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XENON OSCILLATION CALCULATION BY MODAL ANALYSIS

D. RANDALL

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Savannah River Laboratory

Aiken, South Carolina

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XENON OSCILLATION CALCULATION BY MODAL ANALYSIS

by

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ABSTRACT

A one-dimensional, one-group modal analysis treatment of xenon spatial oscillations is presented. The one-dimensional, one-group diffusion theory flux distribution equation and the equations for the rates of formation of iodine and xenon are solved with the usual linearization technique. Time derivatives are eliminated by assuming that the perturbations vary exponentially with time.

The solution yields threshold reactor-average fluxes (if any) and corresponding oscillation periods at threshold, periods for oscillations (if any; whether growing or decaying), perturbation flux amplitude ratios, and damping factors at any specified reactor-average flux levels. Centrally flattened (to any extent) fundamental fluxes, sinusoidal for the axial case or J_0 for the radial case, are permitted, and the effect of a temperature coefficient may be included.

CONTENTS

	<u>Page</u>
Introduction	5
Summary	7
Discussion	9
Fundamental Equations	9
Perturbation Solution	10
Flux-Shape Eigenfunction Expansion	12
Evaluation of Axial Flux-Shape Eigenvalues	18
Evaluation of Radial Flux-Shape Eigenvalues	21
Reactor-Average Fluxes	26
Oscillation Periods and Perturbation Flux Transients. .	30
Approximations	36
Equilibrium Xenon	36
Truncation	36
References	37
Appendix: Reactor-Geometry Eigenfunction Expansion . . .	38

.LIST OF FIGURES

<u>Figure</u>	<u>Page</u>
1 Threshold Flux $\bar{\phi}_1$ Versus $f(\bar{\phi}_1)$	17
2 Postulated Axial Flux Shapes	18
3 Axial and Radial Eigenvalues as a Function of Flux Flattening	20
4 Radial-Azimuthal Oscillations	22
5 Postulated Radial Flux Shapes	22
6 Comparison of Axial and Radial Eigenvalues for a Right-Cylindrical Reactor	25
7 Ratio of $\bar{\phi}_1$ to Reactor-Average Flux as a Function of Flux Flattening	29
8 Schematic Representation of the Behavior of the Perturbation Amplitude Ratio and the Roots of the Time-Dependent Perturbation Flux Equation . . .	32
9 Oscillation Periods Versus Threshold $\bar{\phi}_1$	35

INTRODUCTION

Two types of xenon oscillation are possible. The fundamental instability, which occurs when the reactor power rises or falls everywhere as a function of time, is of no concern in this report. But a second type of instability can arise even when the reactor is operated at constant total power: a large thermal reactor operating at a sufficiently high flux level is subject to spatial oscillations in the power distribution that are associated with a periodic redistribution of xenon poison.[†]

If, for example, some perturbation raises the flux near the top of the reactor and depresses the flux near the bottom, the xenon will burn out more rapidly at the top than at the bottom, whereas the local rates of formation by iodine decay remain nearly constant for some time. The reactivity therefore increases at the top and decreases at the bottom, amplifying the flux tilt. The power peak at the top is limited by the control devices that maintain constant total power and by the amount of reactivity that can be added by burning out the xenon. Ultimately, the reactivity distribution is reversed by the growth of xenon from the iodine that is now forming more rapidly at the top than at the bottom, and by the decay of xenon at the bottom; consequently, the power now peaks at the bottom of the reactor. At sufficiently high flux levels, these oscillations may persist or grow unless damped out by appropriate control rod movement.

These spatial oscillations can occur only at such high flux levels that the rate of xenon burnout is important relative to the xenon decay rate, and in large reactors in which small local reactivity changes cause extended changes in the power distribution.

Control of such oscillations requires instrumentation adequate to detect changes in the power distribution, and a control system with sufficient flexibility to suppress local peaking.

[†] ^{135}Xe is produced mainly by the 6.7-hr decay (about 30% to the ^{135}Xe ground state, 70% to the 16-min isomer) of ^{135}I (which results from the 29-sec decay of the primary fission product ^{135}Te), and to a lesser extent as a direct product of fission; subsequently ^{135}Xe decays with a 9.2-hr half-life to ^{135}Cs or is burned out as a result of its enormous thermal neutron cross section (about 3×10^6 barns).

By assuming power distributions proportional to the fundamental solution of the boundary value problem associated with the shape of the reactor, Ward¹ devised a perturbation method for computing the threshold flux for xenon spatial oscillations. St. John and Randall extended Ward's approach to reactors with axial² and radial³ power distributions, which may be flattened in the middle to any extent. This report reviews the modal analysis treatment in significantly greater detail than was presented in the literature.^{2,3}

SUMMARY

Solutions for the equations describing the departure from equilibrium flux shape are expanded in terms of the eigenfunctions of the boundary value problem for the unperturbed buckling distribution. These modifications allow estimation of thresholds for practical cases in which the effect of a power (temperature) coefficient of reactivity is included.

The following assumptions and features are basic to the present treatment:

- One-energy-group diffusion theory.
- One dimension only (axial or radial).
- First harmonic only, no higher perturbation modes considered.
- Centrally flattened (to any extent) sinusoidal or J_0 fundamental flux; saddle-shaped or humped flux shapes not considered.
- Temperature coefficient allowed.
- Flux-shape (rather than reactor geometry) eigenvalues used.
- Direct fission formation of ^{135}Xe included.
- Limited to small perturbations.

Solution yields:

- Threshold reactor-average fluxes (if any) and corresponding oscillation periods.
- Periods of decaying, steady, or growing oscillations (if any) at any specified reactor-average flux levels. Perturbation flux damping factors and amplitude ratios of successive perturbation flux peaks are also computed for each specified reactor-average flux.

This calculational technique has been incorporated into a computer code, XENO-1747, that operates on the IBM 360/65. The computer code, although available, is not described in this report.

One-dimensional calculations, of which this modal analysis is one form, are limited in accuracy because of uncertainty in the neutron flux distribution required to represent the three-dimensional reactor.

DISCUSSION

FUNDAMENTAL EQUATIONS

Since xenon oscillations are most likely to occur in large thermal reactors, the distribution of thermal flux is treated to a good approximation by one-group diffusion theory

$$\nabla^2 \phi + B^2 \phi = 0 \quad (1)$$

where ϕ is the neutron flux $[(n/(cm^2 \text{ sec}))]$, and B^2 is the space- and time-dependent material buckling (cm^{-2}) of the hot, poisoned reactor. Equation 1 implies that the oscillations are slow compared with the effects of the delayed neutrons, so that at any time the flux shape is determined primarily by the spatial distribution of B^2 .

The equations governing the concentration of ^{135}Xe at any point in the reactor are

$$\frac{dI}{dt} = \gamma_I \Sigma_f \phi - \lambda_I I - \sigma_I I \phi \approx \lambda_I \Sigma_f \phi - \lambda_I I \quad (2)$$

$$\frac{dX}{dt} = \lambda_I I + \gamma_X \Sigma_f \phi - \lambda_X X - \sigma_X X \phi \quad (3)$$

for the net rates of formation of precursor ^{135}I and ^{135}Xe , respectively. In Equations 2 and 3, I and X represent the atoms/cm³ of ^{135}I and ^{135}Xe , γ_I and γ_X are the respective fractional fission yields, λ_I and λ_X are the respective decay constants (sec^{-1}), σ_I and σ_X are the respective microscopic thermal neutron cross sections (cm^2), and Σ_f is the macroscopic thermal neutron fission cross section of the fuel material in the reactor.

