Validity of Spectral Distribution Methods in Spectroscopy*

F. S. Chang

Department of Physics and Astronomy

University of Rochester

Rochester, New York

and

A. Zuker

CEN Saclay, Service de Physique Theorique

BR No°2, 91-Gif Sur Yvette, France

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Abstract:

The accuracy with which various spectral distributions reproduce the energies in many-particle spectra has been studied by comparing the results with those from an exact (ds)-shell calculation of dimensionality ranging up to 1200. Also investigated has been the splitting of the deviations, between the distribution and exact spectra, into a fluctuating part and a part which varies slowly with energy, as well as the accuracy with which the distribution method reproduces the orbit occupancies over the entire range of excitation energies as well as the level density and spin-cut-off factor. Agreement in all cases is good.

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* Supported in part by the U. S. Atomic Energy Commission.

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I Introduction

Since the spectral distribution method was introduced a few years ago, there has been a wide variety of applications and results.⁽¹⁻⁵⁾ The problem of the accuracy of the method and its applicability to answering further kinds of questions has become increasingly important. In this paper we compare results for various kinds of distributions with those of large (ds)-shell calculations which have been made for other purposes.⁽⁶⁾ We examine the accuracy with which the energies of the ground state and low-lying states can be determined by means of distributions, as well as level density and orbital occupancies, and we study as well the nature of the fluctuations in the exact eigenvalues in order to determine the irreducible errors associated with the distributions.

II The Calculations

The Shell-model calculations are in the (ds)-shell with renormalized and partially renormalized versions of the interaction of Kahana, Lee, and Scott,⁽⁷⁾ which is one type of "realistic" effective interaction. Two sets of single particle energies have been used. (a) $\epsilon_{5/2}=0$, $\epsilon_{1/2}=0.87$, $\epsilon_{3/2}=5.10$ MeV and (b) $\epsilon_{5/2}=0$, $\epsilon_{1/2}=0.87$, $\epsilon_{3/2}=7.00$ MeV. The exact calculation has used the Oak Ridge-Rochester Shell-Model code.⁽⁸⁾ The accuracy of matrix construction and diagonalization for dimensionality up to 1200 has been studied and found to be satisfactory.

In the spectral distribution method one considers the distribution throughout the Hamiltonian eigenstates of the intensity of a set of basis states defined by certain exact symmetries. The distribution is found for the many-particle cases to be close to Gaussian. The essential requirement

of the method is that we should be able to calculate the low-order cumulants (centroid and width, as well as the "skewness" and "excess" for correcting the Gaussian) for the chosen symmetries in a given space. We need also a reasonable way to construct approximate eigenvalues which correspond to the continuous distribution so produced.

We have considered a decomposition of the overall spectroscopic space (that defined by (ds) in the present case) in terms of particle number (m) (which we call "scalar"), spherical-orbit configuration ($\vec{m}$), particle number and isospin (m,T), configuration and isospin ($\vec{m},T$), particle-number, angular momentum and isospin (m,J,T), and configuration, angular-momentum and isospin ($\vec{m},J,T$). The first four of these subspaces can be described in terms of irreducible representation of a single group, and for these cases there exists⁽³⁾ a group-theoretical decomposition method for obtaining the centroid and width. On the other hand the last two subspaces cannot be described in this way since there enters for these an intermediate chain of groups. To directly calculate the cumulants from the matrix elements is then much more complicated, and a really satisfactory working computer program is not yet available. For the present purposes however we can obtain them directly from the eigenvalues and eigenfunctions.

A Gaussian (two-moment) distribution is uniquely defined, but for a general finite set of more than two moments there is no unique way of forming a distribution. We make use of the Gram-Charlier frequency function^(2,9)

$$x = \frac{E - \bar{E}}{\sigma}$$

$$f(E) = \frac{D}{\sqrt{2\pi}\sigma} \exp(-\frac{1}{2} x^2) [1 + \frac{\gamma_1}{3!} (x^3 - 3x) + \frac{\gamma_2}{4!} (x^4 - 6x^2 + 3) + \dots]$$

