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A STUDY OF INTERMEDIATE ANGLE π^+p ELASTIC
SCATTERING FROM 3.0 TO 5.0 GeV/c²

MASTER

BY

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TABLE OF CONTENTS

<u>Chapter</u>		<u>Page</u>
I.	INTRODUCTION.	1
II.	EQUIPMENT	1
	A. Beam Transport System	1
	B. Experimental Layout	3
	C. Fast Logic.	6
	D. Slow Logic.	9
III.	PROCEDURES.	11
	A. Experiment.	11
	B. Monitoring.	13
IV.	ANALYSIS OF DATA.	16
	A. Reconstruction of Events.	16
	B. Various Cuts and Constraints.	17
	C. Corrections to the Data	25
V.	RESULTS	30
VI.	THEORY.	42
	A. Optical Model	42
	B. Regge Exchange Model.	45
	C. The Effective Trajectory.	51
	REFERENCES.	60

LIST OF FIGURES

Figure	Page
1. Beam Transport System.	2
2. Experimental Layout (The numerical values are in inches) . . .	4
3. Slave Spark Gap Driver Circuit	7
4. Fast Logic (All type Numbers Refer to EG&G Logic Modules). . .	8
5. Slow Logic (Model Numbers Refer to Scientific Accessories Corporation).	10
6. Cerenkov Counter Pressure Curve at 5.0 GeV/c for Positive Beam.	12
7. Folded Two Body Kinematic Curve for π^+p Elastic Scattering at 3.5 GeV/c.	19
8. Detection Efficiency at 3.0 GeV/c with Counter F1 In Coincidence and In Anticoincidence	20
9. Coplanarity Distribution for all Reconstructable Events at 3.0 GeV/c for $\pi^+p \rightarrow \pi^+p$ Reaction.	23
10. DL Distribution for a Bin $1.090 \leq t \leq 1.190$ (GeV/c) ² at 3.0 GeV/c for $\pi^+p \rightarrow \pi^+p$ Reaction, Where DL = Measured Opening Angle ($\theta_1 + \theta_2$) - Kinematically Predicted Opening Angle ($\theta_1 + \theta_2$)	24
11. Differential Cross Sections for π^+p Elastic Scattering (mb/(GeV/c) ² vs. t)	40
12. Differential Cross Sections for π^-p Elastic Scattering (mb/(GeV/c) ² vs. t)	41
13. Optical Model Fit to π^+p Elastic Scattering at 3.0 GeV/c for all Angles.	46
14. Fitted and Extrapolated Curves for π^+p Polarization at 5.5 GeV/c from Cyclic Residue Model	49
15. π^+p Regge Exchange Model Fits with P, P' and ρ Poles	52
16. π^-p Regge Exchange Model Fits with P, P' and ρ Poles	53
17. π^+p Effective Trajectory vs. t as Determined from $d\sigma/dt = P(t) s^{2\alpha(t)-2}$	55

<u>Figure</u>	<u>Page</u>
18. π^-p Effective Trajectory vs. t as Determined from $d\sigma/dt = F(t) s^{2\alpha(t)-2}$	56
19. π^+p Effective Trajectory vs. t as Determined from $d\sigma/dt = (1/P_1^2) f(t) v^{2\alpha(t)}$	58
20. π^-p Effective Trajectory vs. t as Determined from $d\sigma/dt = (1/P_1^2) f(t) v^{2\alpha(t)}$	59

LIST OF TABLES

<u>Table</u>	<u>Page</u>
I. Number of Events with F1 In Coincidence and In Anticoincidence at Different Momentum	14
II. Number of Sparks in Each Wand.	15
III. Number of Tapes Taken Under Various Conditions	21
IV. Electron Content in the Beam	27
V. Accidentals.	29
VI. $\pi^+ p$ Differential Cross Sections.	31
VII. $\pi^- p$ Differential Cross Sections.	35

A Study of Intermediate Angle $\pi^\pm p$ Elastic
Scattering from 3.0 to 5.0 GeV/c

Han Won Paik

ABSTRACT

We present a study of a $\pi^\pm p$ elastic scattering experiment performed at the Argonne National Laboratory Zero Gradient Synchrotron. Both positive and negative pions at incident momenta of 3.0, 3.5, 4.0 and 5.0 GeV/c were incident on a liquid hydrogen target, and the scattered particles were detected by arrays of scintillation counters and wire spark chambers. The geometry of the experimental layout was such as to cover an angular range from 55° to 130° in the center of mass frame. Both $\pi^+ p$ and $\pi^- p$ angular distributions are characterized by the presence of a non-shrinking forward diffraction peak, a dip or shoulder near $t = -0.9$ (GeV/c) 2 and a deep minimum near $t = -3.0$ (GeV/c) 2 , all of which were previously observed in experiments at momenta above 2.5 GeV/c. In addition the data show the presence of a shoulder for $\pi^+ p$ and indicate another minimum for $\pi^- p$. Both occur approximately at a constant value of $\theta_{c.m.} = 125^\circ$. The experimental results can be described either using an optical model or in terms of a t -channel Regge exchange model.

