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DECATURN BEAM EXTRACTION FROM SYNCHROTRONS

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1. Introduction

Three stages of acceleration are customary in present synchrotrons; more stages are likely to be used in the multi-hundred BeV accelerators. One or two of these additional stages may well be a "booster" synchrotron, either of the fast-cycling or the slow-cycling type. In the latter case, the beam has to be extracted over something like 5 to 30 turns. For this "decaturn" extraction, two methods come to mind: reverse multiturn injection and resonant extraction. Some of their properties are compared here, including the extraction efficiency, beam emittance, dilution in phase space and hardware requirements. The result is that efficient decaturn beam extraction looks promising enough not to use the need for it as a major argument against a slow-cycling booster.

2. Reverse Multiturn Injection

Multiturn injection is now standard practice both at the AGS and the CPS. It is achieved by distorting the closed orbit so that it grazes the septum inflector placed at the edge of the useful aperture and then collapsing the distortion to zero during the injection process.¹

In the reverse process, the distortion would be increased, ideally in a number of steps designed to "slice off" N turns of equal intensity (Fig. 1).

* On leave from CERN, Geneva, Switzerland.

1. M.Q. Barton, "Multiturn Injection in the AGS", BNL Accelerator Dept. Int. Rpt. AADD-57, December 18, 1964.

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(In practice, a continuous increase is probably good enough if the first turn is discarded.) For a rough estimate, let us assume that a fraction

$$L = (N-1)t/w \quad (1)$$

of the particles falls on the septum of thickness t and is lost, where w is the beam width in the horizontal plane. For $t = 0.2$ mm (which would appear to be conservative using a low-duty-cycle two-stage magnetic deflector) Eq. (1) leads to the following number for various values of w :

w (mm)	5				10			
N	5	10	20	30	5	10	20	30
L	0.16	0.36	0.76	> 1.0	0.08	0.18	0.38	0.58

w (mm)	20				30			
N	5	10	20	30	5	10	20	30
L	0.04	0.09	0.19	0.29	0.02	0.045	0.095	0.145

From this table it is immediately obvious that in the case $w = 5$ mm a few turns is the maximum number for which this basic scheme could be used, whereas in the case $w = 30$ mm, dozens of turns could possibly be envisaged.

To extract a beam of small width over a higher number of turns, one could think of first defocusing the beam horizontally at the location of the septum (by tuning ν_r , the number of betatron oscillations per turn, close to a half-integral value and energizing a suitably located quadrupole lens).^{*} The limitation of this procedure has to do with the requirements for the two kickers, particularly if working at an energy as high as 30 BeV. At that energy level, present kickers (operated with a constant flat-top of the current pulse) just manage to provide deflections of about 25 mm for about 3 μ sec. Even if the rise

* Because of presumably more uniform extracted beam intensity, this scheme would probably be preferred to employing an rf quadrupole run at the second harmonic of the betatron frequency as used for a slightly different purpose at the PPA.

time requirements could be relaxed, programmed increase of the amplitude to a final value higher by a factor of two, and increase of the duty cycle by an order of magnitude, do not look very attractive.

Instead, resonant extraction appears preferable in this case. On comparing the particle orbits for reverse injection with those pertaining to resonant extraction (Fig. 2), one notices that part of the kicker energy is expended merely to keep the inside of the vacuum chamber opposite the septum free from beam while the extraction is going on. Stated differently, the kicker has to provide the full deflection $\Delta\varphi$ in a single kick and, therefore, has to have the full strength required to obtain a displacement step

$$x_{s+1} - x_s = \Delta\varphi \beta_k^{\frac{1}{2}} \beta_s^{\frac{1}{2}} \sin \theta_\beta, \quad (2)$$

where $\beta_k^{\frac{1}{2}}$ and $\beta_s^{\frac{1}{2}}$ are the values of the amplitude function at the kicker and septum positions, respectively, and θ_β is the phase shift of the betatron oscillations between these two positions.

In contrast, in resonant extraction, one has

$$x_{s+1} - x_s = \sum_{i=0}^{i=a} \Delta\varphi(i) \beta_l^{\frac{1}{2}} \beta_s^{\frac{1}{2}} \sin \theta_\beta \quad (3)$$

where a is the order of the resonance ($a = 2$ for half-integer resonance, etc.) and l refers to the position of the perturbation, usually a lens. Now two or three turns are available to reach the same amplitude (in half and third integer resonance, respectively), making for weaker elements [despite the fact that $\sin \theta_\beta$ in Eq. (3) will usually be smaller than in Eq. (2)]. Also, a suitably designed lens carries less magnetic flux than the equivalent dipole magnet, and is therefore easier to pulse. As an additional advantage, the amplitude increase occurs automatically, at constant excitation, the field gradient having taken the role of the increase of kicker excitation. The problem is then essentially to select the right gradient values corresponding to the desired rate of peel-off.