where

$d \equiv$ dimensionality

$\bar{E} \equiv$ centroid energy

$\mu_p \equiv p^{\text{th}}$ central moment

$\sigma = \mu_2^{1/2} \equiv$ width

$\gamma_1 = \mu_3/\sigma^3 \equiv$ skewness

$\gamma_2 = (\mu_4/\sigma^4) - 3 \equiv$ excess

There is also no unique way to reproduce a discrete spectrum from the continuous distribution. In this paper we adopt the method first used by Ratcliff.⁽²⁾ If the i^{th} eigenstate has degeneracy D_i then the distribution eigenvalue E_i is defined by

$$\int_{-\infty}^{E_i} r(E) dE = \sum_{j=1}^{i-1} D_j + \frac{1}{2} D_i$$

Since the exact results are given in the (m, J, T) -space, we first discuss the distribution in that space and leave the others until later. In Table 1 we first give the dimension, centroid energy, width, skewness and excess for all 16 cases for which the exact results are available. First we notice that in every case $|\gamma_1|, |\gamma_2| \leq 0.4$. These small values define the distributions as basically Gaussian. Note that the skewness is always positive, tending to reduce the density at low energies and to increase it at high energies. Consequently it will lift up the ground state and give an asymmetrical distribution. The values of γ_1 and γ_2 are such that the skewness provides a more important correction than does the excess. We shall later give evidence that higher cumulants have little effect on the overall form of the distributions, affecting only the fluctuations which

are small.

III Ground State Energies (m,J,T)

The most crucial test for the accuracy of a distribution is the calculation of the ground state energy, for it is here that the largest error occurs. In the last three columns of Table 1 we list the difference between the exact and distribution ground state energies. In most of the cases the two-moment (Gaussian) distribution produces too low an energy, it is often over corrected by a positive skewness, and then is lowered again by the excess. When we use the four-moment distributions, the difference is always less than 3 MeV, and this value appears to be fairly independent of dimensionality for the range considered. Both these findings are extremely encouraging. Measuring the ground state energy differences as a percentage of spectrum span, the value of which is also given in the table, the error is always less than 3% when we have enough particles (the spectrum tends to be Gaussian when there are many particles, so it is not surprising that the error is, in one case, larger for $(ds)^4$). This is also pleasing, for the spectrum span does not increase very rapidly with dimensionality. In our sixteen cases the dimensionalities increase by a factor of 20, but the spectrum span only by a factor 2. It appears from other calculations that the spectrum span increases even more slowly in more complicated cases; for example⁽¹⁰⁾ in Cu^{63} four-shell (fp) calculations the total dimensionality for $T=5/2$ is 5×10^9 and the spectrum span is 108 MeV. When we open up the $g_{9/2}$ shell, allowing $(g_{9/2})^2$, the dimensionality becomes 10^{14} ; but the spectrum span

increases only to 160 MeV. We cannot really discuss these extremely large cases here (their application is in the level-density domain) but the results from the present cases, as well as the larger ones, give us reason to believe that we shall get good accuracy even in enormous spaces, provided that they are appropriately partitioned.

IV Low-lying States

In a few cases in Table 1, the corrections are such that the ground-state energy error for the four-moment distribution is larger than that of the two-moment distribution. But when we consider the excited states and also the average deviations over the entire spectrum, we see that the errors decrease markedly as we take account of three-moment and four-moment corrections. Figure 1 shows for three cases the exact low-lying spectrum compared with the two, three, four-moment distribution results. The cases represent typical even-odd, odd-odd, and even-even nuclei, and all three cases have approximately the same dimensionality. The first thing we notice is that the high-moment corrections indeed improve the low-lying spectrum. The error for the first excited states reduces from -2.9 MeV to 0.1 MeV for the even-odd case, from -4.4 MeV to -0.6 MeV for the odd-odd, and from -1.6 MeV to -1.0 MeV for the even-even case. The errors for higher states are smaller to begin with and improve even more significantly. It is clear from the figure that the accuracy depends on the nature of the spectrum, and in fact this dependence is much stronger than the dependence on the dimensionality. The distribution reproduces the even-odd and the odd-odd spectrum much better than the even-even one. It has special difficulty with the $J=T=0$ cases, since in such a case the

ground states are strongly depressed, and this causes an unusually large change of densities between the first two or three states and the higher ones. We also notice that the error for excited states is almost always smaller than that for the ground state, and decreases rapidly as we go higher. For all 16 cases, when we use the four-moment distribution, the error of any states higher than the second excited state is always smaller than 1 MeV compared with the 3 MeV error for the ground state. The fact that we can calculate low-lying excited-state energies with high accuracy will prove significant in applications to studies of the level density. Instead of using the ground state for a reference we can always use a more accurately fixed low-lying excited state.