The steady-state concentrations of ^{135}I and ^{135}Xe are given by

$$I^0 = \frac{\gamma_I \Sigma_f \phi^0}{\lambda_I} \quad (\text{for } \sigma_I \phi I \ll \gamma_I I) \quad (4)$$

$$X^0 = \frac{\lambda_I I^0 + \gamma_X \Sigma_f \phi^0}{\lambda_X + \sigma_X \phi^0} \quad (5)$$

Eliminating I^0 ,

$$X^0 = \frac{(\gamma_I + \gamma_X)\Sigma_f\phi^0}{\lambda_X + \sigma_X\phi^0} \quad (6)$$

The equilibrium concentration of ^{135}Xe at very high flux levels is then

$$X_{\text{sat}}^0 = \frac{(\gamma_I + \gamma_X)\Sigma_f}{\sigma_X} \quad (7)$$

Then defining

$$\begin{aligned} i &= I/X_{\text{sat}}^0 \\ x &= X/X_{\text{sat}}^0 \\ \bar{\gamma}_I &= \gamma_I/(\gamma_I + \gamma_X) \\ \bar{\gamma}_X &= \gamma_X/(\gamma_I + \gamma_X) \end{aligned} \quad (8)$$

Equations 2 and 3 become

$$\frac{di}{dt} = \bar{\gamma}_I\sigma_X\phi - \lambda_I i \quad (9)$$

$$\frac{dx}{dt} = \lambda_I i + \bar{\gamma}_X\sigma_X\phi - \lambda_X x - \sigma_X x\phi \quad (10)$$

PERTURBATION SOLUTION

Small perturbations are assumed, and Equations 9 and 10 are linearized. The effect of the power coefficient α_T is included, and the solutions are expanded in terms of the eigenfunctions of the boundary value problem for the actual buckling distribution corresponding to fundamental flux profiles of any degree of flatness (i.e. from cosine or J_0 to totally flat axially or radially). For small perturbations (primed) about steady-state conditions (starred),

$$\begin{aligned} \phi &= \phi^* + \phi' \\ B^2 &= B^{2*} + a_T\phi' + a_X x' \\ x &= x^* + x' \\ i &= i^* + i' \end{aligned} \quad (11)$$

Here a_T and a_X are coefficients that express the dependence of buckling on changes in flux and ^{135}Xe concentration. Substituting Equations 11 into Equations 1, 9, and 10 results in

$$\begin{aligned} \nabla^2 \phi^* + \nabla^2 \phi' + B^{2*} \phi^* + a_T \phi^* \phi' + a_X \phi^* x' + B^{2*} \phi' \\ + a_T \phi'^2 + a_X x' \phi' = 0 \end{aligned} \quad (12)$$

$$\frac{di'}{dt} + \frac{di^*}{dt} = \bar{\gamma}_I \sigma_X \phi^* + \bar{\gamma}_I \sigma_X \phi' - \lambda_I i^* - \lambda_I i' \quad (13)$$

$$\begin{aligned} \frac{dx'}{dt} + \frac{dx^*}{dt} = \lambda_I i^* + \lambda_I i' + \bar{\gamma}_X \sigma_X \phi^* + \bar{\gamma}_X \sigma_X \phi' - \lambda_X x^* - \lambda_X x' \\ - \sigma_X x^* \phi^* - \sigma_X x^* \phi' - \sigma_X x' \phi^* - \sigma_X x' \phi' \end{aligned} \quad (14)$$

These equations are linear for small perturbations, for which $x' \phi' \approx \phi'^2 \approx 0$. Subtracting the original unperturbed equations [i.e., Equations 1, 9, and 10 with $i = i^*$, $x = x^*$, $\phi = \phi^*$, and $B^2 = B^{2*}$] from Equations 12, 13, and 14 leads to

$$\nabla^2 \phi' + B^{2*} \phi' + a_T \phi^* \phi' + a_X \phi^* x' = 0 \quad (15)$$

$$\frac{di'}{dt} = \bar{\gamma}_I \sigma_X \phi' - \lambda_I i' \quad (16)$$

$$\frac{dx'}{dt} = \lambda_I i' + \bar{\gamma}_X \sigma_X \phi' - \lambda_X x' - \sigma_X x^* \phi' - \sigma_X x' \phi^* \quad (17)$$

The time derivatives are eliminated by searching for solutions of the form[†]

$$\begin{aligned} \phi' &= \phi'' e^{\omega t} \\ x' &= x'' e^{\omega t} \\ i' &= i'' e^{\omega t} \end{aligned} \quad (18)$$

Substituting Equations 18 into Equations 15, 16, and 17 yields

$$\nabla^2 \phi'' + B^{2*} \phi'' + a_X \phi^* x'' + a_T \phi^* \phi'' = 0 \quad (19)$$

$$\omega i'' = \bar{\gamma}_I \sigma_X \phi'' - \lambda_I i'' \quad (20)$$

$$\omega x'' = \bar{\gamma}_X \sigma_X \phi'' + \lambda_I i'' - \lambda_X x'' - \sigma_X \phi^* x'' - \sigma_X x^* \phi'' \quad (21)$$

[†] Identical results are obtained by taking Laplace transforms of Equations 15, 16, and 17.

FLUX-SHAPE EIGENFUNCTION EXPANSION

The perturbation amplitudes ϕ'' , i'' , and x'' can be expanded in terms of the orthonormal eigenfunctions g_i of the boundary value problem which has the unperturbed flux shape[†] as its fundamental solution:

$$\phi'' = \sum_i A_i g_i \quad i = 0, 1, 2, \dots \quad (22)$$

$$x'' = \sum_i \chi_i g_i \quad i = 0, 1, 2, \dots \quad (23)$$

$$(\nabla^2 + B^{2*})g_i + \mu_i^2 g_i = 0 \quad i = 0, 1, 2, \dots \quad (24)$$

Here μ_i^2 is the added uniform buckling necessary to produce the i^{th} mode; $\mu_0^2 = 0$; and g_0 is the fundamental flux eigenfunction. The corresponding A_0 is taken equal to zero, in accord with the usual way of controlling a reactor at constant total power level.^{††}

Eliminating the iodine terms between Equations 20 and 21 yields

$$\omega x'' = \bar{\gamma}_x \sigma_x \phi'' + \frac{\lambda_I \sigma_x \bar{\gamma}_I \phi''}{\omega + \lambda_I} - \lambda_x x'' - \sigma_x \phi^* x'' - \sigma_x x^* \phi'' \quad (25)$$

Substituting Equations 22 and 23 into Equation 19 gives

$$\sum_i (\nabla^2 + B^{2*})A_i g_i + a_T \phi^* \sum_i A_i g_i + a_x \phi^* \sum_i \chi_i g_i = 0$$

so that by Equation 24,

$$\sum_i \left[-\mu_i^2 A_i g_i + a_T \phi^* A_i g_i + a_x \phi^* \chi_i g_i \right] = 0 \quad (26)$$

[†] See Appendix for expansion in terms of reactor-geometry modes.

^{††} If A_0 were not zero, we would be dealing with the "ordinary" xenon instability in which the flux changes everywhere in the same direction with time.

Substituting Equations 22 and 23 into Equation 25 results in

$$\sum_i \left[\left(\sigma_x \bar{Y}_x + \frac{\lambda_I \bar{Y}_I \sigma_x}{\omega + \lambda_I} - \sigma_x x^* \right) A_i g_i - \left(\omega + \lambda_x + \sigma_x \phi^* \right) X_i g_i \right] = 0 \quad (27)$$

The unknown coefficients A_i and X_i may be determined (within an arbitrary normalizing constant) by multiplying Equations 26 and 27 by g_j ($j = 0, 1, 2, \dots$) and integrating[†] over the reactor volume: taking the integral of the infinite sum equal to the sum of the integrals,

$$\sum_i \left[\int (a_T \phi^* - \mu_1^2) A_i g_i g_j dV + \int a_x \phi^* X_i g_i g_j dV \right] = 0$$

$$\sum_i \left[\int \left(\sigma_x \bar{Y}_x + \frac{\lambda_I \bar{Y}_I \sigma_x}{\omega + \lambda_I} - \sigma_x x^* \right) A_i g_i g_j dV - \int \left(\omega + \lambda_x + \sigma_x \phi^* \right) X_i g_i g_j dV \right] = 0$$

Rearranging, then using the orthonormal properties

$$\begin{aligned} \int g_i g_j dV &= 0 \quad \text{for } i \neq j \\ &= 1 \quad \text{for } i = j \end{aligned}$$

and defining

$$\left. \begin{aligned} \int \phi^* g_j^2 dV &= \bar{\phi}_j & \int \phi^* g_i g_j dV &= \bar{\phi}_{ij} \\ \int x^* g_j^2 dV &= \bar{x}_j & \int x^* g_i g_j dV &= \bar{x}_{ij} \end{aligned} \right\} \quad (28)$$

these two sets of equations are reduced to

$$[a_T \bar{\phi}_j - \mu_j^2] A_j + [a_x \bar{\phi}_j] X_j + \sum_{i \neq j} [a_T \bar{\phi}_{ij} A_i + a_x \bar{\phi}_{ij} X_i] = 0 \quad (29)$$

$$\begin{aligned} & \left[\sigma_x \bar{x}_j - \sigma_x \bar{Y}_x - \frac{\lambda_I \bar{Y}_I \sigma_x}{\omega + \lambda_I} \right] A_j + \left[\omega + \lambda_x + \sigma_x \bar{\phi}_j \right] X_j \\ & + \sum_{i \neq j} \left[\sigma_x \bar{x}_{ij} A_i + \sigma_x \bar{\phi}_{ij} X_i \right] = 0 \end{aligned} \quad (30)$$

[†] Assuming a_T and a_x do not vary with position; it may be necessary to compute averaged values of the related quantities α_T and α_x .