Summing up, it would then appear that use of a deformation with increasing amplitude provided by two programmable fast deflectors could be a suitable method

for extraction over five turns and up to ten in favorable circumstances.* (This includes the case of a two-turn extraction which is special in that at $\nu_r \approx$ half-integral value two turns of equal intensity can be obtained with a constant flat-top of the excitation pulse. If on the first turn the beam is sliced exactly into half and the first half extracted, the second half will, on the next turn, automatically end up in the ejector magnet aperture.) Presumably ν_r need not necessarily be tuned to a half-integral value though this point would have to be studied more in detail, including the implication on the program for the excitation current, and the extraction efficiency.

Adjusting the steps for equal particle intensity of each of the N turns ejected, the emittance E of the extracted beam should be with Eq. (1)

$$E = E_0 (1-L)/N \quad (4)$$

if the phase-space density were constant. Any density variation will translate itself (reciprocally) into an emittance variation. Scattering by the septum will tend to increase the emittance. A further dilution of about 1.6 would occur if rectangular "slices" were to be fitted into circumscribed acceptance ellipses. Alternatively, throwing away the corners and fitting an ellipse inside would lead to no dilution but about 20% beam loss.²

3. Resonant Extraction

3.1 Principle

Having been rather slow in its recognition and development, resonant extraction is at present the method generally used for multiturn extraction³ (hundreds to hundreds of thousands of turns). It consists in setting up a controlled resonance between the revolution of a given particle around the synchrotron and the betatron oscillations about its closed orbit. The resultant forced oscillations are either linear (using a linear perturbation, i.e., quadrupole lens) or non-linear (using non-linear perturbations like sextupole or octupole lenses).

*The same conclusion has just been reached by A.W. Maschke (private communication) who also states that even operation at the half-integral stopband should not present a problem.

2. J.M. Peterson, private communication.

3. K.H. Reich, "Beam Extraction Techniques for Synchrotrons", Progress in Nuclear Techniques and Instrumentation, Vol. II, in press.

The latter elements, in addition to providing the driving force, also make the frequency depend on the amplitude of the oscillation thereby creating stable and unstable regions. Multiturn ejection is then achieved by shifting the particles sequentially from the stable to the unstable region(s). This shifting can be sped up for the use under discussion by means of the rf knockout technique, leading to stimulated slow extraction.⁴ However, a linear resonance was studied here for the following reasons:

- It is naturally fast, practically all particles becoming unstable simultaneously (beam extraction in 6 revolutions - 36 nsec was achieved by this means at the electron synchrotron of Iowa State University).
- The emittance is smaller because the phase-space trajectories for the last few turns are less distorted than with higher order lenses.
- The growth rate of the turn-to-turn separation being not faster than exponential, the intensity distribution across the beam is somewhat more advantageous, i.e., a smaller fraction of particles is lost on the septum.
- It is easier to calculate and to understand than the non-linear resonances.

(A similar reasoning has independently led the Berkeley Group to initiate a study of a linear resonance for the same purpose.²)

If only one type of periodic perturbation is used, the equation of motion becomes, in the case of interest here

$$x'' + \nu_r^2 x = kx \cos p\theta \quad (5)$$

where $x'' = d^2x/d\theta^2$ with θ as the azimuthal angle, x is the radial displacement from the equilibrium position, ν_r the number of radial betatron oscillations per revolution, k the quadrupole strength in the suitable units, and $p = 2\nu_r = \text{integer}$. Thus, resonance is possible for $\nu_r \approx \text{integer}$ or $\nu_r \approx \text{half-integer}$. However,

4. D.S. Robertson, "Stimulated Slow Extraction", BNL Accelerator Dept. Internal Report AADD-97, February 14, 1966.

in the first case the beam is ejected simultaneously to the outside and inside and the closed orbit also becomes unstable. Use of the half-integer resonance would hence seem preferable; this was therefore studied exclusively.

As there will then be $2\nu_r$ half-waves set up around the synchrotron, the corresponding number of driving quadrupoles could be used. However, a single quadrupole of $2\nu_r$ times the strength will do the same job. Thus, the number to select is essentially a problem of convenience and economics. A fundamental difference exists, however, between this case and the one of using two different types of lenses corresponding to adding a term $-kx \sin p\theta$ to the right-hand side of Eq. (5). Consider first the action of a single type of lens (Fig. 3). A particle arriving at the lens at point A will experience a kick

$$|\vec{AB}| = kx_A = kr_1 \cos |\alpha_1| \quad (6)$$

This kick can be decomposed into an increase of radius (see Appendix A)

$$\Delta r = r_2 - r_1 \approx r_1 k \sin |\alpha_1| \cos |\alpha_1| \quad (7)$$

and a phase shift

$$\Delta\theta_\beta \approx \tan(\Delta\theta_\beta) \approx -k \cos^2 \alpha_1 \quad (8)$$