Consider now the spacing between different J spectra. We have only one case in the (ds)-shell, that involving the $Mg^{24} 0^+ - 2^+$ spacing. The experimental result is +1.4 MeV, the exact calculation gives +1.8 MeV. The results for the 2,3,4-moment distributions are -2.4, -2.0, +1.3 MeV respectively. Comparing this result with that of Hartree-Fock theory, we observe that the absolute H-F error, which is in the order of 10 MeV, is far larger than the distribution-theory error; but since the 0^+ and 2^+ H.F. states both derive from the same intrinsic state, we get a good $0^+ - 2^+$ H.F. spacing. The $0^+ - 2^+$ distribution states however arise from separate calculations, each with its own errors and fluctuations; and it is unclear whether, even with a four-moment distribution, we should expect a good spacing result for the even-even nuclei. The Mg^{24} result may then be better than we have a right to expect and may not be typical of the ($J=T=0$) nuclei. For other nuclei, odd nuclei in particular, we

would be more optimistic a priori.*

V Deviations and Fluctuations

Moving from the low-lying spectrum to a consideration of the overall deviations will give us a better understanding of the quality of the distribution theory. We can measure the deviation in an absolute energy unit, $\Delta_i = E_i(\text{distribution}) - E_i(\text{exact})$ or in the unit defined by the local spacing $\delta_i = \Delta_i / (\text{local spacing}) = \Delta_i \times \rho(E_i)$ where $\rho(E_i)$ is the density at energy E_i . Table 2 shows the mean square values of both Δ_i and δ_i , the deviations being taken with respect to the 2,3 and 4-moment distributions. First we notice that the skewness always make a significant improvement usually more than 90% of the total correction, while the excess gives only a minor one. The mean square values of Δ_i slowly decrease when the dimensionality increase. The mean square value of δ_i for the two-moment distribution increases with the dimensionality, for the three and four moment distribution, they fluctuate instead in the neighborhood of one spacing unit.

* Complete three-shell calculations are available⁽¹¹⁾ for Cu^{63} , the dimensionalities ranging up to 273. The four-moment distribution in this case does give the proper ordering for the lowest state of each J and in fact for the first excited state as well. The deviations between corresponding distribution and exact energies are of the order of one half the fixed-J spacing. The two-moment distribution gives larger deviations and some level cross-overs.

When we try to determine the ground-state energy or that of some low-lying state as the zero point for a level-density calculation, the only deviation we are interested in is the absolute deviation, which decreases with the excitation energy, and reaches its smallest value near the centroid. It is however better not to argue that the distribution theory produces a better spectrum near the centroid than near the ground state. Instead one should realize that, as is well known in the statistical theory of spectra, the local spacing unit, which is strongly energy dependent, is the natural one to use, the δ_1 then supplying the proper measure for studying the overall deviations and fluctuations. The deviations can be classified as secular deviations and as fluctuations (noise). The former, which vary slowly from level to level, can be eliminated by adopting a different smooth distribution (taking a few more moments for example) while the latter cannot be so corrected and give then a good measure of the accuracy which can be achieved. The deviations which show a random sign behavior and locally average to zero may be recognized as fluctuations. Figure 2 shows for a typical case the deviations δ_1 as a function of excitation energy. The two-moment distribution has a simple structure. For the first 27 MeV (160 states) they are all negative, (the distribution producing lower energies than the exact ones); for the next 27 MeV (820 states) the deviations are all positive; and for the last 30 MeV (220 states) negative again. The values are in the range of -7 to +13 units. We see immediately that a strong skewness correction is needed. When we take account of the skewness (γ_1) the deviations show a much more complicated structure, with values ranging from -3 to

+3 units. With the excess (γ_2) correction, as we can see in Fig. 2, the deviations are seen to be close to fluctuations, with values in the range of -1.8 to +1.8 spacing units. We should notice here that the deviations are actually cumulative, in other words we accumulate all the deviations starting from ground state up. The fact then that throughout the whole range of energy the cumulative deviations have approximately the same value, one to two spacing units for the four-moment distributions, indicates that these deviations are quite close to fluctuations. Any appreciable secular deviations would give rise to a strong departure from this pattern.

