ must satisfy the heat conduction also since our solution $T(r, \theta)$ is a linear form. The general solution for an annular passage is ⁽¹⁾,

$$T_1(r, \theta) = \sum_{n=1}^{\infty} \left[\left(C_n r^n + \frac{D_n}{r^n} \right) \cos n\theta + \left(E_n r^n + \frac{F_n}{r^n} \right) \sin n\theta \right] + D_o \ln r + C_o \quad (16)$$

The boundary conditions on $T_1(r, \theta)$, equations (12) and (15), are applied to determine the constants in equation (16). From this we find

$$C_n = \frac{a^n f_n^{(1)}}{r_o^{2n} - a^{2n}} \quad (17)$$

$$D_n = r_o^{2n} a^n f_n^{(1)} \quad (18)$$

$$C_o = \frac{f_o^{(1)}}{\ln(a/r_o)} \quad (19)$$

$$E_n = \frac{a^n f_n^{(2)}}{a^{2n} - r_o^{2n}} \quad (20)$$

$$F_n = \frac{a^n r_o^{2n} f_n^{(2)}}{r_o^{2n} - a^{2n}} \quad (21)$$

$$D_o = \frac{f_o^{(1)} \ln r_o}{\ln(r_o/a)} \quad (22)$$

where $f_n^{(1)}$ and $f_n^{(2)}$ are the Fourier coefficients of the function $A/a f(\theta)$ defined by

$$f_o^{(1)} = \frac{A}{2\pi a} \int_o^{2\pi} f(\theta) d\theta \quad (23)$$

$$f_n^{(1)} = \frac{A}{\pi a} \int_o^{2\pi} f(\theta) \cos n\theta d\theta (n = 1, 2, \dots) \quad (24)$$

and

$$f_n^{(2)} = \frac{A}{\pi a} \int_o^{2\pi} f(\theta) \sin n\theta d\theta \quad (25)$$

The temperature perturbation $T_1(r, \theta)$ is thus determined. It was determined by using the general solution for two concentric annular passages. The coefficients in the solution for $T_1(r, \theta)$ are determined by the shape of the perturbed boundary, $f(\theta)$.

Leaving the constants A and B undetermined in equation (3) allows one to choose the best fit for $T_0(r, \theta)$. The crux of the problem is writing equation (3) in general form then proceeding to obtain boundary conditions on $T_1(r, \theta)$ by matching coefficients of ϵ . Having done this, one proceeds to solve the heat conduction equation for $T_1(r, \theta)$ for the general symmetrical boundaries but with boundary conditions generated on $T_1(r, \theta)$ due to the irregular shaped boundary.

If it happens that $T_i = T_i(r_p, \theta)$ then T_i will be a function of ϵ (since r_p is a function of ϵ) and we proceed in a similar manner. If both boundaries are perturbed one may solve the problem of each perturbed boundary separately and add the solutions to give the net effect due to the combined problem. This is due to the linearity of the heat conduction equation. For most applications only one term in the series for $T_1(r, \theta)$ will be sufficient to give an accurate correction. If the boundary perturbation is extreme, one may have to resort to higher order terms in ϵ in all the description of the boundary curve, equation (1). However, since all higher order solutions $T_n(r, \theta)$ satisfy the heat conduction equation for the symmetric case (these are easily determined from standard references), only the boundaries condition on $T_n(r, \theta)$ are needed to find the Fourier coefficients for the solution $T_n(r, \theta)$. Transient problems are also amenable to this technique. The method would appear useful in calculating corrections to thermocouples due to boundary irregularities.

References

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