On the next turn around, the particle will arrive at the lens at A' and corresponding action will take place. The important point to note is that while Δr is positive in both cases (as wanted), $\Delta\theta_\beta$ is negative throughout. Thus, to make this system resonate (using a defocusing lens), the unperturbed system has to have a slight phase advance, i.e., ν_r must be integer + 0.5 + ϵ where ϵ depends on the strength of the perturbation (see below) and is of the order of 0.01 in the cases of interest here. (Conversely, a system using a focusing lens(es) would operate with ν_r slightly below the half-integer ν_r value). Now consider adding a focusing lens about $\lambda_\beta/4$, i.e., $\theta_\beta = 90^\circ$ further downstream. Its action will bring the particle under consideration from C to D. Thus, while Δr is again positive as wanted, $\Delta\theta_\beta$ is now also positive. Hence,

using a doublet consisting of a focusing and a defocusing lens will tend to make the resultant phase shift zero, the more so the smaller the kick. Thus, while not required to operate exactly on the half-integral ν_r value (in which case $|\alpha_1| + |\Delta\theta_{\beta 1}/2| \neq |\alpha_2| + |\Delta\theta_{\beta 2}/2|$) there is the possibility in this case to do so. In view of these differences, both the system with one type of lens (singlet) and with a lens doublet will be discussed.

S.C. Snowdon has applied perturbation theory to the half-integral case⁵ and J.Claus has derived some of the more interesting relations for the single lens case from the phase plane geometry.² Here the matrix method is used as this permits directly to draw heavily on the existing body of AG synchrotron theory.⁶ Once the resonance is fully established, the phase vector is an eigenvector

$$r_{n+1} = \lambda r_n \quad (9)$$

where the Floquet multiplier λ is a negative real number, with $|\lambda|$ either > 1 (trajectories along outgoing branches of hyperbola) or < 1 (trajectories along incoming branches of hyperbola) and $\lambda_1 \lambda_2 = 1$, the minus sign corresponding to the 180° phase shift on each turn. Because of this phase shift one is interested in the amplitude after every second turn. From Eq. (9) one has immediately

$$r_n / r_{n-2} = \lambda^2,$$

i.e.,

$$r_n - r_{n-2} = (\lambda^2 - 1) r_{n-2} = \frac{\lambda^2 - 1}{\lambda^2} r_n \quad (10)$$

For efficient extraction $w_1 = r_n - r_{n-2}$ must be large compared to the septum thickness and small compared to the width of the vacuum chamber, say $10 < w_1 < 30$ mm. Eq. (10) will then give the permissible range of λ values for maximum and minimum values of r_n . As an example, $\lambda^2 = 1.23$ and $r_{n-2} = 55$ mm gives $w_1 = 12.7$ mm. As it turns out, this is about the right value of λ for 25-turn extraction so that the wanted range of w_1 just happens to be roughly right for decaturn extraction.

5. S.C. Snowdon, private communication.

6. E.D. Courant and H.S. Snyder, Ann. Phys. 3, 1 (1958).

The following consideration may serve as a first guide to select a particular value of λ within this range. If r_{01} is the smallest initial amplitude of particles to be extracted and r_{02} the largest, the respective number of turns to reach r_n are $r_{01}\lambda^{n_1}$ and $r_{02}\lambda^{n_2}$. Hence

$$\log \lambda = (\log r_{02} - \log r_{01}) / (n_1 - n_2) \quad (11)$$

As an example, for $r_{01} = 0.35$ mm, $r_{02} = 4.4$ mm and $N = n_1 - n_2 = 25$; $\lambda = 1.107$.

The lens strength k required for this value of λ is (see Appendix B)

$$k_d^2 = - (\lambda^2 + 2\lambda \sin^2 \theta_\beta + 1) / \lambda \sin^2 \theta_\beta + 2 \cot^2 \theta_\beta \quad (12a)$$

for a system with a lens doublet, which becomes for $\theta_\beta = \pi/2$

$$k_d^2 = - (1 + \lambda)^2 / \lambda \quad (12b)$$

and

$$k_s = (\lambda^2 - 2\lambda \cos \theta_\beta + 1) / 2\lambda \sin \theta_\beta \quad (13)$$

for a singlet where $2k$ is in all cases the total strength of all lenses (e.g., one lens of strength $2k$, two lenses of strength k , etc.) and θ_β is the phase shift between subsequent kicks, $\theta_\beta \approx \pi/2$ in the case of Eq. (12a) and $\theta_\beta \approx \pi$ in the case of Eq. (13). As already stated, inspection of these equations shows immediately that the doublet system can work with $\theta_\beta = 0$, i.e., almost exactly on the half-integral value of ν_r , whereas the singlet system requires $\sin \theta_\beta \neq 0$ (i.e., $\theta_\beta \gtrless \pi$) and k of opposite sign as $\sin \theta_\beta$ (λ being negative). Note that from the point of view of efficient use of lens strength both systems are equivalent if $k_s = \sin \theta_\beta$. (In that case $|\alpha| + |\Delta\theta_\beta/2| = \pi/4$ in Fig. 3.) From Eq. (12b) with $\lambda = 1.107$ one obtains $k_d \approx 0.1$.