To test this we can proceed a little differently and ignoring the representation of the densities by means of moments, use the locally averaged density itself to define the approximate spectra and hence the fluctuations, this procedure has been used by Wong and French⁽¹²⁾ in a more general study of level density fluctuations. By doing this for the $(ds)^8_{J=2, T=0}$ case, we have found that the residue deviations are in the range (-1 to +1) units, with mean square value 0.75. This is in general agreement with results of Dyson and Mehta⁽¹³⁾ for the random ensembles. For the same case, the four-moment distributions have deviations in the range (-1.8 to +1.8) units, with mean square value 1.04 units, similar comparison emerges for other cases. We can thus conclude that the four-moment deviations do have a secular component but are mostly fluctuations. Consequently further corrections with higher cumulants will in most cases not provide significant improvement to the average behavior.

VI $(\vec{m}, J, T)$ Distributions

Usually, we can improve a Gaussian distribution by using higher cumulants or finer averages; each will put more structure into the distribution, and the second modification in particular will make it possible to study further quantities such as orbital occupancies. We therefore consider the difference between the results for the (m, J, T) distribution and its refinement $(\vec{m}, J, T)$. We use the $(ds)^{12}$ ($J=0$, $T=0$, $d=839$) case as an example. The $(\vec{m}, J, T)$ cumulants are calculated from eigenvalues and eigenfunctions (they could equally well be derived from matrices). Figure 3 compares the low-lying spectra produced by these distributions and by the exact calculations if only two moments are taken into account. $(\vec{m}, J, T)$ produces a better spectrum than does (m, J, T) , except for the lowest three states; but the three- and four-moment spectra are almost identical. As shown on the last two lines of table 2, the two moment $(\vec{m}, J, T)$ gives considerably smaller deviations than does (m, J, T) . For the four-moment we get a small improvement with $(\vec{m}, J, T)$. The change is in the opposite direction for the three-moment case; this may indicate that the skewness corrections for the configurations give a scalar excess which is far removed from the true one.

The results are not really surprising, since to use a finer distribution is one way to provide higher-cumulant corrections. For example if we add the set of two-moment $(\vec{m}, J, T)$ distribution, the overall distribution has the same centroid and width as the (m, J, T) one but is not a Gaussian; it has a skewness, an excess, and higher cumulants. In general a distribution constructed by summing up a set of K -moment $(\vec{m}, J, T)$ distributions, will have the same lowest K cumulants as the K -moment (m, J, T) distribution, but, unlike (m, J, T) , it will have higher cumulants. In the case shown in Table 2, the skewness and excess of the combined two-moment $(\vec{m}, J, T)$ distribution are 0.22 and 0.14 respectively. Although these are different from the

exact values (0.14 and 0.11 respectively), they obviously represent an improvement over the two-moment (m, J, T) which takes these quantities to be zero. This explains why the two-moment $(\vec{m}, J, T)$ distribution is significantly better than the two-moment (m, J, T) . On the other hand, the only difference between four-moment $(\vec{m}, J, T)$ and (m, J, T) comes from cumulants of order higher than four. The fact that the $(\vec{m}, J, T)$ four-moment distribution is not much better than the four-moment (m, J, T) is in a sense equivalent to saying that the cumulants higher than the fourth provide only minor corrections. This reinforces the conclusion arrived at in the last section.

With the $(\vec{m}, J, T)$ distribution we can very easily calculate the orbital occupancies as a function of the excitation energies. The method is discussed in Ref. 3. Figure 4 shows a comparison of the exact occupancies calculated from eigenvalues and eigenfunctions and those calculated by means of the distributions. The agreement is extremely good except for the few lowest and highest states, which indicates that matters more subtle than the energies are well looked after by the distribution theory. The study of orbital occupancies and related phenomena forms a large topic,^(3,4) which will be discussed in detail in a later paper by the first-named author.