If the summation is cut off at n terms (n values of j), there are $2n$ linear equations (Equations 29 and 30) in the $2n$ unknown expansion coefficients A_j and X_j . For these homogeneous equations to have a nontrivial solution, Cramer's rule requires the determinant of the coefficients to be zero; the resultant polynomial equation of degree $2n$ determines possible values of ω . Since the cross terms $\bar{\phi}_{ij}$ and $\bar{x}_{ij}$ are small because of the orthogonality of the g_i 's (being actually equal to zero if the unperturbed flux distribution is flat), they can be ignored. For $n = 1$, possible values of ω are given by

$$\begin{vmatrix} (a_T \bar{\phi}_1 - \mu_1^2) & (a_X \bar{\phi}_1) \\ \left(\sigma_X \bar{x}_1 - \sigma_X \bar{y}_X - \frac{\lambda_I \bar{y}_I \sigma_X}{\omega + \lambda_I} \right) & (\omega + \lambda_X + \sigma_X \bar{\phi}_1) \end{vmatrix} = 0$$

Thus

$$\omega^2 + C_1 \omega + C_2 = 0 \quad (31)$$

where

$$C_1 = \lambda_I + \lambda_X + \sigma_X \bar{\phi}_1 - \frac{a_X \sigma_X \bar{\phi}_1 (\bar{x}_1 - \bar{y}_X)}{a_T \bar{\phi}_1 - \mu_1^2} \quad (32)$$

and[†]

$$C_2 = \lambda_I \left[\lambda_X + \sigma_X \bar{\phi}_1 + \frac{a_X \sigma_X \bar{\phi}_1 (1 - \bar{x}_1)}{a_T \bar{\phi}_1 - \mu_1^2} \right] \quad (33)$$

If ω is real or complex with a real part, any perturbation will either grow or decay, with or without oscillation as described on page 30. If ω is a pure imaginary, oscillations of constant amplitude are possible, as discussed below.

The threshold for sustained oscillations is given by the condition that the damping term $C_1 \omega$ is zero. For C_1 to be zero,

$$\mu_1^2 = a_T \bar{\phi}_1 - \frac{a_X (\bar{x}_1 - \bar{y}_X)}{1 + (\lambda_I + \lambda_X) / \sigma_X \bar{\phi}_1} \equiv a_T \bar{\phi}_1 - a_X f(\bar{\phi}_1) \quad (34)$$

[†] After replacing $\bar{y}_I$ by $1 - \bar{y}_X$

in which $\bar{x}_1$ can be approximated by the steady-state value of xenon for the average flux $\bar{\phi}_1$ (Equation 6 divided by Equation 7):

$$\bar{x}_1 = \frac{1}{1 + \lambda_x / \sigma_x \bar{\phi}_1} \quad (35)$$

This is a good approximation because $\bar{\phi}_j$ and $\bar{x}_j$ are averages taken over the same weighting functions.

The eigenvalues μ_j^2 defined in Equation 24 depend only on the flux shape and reactor size. First-harmonic values of μ_1^2 have been derived for axial and radial oscillations, as discussed on pages 18 and 21. Combining factors that depend upon a particular reactor, and substituting $a_x = \alpha_x / M^2$, $a_T = \alpha_T / M^2$, and $\Lambda_1 = \mu_1^2 / B_g^2$,

$$f(\bar{\phi}_1) = - \frac{M^2 \mu_1^2}{\alpha_x} + \frac{\alpha_T}{\alpha_x} \bar{\phi}_1 = - \frac{M^2 B_g^2 \Lambda_1}{\alpha_x} + \frac{\alpha_T}{\alpha_x} \bar{\phi}_1 \quad (36)$$

where α_x is the negative reactivity change due to saturated xenon, α_T is the power (temperature) coefficient in terms of reactivity change per unit flux, B_g^2 is the geometric buckling, and M^2 is the migration area. This expression, which can be written as

$$\frac{1}{1 + a_1 / \bar{\phi}_1} - a_2 = \frac{a_4 - a_5 \bar{\phi}_1}{1 + a_3 / \bar{\phi}_1}$$

is a cubic in the unknown threshold flux, viz.

$$a_5 \bar{\phi}_1^3 + [1 - a_2 - a_4 + a_5(a_1 + a_3)] \bar{\phi}_1^2 + [a_1 a_3 a_5 - a_1 a_2 - a_1 a_4 - a_3 a_4] \bar{\phi}_1 - a_1 a_3 a_4 = 0 \quad (37)$$

Here

$$\begin{aligned} a_1 &= \lambda_x / \sigma_x \\ a_2 &= \bar{\gamma}_x \\ a_3 &= (\lambda_I + \lambda_x) / \sigma_x \\ a_4 &= - M^2 \mu_1^2 / \alpha_x \\ a_5 &= - \alpha_T / \alpha_x \end{aligned}$$

If $\alpha_T = 0$, this reduces to the quadratic

$$\left[1 - a_2 - a_4\right] \bar{\phi}_1^2 + \left[-a_1 a_2 - a_1 a_4 - a_3 a_4\right] \bar{\phi}_1 - a_1 a_3 a_4 = 0 \quad (38)$$

Solution of Equation 37 or Equation 38 determines all real (positive or negative) and complex (if any) threshold flux roots. When $\alpha_T \geq 0$, these equations can yield only one real, positive value for $\bar{\phi}_1$; but if $\alpha_T < 0$, there will in general be two real, positive values of the threshold flux, representing lower and upper bounds on the region of instability. The upper bound arises because xenon burnout saturates with increasing flux while the effect of the power coefficient does not.

Figure 1 shows a plot of $\bar{\phi}_1$ versus

$$f(\bar{\phi}_1) \equiv \frac{\frac{1}{1 + \lambda_x / \sigma_x \bar{\phi}_1} - \bar{\gamma}_x}{1 + (\lambda_I + \lambda_x) / \sigma_x \bar{\phi}_1} \quad (39)$$

for $\lambda_I = 2.94 \times 10^{-5} \text{ sec}^{-1}$, $\lambda_x = 2.10 \times 10^{-5} \text{ sec}^{-1}$, $\gamma_I = 0.0586$, $\gamma_x = 0.003$ (i.e. $\bar{\gamma}_x = 0.0487$), and $\sigma_x = 3.08 \times 10^{-18} \text{ cm}^2$. $f(\bar{\phi}_1)$ is negative for real, positive flux levels less than $\phi_{\min}$, where, for these values,

$$\phi_{\min} = \frac{\lambda_x \bar{\gamma}_x}{\sigma_x (1 - \bar{\gamma}_x)} = 3.5 \times 10^{11} \text{ n/(cm}^2 \text{ sec)} \quad (40)$$

at which the burnout of xenon just equals the rate of direct fission formation (i.e. second and fourth terms equal on the right-hand side of Equation 10, with $x = \bar{x}_1$ and $\phi = \bar{\phi}_1$, so that $\bar{x}_1 = \bar{\gamma}_x$). Below this flux level, xenon oscillations of constant or increasing amplitude cannot occur in a reactor of any size; only decaying xenon transients (steady or oscillatory, as discussed later) are possible. The value of $f(\bar{\phi}_1)$ increases monotonically with increasing flux level and saturates at high flux level to the value $(1 - \bar{\gamma}_x)$, i.e. about 0.95.

The error involved in using Equation 35 in the expression for $f(\bar{\phi}_1)$, i.e. Equation 39, solid line of Figure 1, rather than $\bar{x}_1 = \int x \cdot g_1^2 dV$ (which yields the dashed line for a sine fundamental and the same solid line for a flat fundamental) is discussed on page 36.

FIGURE 1 Threshold Flux $\bar{\phi}_1$ Versus $f(\bar{\phi}_1)$

EVALUATION OF AXIAL FLUX-SHAPE EIGENVALUES

Equation 24 for the axial direction of a reactor which is radially infinite and uniform is

$$\left(\frac{d}{dz^2} + B^{2*}\right)g_i + \mu_i^2 g_i = 0$$

B^{2*} is taken regionwise constant and such as to produce a centrally flattened symmetric fundamental flux shape as shown in Figure 2. If the series solution is truncated after the first harmonic, as discussed following Equation 30 and on page 36, then μ_1^2 can be computed as a function of fundamental flux flattening $(H-a)/H$. Flux shapes ranging from a pure sine to a perfectly flat fundamental are considered.