3.2 Intensity Distribution

If one assumes constant phase-space density, the fractional intensity extracted on each turn can be computed analytically.² Here a more realistic density distribution is used based on measurements at the CPS.⁷ It results from a uniform distribution of betatron oscillation phases and a distribution of amplitudes according to

$$g(r) = \text{constant } r e^{-0.22r^2} \quad (14)$$

Rather than using this continuous distribution, it was divided into 10 "slices", $g(r)dr$ and the central r value of each "slice" was employed in the computation (Fig. 4). The intervals were selected such that their contribution to the beam intensity $I = \text{constant} \int g(r)dr$ was equal. Thus individual results all have equal weight and the intensity on each turn is simply proportional to the sum of the model particles extracted on that turn. In the particular case studied, the model internal beam was composed of 160 particles having the amplitudes

No. of "slice"	1	2	3	4	5	6	7	8	9	10
Amplitude in mm	0.35	0.85	1.15	1.42	1.66	1.91	2.20	2.52	3.0	4.4

and 16 different phase angles per amplitude, 0° , 22.5° , etc. to 337.5° .

Using a computer program developed by C. Bovet,⁸ these 160 particles were submitted to the effect of a lens doublet of strength $k = \pm 0.1$, the two lenses being spaced by exactly 3.25 wavelengths and v_r being equal to 6.50. A particle was considered extracted when it had reached the displacement $x = 55$ mm, about $\lambda_p/8$ downstream from the defocusing lens. The result (Fig. 5), shows that in addition to the 25 turns anticipated from Eq. (11) (turn 26 containing first particle of slice 10 to turn 51 containing first particle of slice 1) there exists

7. C. Bovet, "Densité transversale de protons dans le faisceau du CPS", CERN Internal Report MPS/Int. DL-65-14, October 27, 1965.

8. C. Bovet, "Programmes Fortran pour le calcul des trajectoires perturbées et des systèmes d'éjection par résonance dans le CPS", CERN Internal Report MPS/Int. DL-65-5, March 12, 1965.

a long tail extending over another 24 turns. (No undue importance should be attached to the several particles outside the broken curve. They merely result from the fact that the septum thickness was specified as infinitely thin whereas the accuracy of the computation is not infinitely good and the total number of particles is small compared to the effect of these statistical fluctuations, leading to an uncertainty of ± 2 in the number of turns required for a particular particle. The early arrival of slice 10 is due to the fact that 4.4 mm is really too high a value to represent this slice, see Fig. 4.)

To understand this tail and to find a cure, the following consideration due to G.R. Lambertson was found useful.⁹ Instead of visualizing the particles being extracted, apply time reversal and picture the septum moving stepwise towards the radially stationary beam until it cuts off slices containing the numbers of particles extracted on each turn (Fig. 6). On this figure, the septum position 0 is obtained by simply transforming the septum back to the position of the upstream lens ($< 1/8$ of a wavelength), and the other positions by applying Eq. (10) to the intersection points of septum and asymptotes. It is then immediately obvious that, even for uniform phase-space density, the early and the late slices are of smaller intensity than the in-between ones. The amplitude distribution of Eq. (14) reinforces this state of affairs.

To obtain equal intensity (in the case of uniform phase-space density), the beam should completely fill the various bins extending up to the limiting hyperbola. In other words, no linear transformation of the beam shape (like prefocusing) will solve the problem on hand. Rather, one answer¹⁰ is to vary λ in such a fashion that the slices become of equal intensity.* Having an analytical expression for the phase-space density, the required ideal lens strength modulation could presumably be calculated. Here only the general validity of the idea was shown by varying k as

$$k = k_0 [1 + (n - n_{01}) B + C \sin \pi (n - n_{02})/G] \quad (15)$$

* J. Claus (private communication) proposes to modulate the closed orbit instead, requiring extra programmed orbit deflectors.

9. G.R. Lambertson, private communication.

10. Proposed by P.F. Meads, Jr. for extraction from the MURA 50-MeV Electron Accelerator (MURA-712) as learned by the author after this report was written.

where $n - n_{01} > 0$, $n - n_{02} > 0$ and B , C and G are constants. In other words, a "ramp" is switched in at turn n_{01} and a sinusoidal modulation of variable amplitude and period at turn n_{02} .

Keeping all other conditions the same, switching in a ramp with $B = 0.1$ at turn 42 has the effect shown in Fig. 7a, while a sinusoidal modulation with $C = 0.42$ and $G = 26$ switched in at turn 24 leads to the distribution shown in Fig. 7b. Finally, adding a different sinusoidal modulation ($C = 0.35$, $G = -13$) at turn 16 and a ramp ($B = 0.03$) at turn 42 results in the distribution shown in Fig. 7c. Using a more sophisticated modulation, something like 90% of the intensity could undoubtedly be squeezed into turns of constant intensity, i.e., the dilution in transverse phase space strongly diminished. If necessary, the changing beam width could presumably be corrected by means of a transport system with programmed magnification.