I. INTRODUCTION

In recent years, as πp elastic scattering data have become available, various models have been proposed to explain their behavior. Diffraction and t -channel Regge pole exchange¹ have successfully described forward scattering, while backward scattering has been fit using direct channel resonances² or the exchange of Reggeized baryon trajectories³. The intermediate angular region can be explained in terms of resonances or in terms of a combination of t - and u -channel exchanges, including cuts at large values of both t and u . Recently a simple absorptive optical model for all scattering angles⁴ was proposed to explain the full angular distribution in πp data. In the intermediate angular region the $\pi^{\pm}p$ data⁵⁻⁸ existing before this experiment were rather sparse. When the data from this experiment are added to existing data, a more stringent test of this elastic scattering model is possible.

In this study we are concerned with the experimental aspect of the work, the characteristics of the data, and their theoretical implications.

II. EQUIPMENT

A. Beam Transport System

The experiment described here was carried out in secondary beam #1 of the Argonne National Laboratory Zero Gradient Synchrotron (ZGS). A schematic view of the beam transport system is shown in Figure 1 (page 2). Protons, accelerated by the ZGS to a momentum of 12.5 GeV/c were deflected from the external proton beam by the bending magnets XB5 and XB6 and struck

POSITION CODE	EQUIPMENT TYPE
XB5	Septum Bending Magnet 2 I 60
XB6	Piccioni Bending Magnet 3.5 I 39.5 (BM-101)
XBA11, XBA12, XBA13	Bending Magnets 18 VI 72 (BM-105)
XQA11, XQA12, XQA13, XQA14	Quad. Magnets 10 Q 36 (QM-102)
TA-1	Copper Target .320" x .320" x 4"
TG	Hydrogen Target
A	Collimator, 1st Focus

FIGURE 1

the copper target TAl to produce a secondary beam of particles with various momenta.⁹ The secondary beam was next focused by quadrupoles XQAl1 and XQAl2. To extract a beam of well defined momentum from this secondary beam, the bending magnet XBA11 and a lead collimator A were employed. The beam thus collimated was further deflected away from the main passage of the external proton beam by two more bending magnets XBA12 and XBA13 and was focused onto the liquid hydrogen target by quadrupoles XQAl3 and XQAl4. The momentum resolution of the incoming beam was approximately 1%, with an equal uncertainty in the absolute value of the central momentum.¹⁰ The ZGS had a beam repetition rate of 1 burst every 3.5 seconds and each beam spill lasted 600 ms. The positive beam intensity varied from 40,000 particles (20,000 pions) per burst at 3.0 GeV/c to 100,000 particles (20,000 pions) at 5.0 GeV/c. The negative beam intensity varied from 20,000 particles (18,000 pions) per burst at 3.0 GeV/c to 55,000 particles (50,000 pions) at 5.0 GeV/c.

A gas threshold Čerenkov counter (not shown in the figure) was placed between the collimator and the bending magnet XBA12 to distinguish positive pions from protons and kaons and negative pions from kaons.

B. Experimental Layout

The experimental layout is shown in Figure 2 (page 4). The incoming beam was detected by scintillation counters B₁ and B₂ which are 2" x 2" x $\frac{1}{8}$ " and 1" x 1" x $\frac{1}{8}$ " respectively. Anticounters A₁, A₂, A₃ and A₄ (A₃ and A₄ were placed at locations above and below the target and are not shown in Figure 2) covered much of the solid angle that was not subtended by the detectors. A₁ and A₂ (not shown in this figure) were located downstream

FIGURE 2

from the target to eliminate the large number of elastic and inelastic events with particles scattered into the forward direction. This allowed a more detailed study of large angle elastic events. Counter AH was an anticounter with a 1 inch diameter hole through which the beam passed. The outgoing particles were detected by one and only one of the F-arm counters (F_1 , F_2 and F_3) and one and only one of the S-arm counters (S_1 , S_2 and S_3). The trajectories of the incoming pion and of the elastically scattered pion and proton were each determined by a set of four wire spark chambers.