VII Other Distributions

The (m, J, T) and $(\vec{m}, J, T)$ distributions used above are difficult to construct directly from the interactions. We therefore turn briefly to others whose construction is simple,⁽³⁾ namely two-moment (m) , (m, T) , $(\vec{m})$, $(\vec{m}, T)$ distributions. With these we can calculate approximate low-lying spectra as long as the (J, T) ordering of the level is known or

assumed (this is necessary since the values of the distribution function which fix the spectra depend on that ordering). Since in each case only certain J values have been treated in the shell-model calculations, we in fact do not know the ordering and must therefore, for comparison, deal with the ground state only. Table 3 shows how the ground-state energies for those simpler distributions compare with the more complex ones and with the exact values. One cannot, of course, expect the (m) and (m,T) distributions to produce good ground-state energies, since to locate a single state out of 10^5 or 10^6 states by means of two calculated quantities is certainly too much to ask. On the other hand in earlier papers^(3,4) it has been shown that the $(\vec{m})$ and $(\vec{m},T)$ distributions may be expected to give reasonable ground-state energies. The errors for those distributions are in fact only 4 MeV, and presumably would be even smaller if we could correct for the skewness and excess. As it is the errors are of about the same size as given by the (m,J,T) distribution.

Besides the ground-state energy, the $(\vec{m})$ and $(\vec{m},T)$ distributions enable us easily to calculate the orbital occupancies⁽³⁾ which in this case would of course be averaged over all J values. Except possibly near the extremes of the spectrum we would not in fact expect any large J -dependence of the occupancies, and this is well borne out by the comparison with the fixed- J and exact occupancies given in Fig. 4. They all agree remarkably well.

We can also calculate the spin-cut-off factor^(3,4) and then, besides the total state density, also the total level-density and the fixed- J level density^(3,4) as well. The method however does not calculate well the fixed- J centroids and so, in Fig. 5, which shows a typical result

compared with the exact density, we have simply put the centroids together. The resultant level-density curves agree so well for this case (and for many others with different (J,T) values and in the (fp) -shell) that we are quite confident of our ability to calculate accurately the spin-cut-off factor and level density. Since the (m,J,T) centroid, as opposed to the width, is relatively easy to calculate, we expect that the weakness of this method can be easily overcome.

VIII. Conclusions

All the evidence shows that the four-moment fixed- (J,T) distributions are remarkably accurate, the average deviation from the exact energies being of the order of one spacing unit, almost all of which can be regarded as fluctuations. We might thus expect to predict the ground-state energy in these (ds) -shell cases with a error of order 4 MeV, since the local spacing near the ground state is of this magnitude; in fact we do rather better than that with a 1-3 MeV error. We have given evidence moreover that the simple four-moment configuration distribution, with no angular-momentum specification, will also produce good ground-state energies. The distribution theory not only produces good spectra, but also the level density, spin-out-off factor, and the orbital occupancies with good accuracy. It is pleasing that it reproduces these quantities which are sensitive to the nature of the interactions.

We are deeply grateful to Prof. J. B. French for his suggestions and discussions and for reading the manuscript. We also wish to thank Prof. S. S. M. Wong for many useful discussions and for certain unpublished results.

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Table Captions

Table 1: Low order parameters and the spectrum span for (m,J,T) spaces, along with their corresponding ground state energy errors.

The energy unit is MeV. $\Delta E_g(k) = E_g(k\text{-moment distribution}) -$

$E_g(\text{exact})$. The numbers in brackets are the ground state energy errors measured as a percentage of the spectrum span.

(a)...(d) label the different interactions and single-particle energies, where (a) is renormalized Kahana, Lee and Scott in-

teraction with $\epsilon_{5/2} = 0$ $\epsilon_{1/2} = 0.87$ $\epsilon_{3/2} = 5.10$ MeV

(b) is same interaction with $\epsilon_{5/2} = 0$ $\epsilon_{1/2} = 0.87$, $\epsilon_{3/2} = 7.00$ MeV.

(c) is a partially renormalized Kahana, Lee and Scott interaction with the single-particle energies of (a)

(d) is same interaction as (c) with single-particle energies as (b).

Table 2: Mean square value of the absolute deviations, and of the deviations measured in units of the local spacing. All but last are for fixed (m,J,T) ; the last line gives results for $(\vec{m},J,T)$.

Table 3: $(ds)^{12}$ Ground-state energy from different distributions.