FIGURE 2 Postulated Axial Flux Shapes

The fundamental flux in Regions I and II is described by

$$\left. \begin{aligned} \phi_I^* &= A^* \sin \pi z/a \\ \phi_{II}^* &= A^* = \text{constant} \end{aligned} \right\} \quad (41)$$

and the corresponding superimposed perturbation flux is given by

$$\phi_I' = A' \sin \left[(\pi/a)^2 + \mu_1^2 \right]^{\frac{1}{2}} z \quad (42)$$

$$\phi_{II}' = C' \sin \mu_1 \left(\frac{H}{2} - z \right) \quad (43)$$

where $0 \leq z \leq$ reactor height H , a is the axial distance over which the fundamental is sinusoidal, and μ_1^2 represents the additional uniform buckling in cm^{-2} (Equation 24) required to produce the first axial flux harmonic g_1 . The perturbation flux buckling is thus $B_I^{2'} = B_I^{2*} + \mu_1^2 = (\pi/a)^2 + \mu_1^2$ in Region I, and $B_{II}^{2'} = B_{II}^{2*} + \mu_1^2 = \mu_1^2$ in Region II.

Applying boundary conditions $\phi_I' = \phi_{II}'$ and $d\phi_I'/dz = d\phi_{II}'/dz$ at $z = a/2$, and defining $\eta \equiv a/H$ as the fraction of the reactor height over which the fundamental flux is curved, lead to

$$\left[\sin \frac{1}{2} \sqrt{\pi^2 + (\mu_1 \eta H)^2} \right] \cdot \left[\cos \frac{\mu_1 H}{2} (1-\eta) \right] + \sqrt{1 + \left(\frac{\pi}{\mu_1 \eta H} \right)^2} \cdot \left[\sin \frac{\mu_1 H}{2} (1-\eta) \right] \cdot \left[\cos \frac{1}{2} \sqrt{\pi^2 + (\mu_1 \eta H)^2} \right] = 0 \quad (44)$$

For a pure sine fundamental, $\eta = 1$, so that $\sin \frac{1}{2} \sqrt{\pi^2 + (\mu_1 H)^2} = 0$ and $\sqrt{\pi^2 + (\mu_1 H)^2} = 2m\pi$; taking $m = 1$ yields the lowest positive root, $\mu_1^2 = 3(\pi/H)^2$, or $\Lambda_1 = 3$.

For a perfectly flat fundamental, $\eta = 0$, and Equation 44 yields an indeterminate expression. Successive application of L'Hospital's rule leads to $\mu_1 H/2 = m\pi/2$, $m = 1, 3, 5, \dots$, so that $\mu_1^2 = (\pi/H^2)$, or $\Lambda_1 = 1$.

Values of μ_1^2 for $0 < \eta < 1$ were obtained by numerical solution of Equation 44. Computed values of $\Lambda_1 = \mu_1^2/(\pi/H)^2$ for this axial case are plotted in Figure 3, together with values of the geometric-mode axial eigenvalue Λ_1' , discussed in the Appendix, and the radial flux-shape eigenvalue, discussed next.

FIGURE 3 Axial and Radial Eigenvalues as a Function of Flux Flattening

EVALUATION OF RADIAL FLUX-SHAPE EIGENVALUES

Equation 24 for the radial direction of a reactor which is axially infinite and uniform is

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right] g_i + \frac{1}{r^2} (B^*{}^2 + \mu_i^2) g_i = 0 \quad (45)$$

Separating the angular dependence yields a solution

$$g_i = G_i \sin n\theta$$

For this illustrative case, the phase angle θ_o has been set equal to zero. Equation 45 reduces to

$$r^2 \left[\frac{\partial}{\partial r^2} + r \frac{\partial}{\partial r} \right] G_i + \left[(B^*{}^2 + \mu_i^2) - n^2 \right] G_i = 0$$

which has a solution

$$G_i \propto J_n(j_{n,i} r/R)$$

where R is the outer boundary.

A variety of radial-azimuthal oscillations are possible, a few of which are shown in Figure 4 for selected zeros $j_{n,i}$ of the Bessel function of the first kind and order n , $J_n(j_{n,i} r/R)$. The flux may be considered to be increasing in the shaded regions, decreasing where not shaded, and vice versa a half-period later. Sketch A illustrates "ordinary" xenon instability, where the flux rises or falls everywhere at the same time, the increase or decrease being a maximum at the center and falling off to zero at the circumference. But it is Sketch B that will be considered in this study, i.e. as on page 18, only the first harmonic is retained.

FIGURE 4 Radial-Azimuthal Oscillations

The fundamental flux shape is assumed to have a region of constant flux in the center (Region II, Figure 5) and a region of uniform buckling giving a J_0, Y_0 flux shape in the outer radial region (Region I, Figure 5). Degrees of fundamental flux flatness all the way from a pure J_0 shape to a completely flat profile are considered.

FIGURE 5 Postulated Radial Flux Shapes

The fundamental flux in Regions I and II is described by

$$\begin{aligned}\phi_I^* &= K^* J_0(Br) + C^* Y_0(Br) \\ &= K^* J_0(b\rho) + C^* Y_0(b\rho)\end{aligned}\quad (46)$$

$$\phi_{II}^* = A^* = \text{constant (zero buckling)}$$

where r is the radial distance from the center of the reactor ($r = 0$) to the edge of the reactor ($r = R$), B^2 is the buckling of the unperturbed flux in Region I, and where for convenience, $\rho = r/R$ and $b = RB$ (ρ and b dimensionless).

The corresponding superimposed perturbation flux is given by

$$\begin{aligned}\phi_I' &= C' J_1(\sqrt{B^2 + \mu_1^2} \cdot r) + D' Y_1(\sqrt{B^2 + \mu_1^2} \cdot r) \\ &= C' J_1(\alpha\rho) + D' Y_1(\alpha\rho)\end{aligned}\quad (47)$$

$$\phi_{II}' = A' J_1(\mu_1 r) = A' J_1(\beta\rho)\quad (48)$$

where μ_1^2 represents the additional uniform buckling in cm^{-2} , Equation 24, required to produce the first harmonic g_1 , and where for convenience, $\beta = \mu_1 R$ and $\alpha = R\sqrt{B^2 + \mu_1^2} = \sqrt{b^2 + \beta^2}$ (α and β dimensionless).

The quantity μ_1 is obtained as a function of η , the fraction of the radial distance over which the fundamental flux is curved, by applying the boundary conditions on ϕ^* and ϕ' and solving the resulting equations. Applying the boundary conditions

$$\left. \begin{aligned}\phi_I^* &= \phi_{II}^* \quad \text{and} \quad \phi_I' = \phi_{II}' \\ \frac{\partial \phi_I^*}{\partial \rho} &= 0 \quad \text{and} \quad \frac{\partial \phi_I'}{\partial \rho} = \frac{\partial \phi_{II}'}{\partial \rho}\end{aligned} \right\} \quad \text{at } \rho = 1-\eta$$

$$\phi_I^* = 0 \quad \text{and} \quad \phi_I' = 0 \quad \text{at } \rho = 1$$

leads to

$$\frac{J_1[b(1-\eta)]}{J_0[b]} = \frac{Y_1[b(1-\eta)]}{Y_0[b]}\quad (49)$$

and

$$\frac{J_1[\alpha(1-\eta)]}{J_1[\alpha]} = \frac{\alpha}{\beta} \cdot \frac{J_1[\beta(1-\eta)]}{J_0[\beta(1-\eta)]} \left\{ \frac{J_0[\alpha(1-\eta)]}{J_1[\alpha]} - \frac{Y_0[\alpha(1-\eta)]}{Y_1[\alpha]} \right\} \quad (50)$$

Equation 49 is solved for b as a function of η ; using $\alpha^2 = b^2 + \beta^2$, Equation 50 is then solved for α and β ; finally, $\mu_1^2 = (\beta/R)^2$ and $\Lambda_1 = \mu_1^2/B_g^2 = \mu_1^2/(2.4048/R)^2 = (\beta/2.4048)^2$.

When $\eta = 1$, $C^* = 0$ and $D' = 0$; then $\alpha = 3.8317$, $\beta = \sqrt{\alpha^2 - b^2} = 2.9831$, $\mu_1 = \beta/R = 2.9831/R \text{ cm}^{-1}$, and $\Lambda_1 = (2.9831/R)^2/(2.4048/R)^2 = 1.5388$.

When $\eta = 0$, $J_1(\beta\rho)$ is a maximum at $\rho = 1$, so that $\beta = 1.8412$; also B^2 in Region I becomes infinite, so that $b = \alpha = \infty$. Hence $\Lambda_1 = (\beta/2.4048)^2 = 0.5862$.

Values of this radial flux-shape Λ_1 are plotted in Figure 3, together with Λ_1 and Λ_1' (Appendix) for the axial case.