3.3 Emittance

The following three points will be discussed separately:

- central momentum particles
- effect of lens strength modulation
- off-momentum particles.

The basic contribution of the central momentum particles at constant lens strength is simply given by Eq. (4). The other effects can best be discussed in terms of a variation of the position of point B in Fig. 3. One has (Appendix C)

$$\tan (|\alpha_1| + |\Delta\theta_\beta|) = 1 + 0.5 k \quad (16a)$$

for a doublet operated at $\nu_r = \text{half integer}$ and

$$\tan (|\alpha_1| + |\Delta\theta_\beta|) = k + (k^2 + 2k \cot \theta_\beta - 1)^{\frac{1}{2}} \quad (16b)$$

for the singlet. Remembering that k is usually small compared to one, one sees from Eq. (16a) that even a 50% modulation of lens strength will not have a catastrophic effect ($\Delta x'_E = 0.025 x'_E$ for $k = 0.1$). If this value should be too large, one can obtain a partial correction by also suitably modulating the strength of the extraction magnet.

To discuss quantitatively the situation with respect to the off-momentum particles, one has to know the relation between momentum spread and ν shift

$$\Delta\nu/\nu = C \Delta p/p \quad (17)$$

where

$$\begin{aligned} C &= -1.08 \text{ for a linear AGS} \\ &= -2.4 \text{ for AGS at medium fields (5 kG)} \\ &= -10.5 \text{ for AGS at high fields (12 kG)} \end{aligned}$$

To get a feeling for the numbers involved, assume $\Delta p/p = \pm 2.5 \times 10^{-4}$. With $C = -2.4$ and $\nu_r = 8.5$, this gives from Eq. (17)

$$\Delta\nu_r = \pm 0.005 \quad (18)$$

or $\Delta\theta_\beta = 0.005 \pi = 0.0157$ in the doublet case (and twice as much in the singlet case). To see the effect of this ν_r shift, Eq. (16a) has to be replaced by Eq. (C5a)

$$\tan(|\alpha_1| + |\Delta\theta_\beta|) = 0.5 k \pm \frac{(k^2 - 2 \cos^2 \theta_\beta)^{\frac{1}{2}}}{2 \cos \theta_\beta - k} \quad (19)$$

With $k = 0.1$ and $\cos \theta_\beta \approx \Delta\theta_\beta = 0.0152$, Eq. (19) becomes

$$\tan(|\alpha_1| + \Delta\theta_\beta) = 0.05 \pm \frac{(0.01 - 2 \times 0.000246)^{\frac{1}{2}}}{0.0314 - 0.1} = 1.47$$

i.e., $\Delta x'_E = 0.4 x'_E$ in this case, which is catastrophically large. [The reason for this large shift is the fact that the stopband created by the two quadrupoles of $k = 0.1$ is only $\Delta\nu = \pm 0.5 \times 0.1/\pi \approx 0.0162$, see Eq. (8) with $\alpha_1 = 45^\circ$.]

The remedy is to use sextupoles for providing the extra phase shift for the off-momentum particles.* Suppose that the sextupoles are distributed in such a manner (either continuously or in suitably arranged pairs) that their influence on the particles with correct momentum cancels. The off-momentum particles, circulating on a different radius, will then experience a differential deflection

$$\sum_{\text{deflections}} = k_s (y_1^2 - y_2^2) \quad (20a)$$

$$\approx 2k_s y \Delta y, \quad (20b)$$

where k_s is the sextupole strength, y_1 and y_2 are the particle displacements with respect to the central orbit (coinciding with the sextupole axis), and $\Delta y = y_1 - y_2$ is the orbit separation given by

$$\Delta y = 2 \alpha R \Delta p/p, \quad (21)$$

α being the momentum compaction factor and R the mean machine radius. As an example, for the CPS with $\alpha = 0.027$, $R = 100$ m and $\Delta p/p = 3.5 \times 10^{-4}$ $\Delta y \approx 2$ mm, i.e., from Eq. (20b) $\sum_{\text{deflections}} = 4 k_s y$. If the deflections are arranged to occur at $x' \approx 0$, providing maximum phase shift,

$$\tan \theta_{\beta_s} \approx \frac{4k_s y}{y} = 4k_s \approx \Delta \theta_{\beta} \quad (22)$$

Continuing the example from the CPS, Eq. (17) gives with $C = -2.2$, $\Delta p/p = \pm 3.5 \times 10^{-4}$ and $\nu_r = 6.5$, $\Delta \nu = \pm 0.005$ or $\Delta \theta_{\beta} = 0.0157$, as before. From Eq. (22) this then gives $k_s \approx 0.004$.