TABLE 1

		Dimen- sion	Cent- roid	Width	Skew- ness	Excess	Spectrum span	$\Delta E_g(2)$	$\Delta E_g(3)$	$\Delta E_g(4)$	
$(ds)^4$	$J=2, T=0$	(a)	56	-0.89	8.04	0.224	0.083	40.6	0.57 (1.4)	1.96 (4.8)	1.83 (4.5)
$(ds)^4$	$J=2, T=0$	(b)	56	1.91	9.50	0.362	0.073	46.6	-1.15 (2.1)	1.43 (3.1)	1.33 (2.8)
$(ds)^5$	$J=\frac{1}{2}, T=\frac{1}{2}$	(a)	109	-3.20	9.01	0.307	0.151	49.4	-2.25 (4.6)	0.54 (1.1)	0.10 (0.2)
$(ds)^5$	$J=\frac{1}{2}, T=\frac{1}{2}$	(b)	109	0.30	10.52	0.399	0.145	57.0	-3.89 (6.8)	0.27 (0.5)	-0.13 (0.2)
$(ds)^5$	$J=\frac{5}{2}, T=\frac{1}{2}$	(a)	223	-4.01	8.41	0.220	-0.086	46.9	-2.13 (4.6)	0.30 (6.4)	0.69 (1.5)
$(ds)^5$	$J=\frac{5}{2}, T=\frac{1}{2}$	(b)	223	-0.63	9.90	0.318	-0.079	54.9	-4.15 (7.6)	0.01 (0.0)	0.35 (0.6)
$(ds)^6$	$J=2, T=0$	(a)	307	-9.05	9.54	0.217	-0.156	51.2	-4.04 (7.9)	-1.01 (2.0)	-0.10 (0.2)
$(ds)^6$	$J=2, T=0$	(b)	307	-4.98	11.07	0.307	-0.159	58.0	-6.67 (11.5)	-1.64 (2.8)	-0.77 (1.3)
$(ds)^6$	$J=2, T=1$	(a)	525	-7.66	9.30	0.262	-0.038	55.1	-5.33 (9.7)	-1.02 (1.8)	-0.74 (1.3)
$(ds)^8$	$J=0, T=0$	(a)	325	-14.91	14.25	0.165	0.188	93.8	1.45 (1.5)	4.90 (5.2)	2.79 (2.9)
$(ds)^8$	$J=0, T=0$	(b)	325	-9.58	16.05	0.252	0.248	105.8	-1.10 (1.0)	4.99 (4.7)	1.93 (1.8)
$(ds)^8$	$J=2, T=0$	(a)	1206	-15.15	13.26	0.138	-0.059	85.4	-2.75 (3.2)	1.15 (1.3)	2.22 (2.6)
$(ds)^{12}$	$J=0, T=0$	(a)	839	-63.38	13.04	0.247	0.195	88.1	-1.69 (1.9)	4.92 (5.6)	1.77 (2.0)
$(ds)^{12}$	$J=0, T=0$	(b)	839	-37.78	14.87	0.308	0.182	68.2	-4.49 (6.6)	4.93 (7.2)	2.10 (3.0)
$(ds)^{12}$	$J=0, T=0$	(c)	839	-64.64	13.03	0.254	0.203	88.0	-1.95 (2.2)	4.84 (5.5)	1.59 (1.8)
$(ds)^{12}$	$J=0, T=0$	(d)	839	-50.28	15.07	0.139	0.114	103.7	0.50 (0.4)	4.49 (4.3)	2.39 (2.3)