Figure 6 shows β ($\mu_1^2 = \beta^2$ in units of R^2) as a function of fundamental flux flattening. For comparison, the corresponding axial eigenvalue is shown for a right-cylindrical reactor ($H = 2R$): since $\Lambda_1 = \mu_1^2/B_g^2$, then $\Lambda_{1R} = \mu_{1R}^2/(2.4048/R)^2$ for radial oscillations, or $\mu_{1R}R = 2.4048\sqrt{\Lambda_{1R}}$; for axial oscillations, $\Lambda_{1Z} = \mu_{1Z}^2/(\pi/H)^2$, so that $\mu_{1Z} = (\pi/H)\sqrt{\Lambda_{1Z}} = (\pi/2R)\sqrt{\Lambda_{1Z}}$, or $\mu_{1Z}R = (\pi/2)\sqrt{\Lambda_{1Z}}$. For example, for $\eta = 1.0$, $\mu_{1R} = 2.4048\sqrt{1.5388} = 2.9831$ and $\mu_{1Z}R = (\pi/2)\sqrt{3} = 2.7207$. In this manner, axial Λ_1 values for all $0 \leq \eta \leq 1.0$ have been multiplied by $\pi/2$ to obtain an axial μ_1 in units of R for a cylindrical reactor with a core diameter equal to the core height. It is apparent from Figure 6 that more buckling must be added to excite the first radial harmonic than to excite the first axial harmonic, assuming core properties such as the temperature coefficient are the same for both axial and radial perturbations; under these conditions, then, the threshold flux for radial oscillation is higher than that for axial oscillation.

FIGURE 6 Comparison of Axial and Radial Eigenvalues for a Right-Cylindrical Reactor

REACTOR-AVERAGE FLUXES

Flux thresholds for both axial and radial oscillations are calculated in terms of the actual reactor-average flux $\bar{\phi}$ by dividing the not-quite-identical $\bar{\phi}_1$ by the ratio

$$\frac{\bar{\phi}_1}{\bar{\phi}} = \frac{\int \phi^* g_1^2 dV / \int g_1^2 dV}{\int \phi^* dV / \int dV}$$

For axial oscillations,

$$\frac{\bar{\phi}_1}{\bar{\phi}} = \frac{\int_0^{a/2} \left[A^* \sin \frac{\pi z}{a} \right] \left[A' \sin \sqrt{\left(\frac{\pi}{a} \right)^2 + \mu_1^2} z \right]^2 dz + \int_{a/2}^{H/2} A^* \left[C' \sin \mu_1 \left(\frac{H}{2} - z \right) \right]^2 dz}{\int_0^{a/2} \left[A' \sin \sqrt{\left(\frac{\pi}{a} \right)^2 + \mu_1^2} z \right]^2 dz + \int_{a/2}^{H/2} \left[C' \sin \mu_1 \left(\frac{H}{2} - z \right) \right]^2 dz} \cdot \frac{\int_0^{a/2} A^* \sin \frac{\pi z}{a} dz + \int_{a/2}^{H/2} A^* dz}{\int_0^{H/2} dz}$$

If the fundamental flux is flat, $a = 0$, and $\bar{\phi}_1/\bar{\phi} = 1$. If the fundamental is sinusoidal, $a = H$, and successive integration by parts leads to $\bar{\phi}_1/\bar{\phi} = 16/15 = 1.0667$. For intermediate values of $\eta = a/H$, define $k = \mu_1 H = \pi \sqrt{\Lambda_1} = \text{constant for a given } \eta$, $\epsilon = \sqrt{\pi^2 + (\eta k)^2}$ and recall that by the boundary conditions discussed on p 23, A' and C' are related by

$$A' \sin \left[\sqrt{\left(\frac{\pi}{a} \right)^2 + \mu_1^2} \left(\frac{a}{2} \right) \right] = C' \sin \left[\mu_1 \left(\frac{H-a}{2} \right) \right]$$

Then analytic integration yields

$$\frac{\bar{\phi}_1}{\bar{\phi}} = \frac{\frac{\eta}{2} \left\{ \frac{1}{\pi} + \frac{1}{2} \left[\frac{\sin \epsilon - 1}{\pi - 2\epsilon} - \frac{\sin \epsilon + 1}{\pi + 2\epsilon} \right] + \frac{1}{4k} \left[\frac{\sin \epsilon/2}{\sin k(1-\eta)/2} \right]^2 \left[k(1-\eta) - \sin k(1-\eta) \right] \right\}}{\left\{ \eta \left[\frac{2}{\pi} - 1 \right] + 1 \right\} \left\{ \frac{\eta}{4} \left[1 - \frac{\sin \epsilon}{\epsilon} \right] + \frac{1}{4k} \left[\frac{\sin \epsilon/2}{\sin k(1-\eta)/2} \right]^2 \left[k(1-\eta) - \sin k(1-\eta) \right] \right\}}$$

(51)

For radial oscillations,

$$\begin{aligned}
\int \phi^* g_1^2 dV &= \int_0^{1-\eta} A^* [A' J_1(\beta \rho)]^2 \rho d\rho + \\
&\quad \int_{1-\eta}^1 [K^* J_0(b\rho) + C^* Y_0(b\rho)] [C' J_1(\alpha \rho) + D' Y_1(\alpha \rho)]^2 \rho d\rho \\
\int g_1^2 dV &= \int_0^{1-\eta} [A' J_1(\beta \rho)]^2 \rho d\rho + \int_{1-\eta}^1 [C' J_1(\alpha \rho) + D' Y_1(\alpha \rho)]^2 \rho d\rho \\
\int \phi^* dV &= \int_0^{1-\eta} A^* \rho d\rho + \int_{1-\eta}^1 [K^* J_0(b\rho) + C^* Y_0(b\rho)] \rho d\rho \\
\int dV &= \int_0^1 \rho d\rho = \frac{1}{2}
\end{aligned}$$

where, as before, $\beta = \mu_1 R$ and $b = BR$, $\alpha = \sqrt{b^2 + \beta^2}$, $\rho = r/R$; and where the coefficients are related by the boundary conditions as follows:

$$A^* = K^* \left\{ J_0[b(1-\eta)] - \frac{J_0[b]}{Y_0[b]} Y_0[b(1-\eta)] \right\} \equiv F_1 K^*$$

$$\text{by } \phi_I^* = \phi_{II}^* \text{ at } \rho = 1-\eta$$

$$C^* = -K^* \frac{J_0[b]}{Y_0[b]} \equiv F_2 K^* \text{ by } \phi_I^* = 0 \text{ at } \rho = 1$$

$$D' = -C' \frac{J_1[\alpha]}{Y_1[\alpha]} \equiv F_3 C' \text{ by } \phi_I' = 0 \text{ at } \rho = 1$$

$$A' = C' \left\{ \frac{J_1[\alpha(1-\eta)]}{J_1[\beta(1-\eta)]} - \frac{J_1[\alpha]}{Y_1[\alpha]} \cdot \frac{Y_1[\alpha(1-\eta)]}{J_1[\beta(1-\eta)]} \right\} \equiv F_4 C'$$

$$\text{by } \phi'_I = \phi'_{II} \text{ at } \rho = 1-\eta$$

Then $\bar{\phi}_1/\bar{\phi}$ can be written as

$$\frac{\bar{\phi}_1}{\bar{\phi}} = \frac{(I_5 + I_3)/(I_4 + I_3)}{(1-\eta)^2 + I_1 + I_2} \quad (52)$$

where

$$I_1 = \frac{2}{F_1} \int_{1-\eta}^1 J_0(b\rho) \rho d\rho$$

$$I_2 = \frac{2F_2}{F_1} \int_{1-\eta}^1 Y_0(b\rho) \rho d\rho$$

$$I_3 = \int_0^{1-\eta} [J_1(\beta\rho)]^2 \rho d\rho$$

$$I_4 = \int_{1-\eta}^1 \left[\frac{1}{F_4} J_1(\alpha\rho) + \frac{F_3}{F_4} Y_1(\alpha\rho) \right]^2 \rho d\rho$$

$$I_5 = \int_{1-\eta}^1 \left[\frac{1}{F_1} J_0(b\rho) + \frac{F_2}{F_1} Y_0(b\rho) \right] \left[\frac{1}{F_4} J_1(\alpha\rho) + \frac{F_3}{F_4} Y_1(\alpha\rho) \right]^2 \rho d\rho$$

If the fundamental flux is flat, $\eta = 0$ and $\bar{\phi}_1/\bar{\phi} = 1$. If the fundamental flux has a pure J_0 shape, $\eta = 1$, $\alpha = 3.8317$, $b = 2.4048$, and $\bar{\phi}_1/\bar{\phi}$ reduces to

$$\frac{\int_0^1 [J_0(b\rho)] [J_1(\alpha\rho)]^2 \rho d\rho}{2 \int_0^1 J_0(b\rho) d\rho} \bigg/ \int_0^1 [J_1(\alpha\rho)]^2 \rho d\rho = 1.3621 \quad (53)$$

by numerical integration.