A few trials were necessary to find the best disposition for the sextupoles, starting from a study of a plot of the wave pattern around the machine. (If the position of the sextupoles is not correct, the asymptotes will be curved as already appears in Fig. 8 for $x > 50$ mm.) Taking the CPS as an example (because of

*Note that in extreme cases the sextupoles may already be required to get the off-momentum particles to resonate, namely if the ν spread from Eq. (17) is larger than the stopband created by the quadrupoles.

the existing computer program) the following arrangement gave reasonable results (which could be improved, though):

Straight Section	70	79	87	1	15	17	17	25	55	65
Type of Element	sept.	sext.	sext.	sext.	quad.	sept.	sext.	sext.	sext.	quad.
Strength		-0.004	-0.002	-0.002	0.1		-0.004	-0.002	-0.002	-0.1

Starting each time with the same initial values (except for Δy), the resulting values of x'_i and x_i in straight section 70 are plotted in Fig. 8 for $v_r = 6.495, 6.500$ and 6.505 without and with the sextupoles in operation. The compensation is so good that Eq. (4) remains a good approximation in all cases.

3.4 Transfer Efficiency

Assuming that the extraction itself can be done with an efficiency of about 95% (using a septum 0.2 mm thick) a further possible loss occurs because one might decide to discard the unfilled first and last turns (Fig. 7). Allowing another 10% for this, it seems reasonable to think that about 85% of the circulating beam could be made available for reinjection. (Note that part of the "lost" 15% could conceivably be used for physics experiments.)

3.5 Synchrotron Oscillations

No thought has been given to the synchrotron oscillations beyond the one that the total duration of the extraction is presumably short compared to a synchrotron oscillation period. It is noted, however, that in contrast to some of the nonlinear extraction schemes, the linear scheme studied (with the sextupole correction applied) expels particles independently of their momentum. The momentum distribution of the extracted beam should therefore be the same as that of the circulating beam.

3.6 Hardware Requirements

In the absence of a definite project not much thought has been given to any details. Basically, the problems encountered should not be overwhelming. A quadrupole with $k = 0.1$ placed in a straight section with $\beta = 22$ m is supposed to give a deflection of $0.1/22 = 0.00455$ mrad at 1 mm. Even at 30 BeV/c this

corresponds only to a gradient of 45.5 gauss/cm $\times$ 1 m, giving a field of 364 gauss at $y = 8$ cm (the present AGS kicker produces 560 gauss $\times$ 1.73 m). If the extraction extends over something like 20 turns, starting with an unusable partly filled one, there is, moreover, no strong objection to lose half a turn or so, i.e., switch the lenses on 5 times more slowly. In the case of the reverse multiturn injection it is noted that the programmed power supplies could probably be adapted from existing klystron modulators.

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Appendix A

Increase of Radius and Phase Shift Produced by Lens

The increase of radius can be found from the relation (Fig. 3)

$$r_2^2 = (r_1 + \Delta r)^2 = (\overline{AB} + \overline{AE})^2 + (r_1 \cos \alpha_1)^2$$

$$\overline{AB} = k r \cos \alpha_1, \quad \overline{AE} = r_1 \sin |\alpha_1|$$

$$(r_1 + \Delta r)^2 = r_1^2 (k^2 \cos^2 \alpha_1 + 2k \cos \alpha_1 \sin |\alpha_1| + \sin^2 \alpha_1 + \cos^2 \alpha_1)$$

giving

$$\Delta r = - r_1 \pm r_1 [\sin^2 \alpha_1 + 2k \sin \alpha_1 \cos \alpha_1 + \cos^2 \alpha_1 (k^2 + 1)]^{\frac{1}{2}}. \quad (A1a)$$

For $k < 1$ (A1a) becomes

$$\Delta r \approx r_1 k \sin |\alpha_1| \cos \alpha_1. \quad (A1b)$$

The phase shift is given by (Fig. 3)

$$\tan \Delta \theta_\beta = \overline{BG} / (r_1 + \overline{AG})$$

$$\overline{BG} = \overline{AB} \cos \alpha_1 = k r_1 \cos^2 \alpha_1$$

$$\overline{AG} = \overline{AB} \sin |\alpha_1| = k r_1 \sin |\alpha_1| \cos \alpha_1.$$

$$\tan \Delta\theta_{\beta} = \frac{kr_1 \cos^2 \alpha}{r_1(1 + k \sin |\alpha_1| \cos \alpha_1)} = \frac{k}{(1 + \tan^2 \alpha_1) \left(1 + k \frac{\tan |\alpha_1|}{1 + \tan^2 \alpha_1}\right)}$$

or

$$\tan \Delta\theta_{\beta} = \frac{k}{1 + \tan^2 \alpha_1 + k \tan \alpha_1} \quad (A2a)$$