TABLE 2

	Mean square value of absolute deviation, in MeV.			Mean square value of the deviation measured in units of the local spacing		
	2-moment	3-moment	4-moment	2-moment	3-moment	4-moment
$(ds)^4$ $J=2$ $T=0$ (a) 0.64	0.49	0.46	0.78	0.48	0.46	
$(ds)^4$ $J=2$ $T=0$ (b) 0.99	0.59	0.57	1.08	0.57	0.56	
$(ds)^5$ $J=\frac{1}{2}$ $T=\frac{1}{2}$ (a) 0.74	0.39	0.36	1.52	0.61	0.61	
$(ds)^5$ $J=\frac{1}{2}$ $T=\frac{1}{2}$ (b) 1.07	0.45	0.43	1.97	0.70	0.74	
$(ds)^5$ $J=\frac{5}{2}$ $T=\frac{1}{2}$ (a) 0.48	0.16	0.17	2.60	0.89	0.82	
$(ds)^5$ $J=\frac{5}{2}$ $T=\frac{1}{2}$ (b) 0.81	0.24	0.23	3.62	1.14	1.05	
$(ds)^6$ $J=2$ $T=0$ (a) 0.57	0.24	0.21	3.45	1.00	0.95	
$(ds)^6$ $J=2$ $T=0$ (b) 0.93	0.34	0.29	4.75	1.44	1.09	
$(ds)^6$ $J=2$ $T=1$ (a) 0.61	0.14	0.13	6.58	1.41	1.31	
$(ds)^8$ $J=0$ $T=0$ (a) 0.70	0.48	0.38	2.36	0.71	0.95	
$(ds)^8$ $J=0$ $T=0$ (b) 1.05	0.60	0.50	3.35	0.83	1.24	
$(ds)^8$ $J=2$ $T=0$ (a) 0.47	0.17	0.16	7.71	1.65	1.04	
$(ds)^{12}$ $J=0$ $T=0$ (a) 0.79	0.34	0.23	9.15	1.81	1.75	
$(ds)^{12}$ $J=0$ $T=0$ (b) 1.09	0.37	0.28	11.12	1.54	2.01	
$(ds)^{12}$ $J=0$ $T=0$ (c) 0.81	0.35	0.24	9.34	1.84	1.78	
$(ds)^{12}$ $J=0$ $T=0$ (d) 0.55	0.29	0.22	5.25	1.06	1.40	
$(ds)^{12}$ $J=0, T=0$ (d) 0.34	0.36	0.18	2.71	2.36	1.07	

TABLE 3

Distribution	(m)	(m,T)	($\vec{m}$)	($\vec{m},T$)	($\vec{m},J,T$) 2-moment	($\vec{m},J,T$) 3-moment	($\vec{m},J,T$) 4-moment	Exact
Ground State Energy	-115.9	-115	-109.0	-108.9	-106.0	-100.1	-103.3	-104.9
Difference from exact result	- 11.0	-10.1	- 4.1	- 4.0	- 1.95	+ 4.84	+ 1.58	0
Width (Typical for Configuration)	10.85	11.38	8.00	8.00	13.04	13.04	13.04	-
Ground state difference measured in the unit of width	- 1.01	- 0.89	- 0.5	- 0.5	- 0.15	+ 0.37	+ 0.12	-
Spectrum Span	106	96	91	90	88	88	88	88.01
Ground State difference measured in the unit of spectrum span	- 0.104	- 0.104	- 0.045	- 0.044	- 0.022	+ 0.055	+ 0.018	0
Dimension	2.704×10^6	2.265×10^5	2.704×10^6	2.265×10^5	839	839	839	-

Figure Captions

- Figure 1. A comparison of exact and 2,3,4-moment distribution spectra. The even-odd case is $(ds)^5$, $J=5/2$, $T=1/2$, $d=223$; the odd-odd case is $(ds)^6$, $J=2$, $T=0$, $d=307$; the even-even case is $(ds)^8$, $J=0$, $T=0$, $d=325$. Note the displacement of the scale for the even-even case.
- Figure 2. The δ deviation as a function of excitation energy for $(ds)^8$, $J=2$, $T=0$, $d=1206$. The continuous curves were produced by applying a 1-MeV square window to the discrete points. Over the 1-MeV interval the deviations are quite uniform, their dispersions being about unity.
- Figure 3. A comparison of the exact low-lying spectrum of $(ds)^{12}$, $J=0$, $T=0$, with the corresponding spectrum produced by (m, J, T) and $(\vec{m}, J, T)$ distributions with 2,3,4-moments.
- Figure 4. The percentage orbital occupancies for $(ds)^{12}$, $J=0$, $T=0$. The continuous curve is obtained from the $(\vec{m}, J, T)$ distributions and the dotted curve from the $(\vec{m}, T)$ distribution.
- Figure 5. Level density for $(ds)^{12}$, $J=0$, $T=0$. The curve was calculated from the $(\vec{m}, T)$ distribution and spin-cut-off factor, the points are obtained by counting the exact results in 1-MeV intervals.

Fig. 1

Fig.2

Fig. 3

Fig. 4

Fig. 5