Values of $\bar{\phi}_1/\bar{\phi}$ as a function of flux flattening are plotted in Figure 7. Those for axial oscillations were obtained by computer evaluation of Equation 51. Those for radial oscillations were calculated from Equations 52 and 53 with a double-precision numerical integration code based on the exact Weddle rule[†] using double-precision Bessel function routines.

FIGURE 7 Ratio of $\bar{\phi}_1$ to Reactor-Average Flux as a Function of Flux Flattening

$$\int_{x_0}^{x_0+6\Delta x} y dx = \frac{\Delta x}{140} \left[41y_0 + 216y_1 + 27y_2 + 272y_3 + 27y_4 + 216y_5 + 41y_6 \right]$$

applied twice per integral I_i

OSCILLATION PERIODS AND PERTURBATION FLUX TRANSIENTS

For the first harmonic in ϕ'' and x'' , a quadratic in ω , Equation 31, is obtained, whose roots ω_1 and ω_2 determine the time dependence of the perturbation flux according to Equation 18:

$$\phi' = \phi'' e^{\omega t} = \phi'' (e^{\omega_1 t} + e^{\omega_2 t}) \quad (54)$$

If ω_1 and ω_2 are real, the perturbation grows steadily with time if either or both roots are positive, or decays steadily if both roots are negative. The roots are given by

$$\omega_1 = -\frac{C_1}{2} + \sqrt{\frac{C_1^2}{4} - C_2}, \quad \omega_2 = -\frac{C_1}{2} - \sqrt{\frac{C_1^2}{4} - C_2} \quad (55)$$

If ω_1 and ω_2 are complex, the perturbation will show oscillations which decay, remain constant, or grow according to the sign of the real part of the root. For the roots

$$\omega_1 = s + j\sigma, \quad \omega_2 = s - j\sigma, \quad j = \sqrt{-1}$$

where

$$s = -\frac{C_1}{2}, \quad \sigma = \sqrt{C_2 - \frac{C_1^2}{4}} \quad (56)$$

Equation 54 becomes

$$\phi' = \phi'' e^{st} [e^{j\sigma t} + e^{-j\sigma t}]$$

which reduces to [†]

$$\phi' = 2\phi'' e^{st} [\cos \sigma t] = 2\phi'' e^{st} \cos \left[\frac{2\pi t}{\tau} \right] \quad (57)$$

The oscillation period τ (sec) is thus

$$\tau = \frac{2\pi}{\sigma} = \frac{2\pi}{\text{Im}\omega} \quad (58)$$

The envelope of these oscillations is clearly proportional to e^{st} , so that

- if $\text{Re}\omega = s < 0$, the oscillation decays;
- = 0, the oscillation has a constant amplitude;
- > 0, the oscillation grows.

[†] Using $e^{\pm jx} = \cos x \pm j \sin x$ and $\cos x = \cos (x+2\pi)$.

A perturbation amplitude ratio may be defined as the amplitude of a perturbation peak divided by the amplitude of the preceding perturbation peak; thus

$$\begin{aligned} \frac{\max \phi'_{n+1}}{\max \phi'_n} &< 1.0 \text{ for decaying oscillations } (s < 0); \\ &= 1.0 \text{ for steady oscillations } (s = 0, \text{ i.e. at threshold}); \\ &> 1.0 \text{ for growing oscillations } (s > 0). \end{aligned}$$

The behavior of this perturbation amplitude ratio is shown schematically in Figure 8 as a function of the nature of the roots of Equation 31. As long as both roots are real ($\omega_1 = s_1$, $\omega_2 = s_2$), the perturbation varies monotonically with time; but as soon as these roots become complex, oscillatory behavior occurs. Figure 8A illustrates an inherently stable reactor condition, in which any perturbation initiated by some spatially asymmetric Δk kick will spontaneously die away. Figure 8B shows the single threshold possible when the temperature coefficient α_T is zero or positive. It should be noted that in the region of steady growth, it is not necessary for both roots to remain positive, as shown; if either one remains greater than zero, the perturbation will grow steadily. Figure 8C illustrates the case when $\alpha_T < 0$, so that two flux thresholds can occur; if the reactor is made more stable, as by decreasing the degree of fundamental flux flattening, the $\max \phi'_{n+1}/\max \phi'_n$ curve will be lowered, so that the two flux thresholds approach each other and, as a special case, become equal; further stabilization would push the entire $\max \phi'_{n+1}/\max \phi'_n$ curve below 1.0 and these would be no threshold at all, as in Figure 8A.

For nonsteady oscillations, the damping factor Δ (positive for decaying oscillations) and the perturbation amplitude ratio follow from Equation 57:

$$\Delta = -\text{Re}\omega = -s \text{ (sec}^{-1}\text{)} \quad (59)$$

$$\frac{\max \phi'_{n+1}}{\max \phi'_n} = e^{s\tau} = e^{-\delta} \quad (60)$$

where the logarithmic decrement δ (positive for decaying oscillations) is defined as

$$\delta = \ln \frac{\max \phi'_n}{\max \phi'_{n+1}} = -\tau s = \tau \Delta \quad (61)$$

FIGURE 8 Schematic Representation of the Behavior of the Perturbation Amplitude Ratio and the Roots of the Time-Dependent Perturbation Flux Equation

Modal analysis can furnish no information about the actual magnitude of the perturbation flux ϕ' or the perturbed flux $\phi^* + \phi'$. Relative magnitudes as a function of time can be obtained by Equation 57, or more generally from Equation 57 applied to times t_1 and t_2 , so that ϕ'' cancels out:

$$\frac{\phi' \text{ at } t_2}{\phi' \text{ at } t_1} = e^{s(t_2-t_1)} \left(\frac{\cos 2\pi t_2/\tau}{\cos 2\pi t_1/\tau} \right) \quad (62)$$

But to actually solve Equation 57 requires a value for ϕ'' , i.e. for the A_i of Equation 22, which are unavailable because it is possible to obtain only the ratios of the unknowns A_i and X_i from the linear, homogeneous[†] Equations 29 and 30.

This is physically reasonable, because the amplitude of the resultant perturbation clearly depends on the magnitude of the disturbance (some asymmetrically located temperature or reactivity change) that caused it. The modal analysis treatment tells only whether or not the system is inherently stable without regard to the nature of the particular incident that upsets it. Perturbation amplitudes can be computed only if some experimental observation (as local total flux change or local change in effluent coolant temperature at a particular time) is available. Otherwise recourse must be had to some numerical space-time approach which utilizes various specific disturbances as input.

In general, oscillations can occur at flux levels not equal to a threshold value, but they will grow or decay with a period $2\pi/\sigma$, where σ is a function of C_1 and C_2 , Equation 56. Just at a threshold, $C_1 = 0$, and sustained oscillations of constant (but not calculable!) amplitude can occur. In this case, $s = 0$ and $\sigma = \sqrt{C_2}$, $C_2 > 0$ always, so that Equation 58 becomes

$$\tau_{th} = \frac{2\pi}{\sqrt{C_2}} \quad (63)$$

where

$$C_{2,th} = \lambda_I \left[\lambda_X + \sigma_X \bar{\phi}_1 + \left(\lambda_I + \lambda_X + \sigma_X \bar{\phi}_1 \right) (1 - \bar{x}_1) / (\bar{x}_1 - \bar{y}_X) \right] \quad (64)$$

[†] In such a set, the equations are not all independent, and Cramer's rule says only that if a non-trivial solution exists, the determinant of the coefficients must be zero; it is powerless to find that solution.

which results from substituting $\alpha_T \bar{\phi}_1 - \mu_1^2$ from Equation 34 into Equation 33. Clearly $C_{2,th}$ in Equation 64 depends only on the physical constants λ_I , λ_X , $\sigma_X^\dagger$, and $\bar{\gamma}_X$ (recall Equation 35 defining $\bar{x}_1$), and not on the reactor properties H (or R), M^2 , η , α_T , or α_X . Hence the oscillation period at threshold, plotted in Figure 9 as a function of threshold flux, is independent of reactor properties. Figure 9 thus applies for any degree of flatness, and for both axial and radial oscillations.