For $\Delta\theta_{\beta}$ and $k \cos \alpha_1 \sin \alpha_1 < 1$, one has

$$\Delta\theta_{\beta} \approx k \cos^2 \alpha_1 \quad (A2b)$$

Appendix B

Lens Strength Required for a Given Growth Factor λ

The phase vector being an eigenvector (once the extraction is started) one has⁶

$$\vec{Mx} = \lambda \vec{x} \quad (B1a)$$

where

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is the transfer matrix for one turn. In other words

$$\begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0 \quad , \quad (B1b)$$

or

$$\lambda^2 - (a + d) \lambda + ad - bc = 0 \quad .$$

Making use of the fact that the determinant is one, the trace becomes

$$\text{Trace} = (\lambda^2 + 1)/\lambda \quad . \quad (B2)$$

1. Doublet Case

The matrix for transfer around the synchrotron, from the center of the defocusing lens to the center of the defocusing lens in this case (using $1/\beta$ radians as unit for the angle instead of radians and similarly for the lens strength⁷),

$$M_D = \begin{pmatrix} 1 & 0 \\ k/2 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta_\beta & \sin \theta_\beta \\ -\sin \theta_\beta & \cos \theta_\beta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} \begin{pmatrix} \cos \theta_\beta & \sin \theta_\beta \\ -\sin \theta_\beta & \cos \theta_\beta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ k/2 & 1 \end{pmatrix}$$

(B3a)

$$= \begin{pmatrix} \cos^2 \theta_\beta - \sin^2 \theta_\beta (1 + k^2/2) & 2 \sin \theta_\beta \cos \theta_\beta - k \sin^2 \theta_\beta \\ \sin^2 \theta_\beta \left(\frac{k^3}{4} - k \right) - \sin \theta_\beta \cos \theta_\beta (2 + k^2/2) & \cos^2 \theta_\beta - \sin^2 \theta_\beta (1 + k^2/2) \end{pmatrix} \quad (\text{B3b})$$

Inserting the trace of (B3b) in (B2) one has

$$2 \cos^2 \theta_\beta - \sin^2 \theta_\beta (2 + k^2) = (\lambda^2 + 1)/\lambda$$

giving

$$k^2 = - \frac{\lambda^2 + 2\lambda \sin^2 \theta_\beta + 1}{\lambda \sin^2 \theta_\beta} + 2 \cot^2 \theta_\beta \quad (\text{B4a})$$

which becomes for $\theta_\beta = \pi/2$

$$k^2 = - (1 + \lambda)^2 / \lambda \quad (\text{B4b})$$

2. Singlet Case

The matrix is now

$$M_S = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \begin{pmatrix} \cos \theta_\beta & \sin \theta_\beta \\ -\sin \theta_\beta & \cos \theta_\beta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \quad (\text{B5a})$$

$$= \begin{pmatrix} \cos \theta_\beta + k \sin \theta_\beta & \sin \theta_\beta \\ 2k \cos \theta_\beta - \sin \theta_\beta (1 - k^2) & \cos \theta_\beta + k \sin \theta_\beta \end{pmatrix} \quad (\text{B5b})$$

Inserting the trace of (B5b) in (B2) one has

$$2 \cos \theta_{\beta} + 2 k \sin \theta_{\beta} = (\lambda^2 + 1) \lambda$$

giving

$$2k = (\lambda^2 - 2\lambda \cos \theta_{\beta} + 1)/\lambda \sin \theta_{\beta} . \quad (B6)$$

If $k = -\sin \theta_{\beta}$, (B6) becomes (for $k^2 \ll 1$ as is usually the case)

$$2k = - [\lambda^2 + 2\lambda (1 - k^2/2) + 1]/\lambda k$$

whence

$$k^2 = (1 + \lambda)^2/\lambda , \quad (B7)$$

in other words, the singlet system is in this case as efficient as the doublet system.

Appendix C

Effect of Modulation of Lens Strength and Particle Momentum

These are discussed in terms of displacement of point B in Fig. 3. If $\overline{AF} = \overline{FB} = \overline{AB}/2$, one has in the doublet case

$$\begin{aligned}\overline{EF} + \overline{FB} &= x'_E + k x_E/2 \\ &= x_E \left(\frac{x'_E}{x_E} + k/2 \right)\end{aligned}\tag{C1}$$

From (B1b) one has

$$(a - \lambda)x_E + b x'_E = 0$$

or

$$\frac{x'_E}{x_E} = \frac{\lambda - a}{b}\tag{C2}$$

which becomes with $\lambda = 0.5 \text{ trace} \pm [(0.5 \text{ trace})^2 - 1]^{1/2}$ [from Eq. (B2)]

$$\frac{x'_E}{x_E} = \frac{\text{trace} - 2a}{2b} \pm \frac{[(0.5 \text{ trace})^2 - 1]^{1/2}}{b}\tag{C3}$$