[†] Actually τ_{th} depends on the *product* $\sigma_X \bar{\phi}_{1,th}$; recalling that $f(\bar{\phi}_1)$ also varies with σ_X only through the product $\sigma_X \bar{\phi}_{1,th}$, where $\bar{\phi}_{1,th}$ is determined by solving $f(\bar{\phi}_1) = -M^2 \mu_1^2 / \alpha_X$ (iteratively corrected, if $\alpha_T \neq 0$, by adding $\alpha_T \bar{\phi}_1 / \alpha_X$ to $-M^2 \mu_1^2 / \alpha_X$, Equation 36), it is clear that $\bar{\phi}_{1,th}$ varies inversely with σ_X . Consequently τ_{th} is independent of σ_X itself. Since τ_{th} is plotted against $\bar{\phi}_{1,th}$ in Figure 9 for the particular value $\sigma_X = 3.08 \times 10^{-18} \text{ cm}^2$, rather than against $\sigma_X \bar{\phi}_{1,th}$, that curve will be shifted up or down if some other value $\hat{\sigma}_X = \sigma_X$ smaller or larger than $3.08 \times 10^{-18} \text{ cm}^2$ is specified. Figure 9 may be used, however, to estimate τ_{th} when a threshold flux $\hat{\phi}_{th}$ has been calculated using this other cross section $\hat{\sigma}_X$ (and the same values of λ_I , λ_X , γ_I , and γ_X), by computing $\bar{\phi}_{1,th} = \hat{\sigma}_X \hat{\phi}_{th} / 3.08 \times 10^{-18}$ and then reading off the corresponding value of τ_{th} .

FIGURE 9 Oscillation Periods Versus Threshold $\bar{\phi}_1$

APPROXIMATIONS

Equilibrium Xenon

$\bar{x}_1$ is calculated by Equation 35, whereas its true value is defined in Equation 28. For a sine fundamental, this true value is

$$\bar{x}_1 = \int_0^H x^* g_1^2 dV = \int_0^H \left[\frac{1}{1 + \frac{\lambda_x}{\sigma_x \phi^*}} \right] \left[\sqrt{\frac{2}{H}} \sin \frac{2\pi z}{H} \right]^2 dz$$

Inserting $\phi^* = A^* \sin \pi z/H$ and letting $\epsilon = \pi z/H$ reduces this to

$$\bar{x}_1 = \frac{2}{\pi} \int_0^\pi \left[\frac{\sin \epsilon}{\sin \epsilon + (\lambda_x / \sigma_x A^*)} \right] [\sin 2\epsilon]^2 d\epsilon$$

Numerical integration followed by substitution in $f(\bar{\phi}_1) = (\bar{x}_1 - \bar{y}_x) / [1 + (\lambda_I + \lambda_x) / \sigma_x \bar{\phi}_1]$,[†] over a range of values of $\bar{\phi}_1$, yields the dashed curve of Figure 1. Corresponding values of $\bar{\phi}_1$ itself are 0.7% low at $\bar{\phi}_1 = 10^{12}$ and 4.0% low at $\bar{\phi}_1 = 10^{14}$.

The error vanishes when the fundamental flux is flat, in which case the solid curve shown in Figure 1, calculated by Equation 37, is exact. For flat flux,

$$\bar{x}_1 = \int x^* g_1^2 dV = x^* \int g_1^2 dV = x^*$$

and $\bar{\phi}_1 = \bar{\phi} = \phi^* = A^* = \text{constant}$, so that the use of Equation 35 is precisely correct in this limit.

Truncation

Only the first terms are used in the expansions of ϕ'' and x'' , Equations 22 and 23, so that the cross terms are necessarily neglected ($\bar{\phi}_{ij}$ and $\bar{x}_{ij}$ do not exist for $i = j = 1$). Just as for the equilibrium xenon approximation, neglecting cross terms introduces no error for a flat fundamental ($\bar{\phi}_{ij} = \int \phi^* g_i g_j dV = \phi^* \int g_i g_j dV = 0$ and $\bar{x}_{ij} = \int x^* g_i g_j dV = x^* \int g_i g_j dV = 0$ when ϕ^* , and hence x^* , is not a function of position).

[†] As defined by Equation 34.

The effect of these two approximations was investigated for a reactor having a sinusoidal flux distribution by carrying two terms in Equations 22 and 23, evaluating the cross terms by numerical integration, determining the $\bar{x}_j$ by numerical integration rather than by $1/(1+\lambda_x/\sigma_x\bar{\phi}_j)$, solving the resulting fourth-degree equation in ω for several values of $\bar{\phi}_1$, and interpolating to $\text{Re}\omega = 0$ to obtain the threshold flux $\bar{\phi} = \frac{15}{16} [\bar{\phi}_1]_{\text{Re}\omega = 0}$. Table I compares threshold fluxes and periods for two typical axial cases. Clearly the combined effect of these approximations is not great, and a single term in the expansions for ϕ'' and x'' is adequate for the calculation of the first-harmonic threshold.

TABLE I

Axial Thresholds for Sine Fundamental Using One versus Two Expansion Terms

$$\alpha_x = 0.03682, \alpha_T = 0$$

H^2/M^2	Threshold Flux, $n/(\text{cm}^2 \text{ sec})$		Oscillation Period at Threshold, hr	
	One Term	Two Terms	One Term	Two Terms
1250	5.3×10^{13}	5.0×10^{13}	22.0	22.1
2500	1.60×10^{13}	1.52×10^{13}	29.9	29.4

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APPENDIX: REACTOR-GEOMETRY EIGENFUNCTION EXPANSION

The computation is considerably simpler if the perturbation amplitudes are expanded in eigenfunctions of the reactor geometry rather than in eigenfunctions of the unperturbed flux shape. Only the axial case is treated here, for illustrative purposes.

In this case, Equation 22, 23, and 24 are written as

$$\phi'' = \sum_i A_i' f_i \quad (A-1)$$

$$x'' = \sum_i X_i' f_i \quad (A-2)$$

$$\nabla^2 f_i + \kappa_i^2 f_i = 0 \quad (A-3)$$

where, in the axial direction,

$$\kappa_i^2 = (i+1)^2 \left(\frac{\pi}{H} \right)^2 \quad (A-4)$$

$$f_i = \sqrt{\frac{2}{H}} \sin (i+1) \frac{\pi z}{H} = \sqrt{\frac{2}{H}} \sin \kappa_i z \quad (A-5)$$

so that
$$\int f_i f_j dV = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}$$

Substituting Equations A-1 and A-2 into Equation 19 yields

$$\sum_i \nabla^2 A_i' f_i + (B^2 + a_T \phi^*) \sum_i A_i' f_i + a_X \phi^* \sum_i X_i' f_i = 0 \quad (A-6)$$

so that by Equation A-3,

$$\sum_i \left[-\kappa_i^2 A_i' f_i + (B^2 + a_T \phi^*) A_i' f_i + a_X \phi^* X_i' f_i \right] = 0 \quad (A-7)$$

Substituting Equations A-1 and A-2 into Equation 25 gives

$$\sum_i \left[\left(\sigma_X \bar{Y}_X + \frac{\lambda_I \bar{Y}_I \sigma_X}{\omega + \lambda_I} - \sigma_X x^* \right) A_i' f_i - (\omega + \lambda_X + \sigma_X \phi^*) X_i' f_i \right] = 0 \quad (A-8)$$

As before, the unknown coefficients A_i' and X_i' may be determined (within an arbitrary normalizing constant) by multiplying Equations A-7 and A-8 by f_j and integrating over the reactor volume:

$$\sum_i \left[\int (B^{2*} + a_T \phi^* - \kappa_i^2) A_i' f_i f_j dV + \int a_x \phi^* X_i' f_i f_j dV \right] = 0$$

$$\sum_i \left[\int \left(\sigma_x \bar{Y}_x + \frac{\lambda_I \bar{Y}_I \sigma_x}{\omega + \lambda_I} - \sigma_x x^* \right) A_i' f_i f_j dV - \int (\omega + \lambda_x + \sigma_x \phi^*) X_i' f_i f_j dV \right] = 0$$

which become

$$\begin{aligned} A_j' \left[\int B^{2*} f_j^2 dV + a_T \int \phi^* f_j^2 dV - \kappa_j^2 \int f_j^2 dV \right] + \sum_{i \neq j} A_i' \left[\int B^{2*} f_i f_j dV \right. \\ \left. + a_T \int \phi^* f_i f_j dV - \kappa_i^2 \int f_i f_j dV \right] + X_j' a_x \int \phi^* f_j^2 dV \\ + \sum_{i \neq j} a_x \left[X_i' \int \phi^* f_i f_j dV \right] = 0 \end{aligned}$$

and

$$\begin{aligned} A_j' \left[\left(\sigma_x \bar{Y}_x + \frac{\lambda_I \bar{Y}_I \sigma_x}{\omega + \lambda_I} \right) \int f_j^2 dV - \sigma_x \int x^* f_j^2 dV \right] \\ + \sum_{i \neq j} A_i' \left[\left(\sigma_x \bar{Y}_x + \frac{\lambda_I \bar{Y}_I \sigma_x}{\omega + \lambda_I} \right) \int f_i f_j dV - \sum_{i \neq j} A_i' \sigma_x \int x^* f_i f_j dV \right. \\ \left. - X_j' (\omega + \lambda_x) \int f_j^2 dV - X_j' \sigma_x \int \phi^* f_j^2 dV - \sum_{i \neq j} X_i' (\omega + \lambda_x) \int f_i f_j dV \right. \\ \left. - \sum_{i \neq j} X_i' \sigma_x \int \phi^* f_i f_j dV \right] = 0 \end{aligned}$$