Inserting (C3) into (C1) and rearranging, one obtains

$$\begin{aligned}\tan \left(|\alpha_1| + |\Delta\theta_\beta| \right) &= (\overline{EF} + \overline{FB})/x_E \\ &= (\text{trace} - 2a)/2b \pm [(0.5 \text{ trace})^2 - 1]^{1/2}/b + 0.5 k\end{aligned}\tag{C4}$$

with (B3b) this becomes for the doublet

$$\begin{aligned} \tan (|\alpha_1| + |\Delta\theta_\beta|) &= 0.5 \pm \frac{\left\{ \left[\cos^2 \theta_\beta - \sin^2 \theta_\beta (1 + 0.5 k^2) \right]^2 - 1 \right\}^{\frac{1}{2}}}{2 \sin \theta_\beta \cos \theta_\beta - k \sin^2 \theta_\beta} \\ &= 0.5 k \pm \frac{\left\{ \cos^4 \theta_\beta - 2 \sin^2 \theta_\beta \cos^2 \theta_\beta (1 + 0.5 k^2) + \sin^4 \theta_\beta (1 + 0.5 k^2)^2 - 1 \right\}^{\frac{1}{2}}}{2 \sin \theta_\beta \cos \theta_\beta - k \sin^2 \theta_\beta} \end{aligned}$$

To get a feeling for the meaning of this relation, we put $\sin \theta = 1$, and neglect terms with $\cos^4 \theta$ or equivalent, giving

$$\tan (|\alpha_1| + |\Delta\theta_\beta|) = 0.5 k \pm \frac{(k^2 - 2 \cos^2 \theta_\beta)^{\frac{1}{2}}}{2 \cos \theta_\beta - k} \quad (C5a)$$

For $\theta_\beta = \pi/2$, i.e., $\cos \theta_\beta = 0$, this gives

$$\tan (|\alpha_1| + |\Delta\theta_\beta|) = 1 + 0.5 k \quad (C5b)$$

In the case of a singlet one has with Eq. (B5b) and (C4)

$$\begin{aligned} \tan (|\alpha_1| + |\Delta\theta_\beta|) &= k + \frac{\left\{ (\cos^2 \theta_\beta + k \sin \theta_\beta)^2 - 1 \right\}^{\frac{1}{2}}}{\sin \theta_\beta} \\ &= k + \left(\cos^2 \theta_\beta + 2k \sin \theta_\beta \cos \theta_\beta + k^2 \sin^2 \theta_\beta - 1 \right)^{\frac{1}{2}} / \sin \theta_\beta \\ &= k + \left(\cot^2 \theta_\beta + 2k \cot \theta_\beta + k^2 - 1/\sin^2 \theta_\beta \right)^{\frac{1}{2}} \\ &= k + \left(k^2 + 2k \cot \theta_\beta - 1 \right)^{\frac{1}{2}} \quad (C6) \end{aligned}$$

(a) A $\lambda_\beta/2$ orbit deformation with increasing amplitude produced by the fast kicker magnets. K is used to slice off the beam by means of a thin septum S.

(b) Cross section at septum, showing "slices" in case of an integral or half-integral number of betatron wavelengths.

Figure 1 - Reverse multiturn injection.

Figure 2 - Standing wave of growing amplitude set up by defocusing lens L is used in resonant extraction to slice off the beam by means of septum S.

Figure 3 - Action of lens(es) on single particle shown in phase space; explanation in text.

Figure 4 - Distribution of betatron amplitudes $g(r)$ used. Slices 1 to 10 contribute equally to total intensity $\int g(r) dr$.

Figure 5

Intensity distribution for turns No. 24 to 75 resulting from action of lens doublet with $k = \pm 0.1$ on model beam made up of 160 particles, having 16 initial phases spaced equally, and 10 amplitudes (central amplitudes of slices 1 - 10 in Fig. 4). The peaked distribution would lead to a undesirable density dilution in transverse phase space if this extracted beam were used for reinjection. The particles are labeled according to the amplitude slice of Fig. 4 to which they belong.

Figure 6

Extraction process as viewed by G.R. Lambertson. Applying time reversal, septum is jumped towards stationary beam, cutting off slices of unequal emittance. (They would be of equal emittance if beam were shaped as shown at right.)

$$n_{01} = 42 \quad B = 0.1 \quad [\text{Eq. (15)}]$$

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Figure 7b

$$n_{o2} = 24 \quad C = 0.42 \quad G = 26$$

Effect of modulating lens strength
[Eq. (15)] on density distribution.
Dilution in transverse phase space
is strongly reduced.

Figure 7c

$$\begin{aligned} n_{o1} &= 42 \quad B = 0.03 \\ n_{o2} &= 16 \quad C = 0.35 \quad G = -13 \end{aligned}$$

Effect of modulating lens strength
[Eq. (15)] on density distribution.
Dilution in transverse phase space
is strongly reduced.

Figure 8

Phase-space trajectories at location of septum magnet without and with sextupole lenses in operation.