Recalling the orthonormality of the f_i and defining

$$\begin{aligned}\int \phi^* f_j^2 dV &= \bar{\phi}'_j & \int \phi^* f_i f_j dV &= \bar{\phi}'_{ij} \\ \int x^* f_j^2 dV &= \bar{x}'_j & \int x^* f_i f_j dV &= \bar{x}'_{ij} \\ \int B^{2*} f_j^2 dV &= \bar{B}^2_j & \int B^{2*} f_i f_j dV &= \bar{B}^2_{ij}\end{aligned}$$

these two sets of equations are reduced to

$$\begin{aligned}& \left[a_T \bar{\phi}'_j + \bar{B}^2_j - \kappa_j^2 \right] A'_j + \left[a_x \bar{\phi}'_j \right] X'_j \\ & + \sum_{i \neq j} \left[(\bar{B}^2_{ij} + a_T \bar{\phi}'_{ij}) A'_i + a_x \bar{\phi}'_{ij} X'_i \right] = 0\end{aligned}\tag{A-9}$$

$$\begin{aligned}& \left[\sigma_x \bar{x}'_j - \sigma_x \bar{y}_x - \frac{\lambda_I \bar{y}_I \sigma_x}{\omega + \lambda_I} \right] A'_j + \left[\omega + \lambda_x + \sigma_x \bar{\phi}'_j \right] X'_j \\ & + \sum_{i \neq j} \left[\sigma_x \bar{x}'_{ij} A'_i + \sigma_x \bar{\phi}'_{ij} X'_i \right] = 0\end{aligned}\tag{A-10}$$

As with Equations 28 and 29, the requirement that the determinant of the coefficients be zero determines possible values of ω . If the fundamental flux shape does not differ too much from the fundamental geometric mode,[†] cross terms can be neglected and only one term carried in Equations A-1 and A-2, yielding

$$\omega^2 + C'_1 \omega + C'_2 = 0\tag{A-11}$$

$$C'_1 = \lambda_I + \lambda_x + \sigma_x \bar{\phi}'_1 - \frac{a_x \sigma_x \bar{\phi}'_1 (\bar{x}'_1 - \bar{y}_x)}{a_T \bar{\phi}'_1 + \bar{B}^2_1 - \kappa_1^2}\tag{A-12}$$

[†] If the fundamental flux shape is equal to the fundamental geometric mode, Equations A-11, A-12, and A-13 are identical to Equations 31, 32, and 33. However, for a flat flux shape, B^{2*} is large near the edge of the reactor and small near the center, so that the cross terms $\bar{B}^2_{ij}$ become increasingly important. More terms in the expansion of the perturbation shape must be carried with Equations A-1 and A-2 than with Equations 22 and 23 for similar accuracy.

$$C_2' = \lambda_I \left[\lambda_x + \sigma_x \bar{\phi}_1' + \frac{a_x \sigma_x \bar{\phi}_1' (1 - \bar{x}_1)}{a_T \bar{\phi}_1' + \bar{B}_1^2 - \kappa_1^2} \right] \quad (A-13)$$

analogous to Equations 31, 32, and 33. At the oscillation threshold, $C_1' = 0$,

$$\kappa_1^2 - \bar{B}_1^2 = a_T \bar{\phi}_1' - \frac{a_x (\bar{x}_1' - \bar{\gamma}_x)}{1 + (\lambda_I + \lambda_x) / \sigma_x \bar{\phi}_1'} \equiv a_T \bar{\phi}_1' - a_x f(\bar{\phi}_1') \quad (A-14)$$

corresponding to Equation 34. Defining $\Lambda_j' = (\kappa_j^2 - \bar{B}_j^2) / B_g^2$,

$$f(\bar{\phi}_1') = - \frac{M^2 B_g^2}{\alpha_x} \Lambda_1' + \frac{\alpha_T}{\alpha_x} \bar{\phi}_1' \quad (A-15)$$

like Equation 36.

Numerical values of the axial geometric-mode eigenvalues are necessary in order to compute the oscillation threshold flux by Equation A-15. For the first harmonic, by virtue of Equation A-4,

$$\Lambda_1' = \frac{\kappa_1^2 - \bar{B}_1^2}{B_g^2} = \frac{(2\pi/H)^2 - \bar{B}_1^2}{(\pi/H)^2} = 4 - \frac{\bar{B}_1^2}{(\pi/H)^2} \quad (A-16)$$

where $\bar{B}_1^2 = \int B^{2*} f_1^2 dz / \int f_1^2 dz = \int B^{2*} f_1^2 dz$ in consequence of the orthonormality of the perturbation flux eigenfunctions. Assuming the same axial flux shapes as earlier (Figure 2),

$$B^{2*} = (\pi/a)^2 \text{ in Region I}$$

$$B^{2*} = 0 \text{ in Region II}$$

then

$$\begin{aligned} \bar{B}_1^2 &= 2 \int_0^{a/2} \left(\frac{\pi}{a} \right)^2 \left(\sqrt{\frac{2}{H}} \sin \frac{2\pi z}{H} \right)^2 dz + 2 \int_{a/2}^{H/2} (0) \left(\sqrt{\frac{2}{H}} \sin \frac{2\pi z}{H} \right)^2 dz \\ &= \frac{4}{H} \left(\frac{\pi}{a} \right)^2 \int_0^{a/2} \left(\sin \frac{2\pi z}{H} \right)^2 dz \\ &= \frac{4}{H} \left(\frac{\pi}{a} \right)^2 \left(\frac{a}{4} - \frac{\sin 2\pi a/H}{8\pi/H} \right) \end{aligned}$$

As before, $\eta = a/H =$ the fraction of the reactor length over which the unperturbed flux is sinusoidal, with the result that

$$\bar{B}_1^2 = \left(\frac{\pi}{H}\right)^2 \left(\frac{1}{\eta} - \frac{\sin 2\pi\eta}{2\pi\eta^2}\right) \quad (A-17)$$

For a pure sine fundamental, $\eta = 1$, so that

$$\bar{B}_1^2 = \left(\frac{\pi}{H}\right)^2 \quad \text{by Equation A-17}$$

$$\therefore \Lambda_1' = 3 \quad \text{by Equation A-16}$$

For a perfectly flat fundamental, $\eta = 0$, and Equation A-17 becomes indeterminate. Rewriting it as

$$\bar{B}_1^2 = \left(\frac{\pi}{H}\right)^2 \left(\frac{2\pi\eta^2 - \eta \sin 2\pi\eta}{2\pi\eta^3}\right)$$

and applying L'Hospital's rule three times leads to

$$\lim_{\eta \rightarrow 0} \bar{B}_1^2 = \left(\frac{\pi}{H}\right)^2 \left(\frac{8\pi^3 \eta \cos 2\pi\eta}{12\pi}\right) = 0$$

$$\therefore \Lambda_1' = 4 \quad \text{by Equation A-16}$$

Values of Λ_1' computed by Equation A-16 and A-17 are plotted, together with Λ_1 , in Figure 3.

Λ_1 and Λ_1' are not very different for $0.6 < \eta < 1.0$, so that the geometric-mode expansion is valid provided the unperturbed flux is no more than about 40% centrally flattened. For a sine fundamental, $B^{2*} = (\pi/H)^2 =$ constant, the added buckling required to excite the first harmonic is $3(\pi/H)^2$, and f_1 is identical with g_1 . But as the fundamental is progressively flattened, the geometric-mode approach breaks down: f_1 becomes an increasingly poor harmonic of the fundamental ϕ^* , because it is a function of reactor geometry only and does not change shape as the fundamental gets flatter. Thus if ϕ' is to be represented by a geometric-mode expansion, more and more terms must be employed; clearly when $\eta = 0$, ϕ' cannot be represented by a sine series with a small number of terms.

Λ and Λ_1' refer to the same physical quantity, namely, the ratio of the added buckling required to excite the first harmonic to the geometric buckling. They are calculated differently because a continuous function is available for the geometric-mode perturbation eigenfunction (Equation A-5), so that its buckling can be calculated directly (Equation A-4); no such single continuous function is available over the whole reactor for the flux-shape-mode perturbation eigenfunction (except of course when $\eta = 1$, in which case $g_1 = f_1$ and $\Lambda_1 = \Lambda_1'$ identically).

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