The geometrical arrangement of the counter arrays of the two final state arms was such as to cover angular intervals from about 5° to 43° in the F-arm and from 24° to 75° in the S-arm, which allowed data acquisition for center of mass scattering angles between 55° and 130° .

There were twelve chambers used in the experiment. Four small chambers (nominal active area of $8" \times 8"$) were mounted in the beam arm, and two medium chambers ($17" \times 23"$) and two large chambers ($28" \times 36"$) were mounted on both the F and S arms. Each chamber had two wire planes separated by a $\frac{1}{2}"$ fiberglass (G10, phenolic) frame. The wires (.005" diameter, beryllium copper) were wound parallel to each other with 1 mm spacing. Over the wire planes, wire mesh was placed on both sides of the large chambers to provide a uniform electric field over the entire active region of the chamber. Two magnetostrictive wands were attached to each chamber to pick up the x and y coordinates of sparks produced in the chamber. (For details on spark chambers and magnetostrictive readout systems see reference 11.) The trigger mechanism for the spark chambers consisted of a master spark gap driver and amplifier which upon receiving a trigger command from the fast logic provided a H.V. output to the slave gaps. A master gap signal caused slave gap break-

down which in turn applied a high voltage pulse to the two chambers connected to each slave gap. A circuit diagram of the slave gap driver is shown in Figure 3 (page 7).

C. Fast Logic

A block diagram of the fast logic is shown in Figure 4 (page 8). Signals from the various counters and anticounters are shown at the top of the diagram. The beam was defined by $CB_1B_2\overline{AH}$, that is, a coincidence between Cerenkov counter C and beam counters B_1 and B_2 and an anticoincidence with AH. The discriminator level of DI was set low to detect the small signal output of the phototube viewing the Cerenkov counter. This introduced spurious beam counts and to estimate these accidental coincidences the signal was deliberately mis-timed by 30 ns. with respect to $B_1B_2\overline{AH}$ and scaled in coincidence (C3) as $B_1B_2\overline{AH}[C]$. It was thus found that about 2% of the beam counts were accidental coincidences.

This fast logic is a modified version of the one which was designed for the large angle p-p elastic scattering experiment.¹¹ One of the modifications was the addition of a voter coincidence module to both the F-arm and the S-arm, which are followed by a NOR module. This modification was to ensure one and only one signal from each of the two final state detecting arms. Thus the trigger of the spark chambers was given by $CB_1B_2\overline{AH}FS\overline{A}$ where F and S stand for a signal from any one of the F-arm and S-arm counters and A stands for no signal from any of the anticounters $A_1 - A_4$. This coincidence was formed at module C6. Whenever an event satisfied the trigger logic, there resulted a trigger signal for the spark gap amplifier, a command to activate various elements in the slow logic system, and the gating off of the fast logic for the preset dead time of the system, which

SLAVE SPARK GAP DRIVER CIRCUIT

FIGURE 3

$\pi^\pm p$

FIGURE 4

was approximately 80 ms.

D. Slow Logic

The schematic of the slow logic system employed is given in Figure 5 (page 10). The trigger caused the accumulation of the quantity NP (the number of the ZGS pulse), the accumulation of the quantity M^1 (the number of accepted events), the accumulation of the quantity B^1 (the total number of beam particles), and the opening of the tagging gate so that the tagging unit could record the counters which detected the scattered particles. It also resulted in a command to the wire spark chamber master control. This pulse initiated the data acquisition and digitizer scanning cycles after a preset period of delay to avoid noise signals generated by the triggered spark chambers. Input signals from a total of twenty-four magnetostrictive readout preamplifiers were fed to the multiple time-digitizer.

Finally the tape transport was commanded to begin. Through the strobe lines the data input units, fixed data, tagging unit, and accumulator scalars were sequentially addressed and the data stored there were transferred to the data bus and fed to the magnetic tape deck where the data were written onto tape. Each tape had approximately 20,000 events written on it containing, in addition to the information mentioned above, such fixed data as run number, data, target condition (full or empty), beam momentum, and F1 counter status (in coincidence or in anticoincidence).

**SPARK CHAMBER LOGIC
FOR PAD EXPERIMENT**
VERSION 1, 2/29/68

FIGURE 5

III. PROCEDURES

A. Experiment

Particles produced at the copper target were momentum selected and passed through a Cerenkov counter filled with Freon 12 gas. Pions were identified and selected in the trigger from other slower particles such as kaons and protons. As the pressure of a gas increases, the index of refraction increases, allowing slower particles to be detected. A pressure curve was run (refer to Figure 6, page 12) at $P_{lab} = 5.0$ GeV/c and 40 psi of pressure was found to be optimum in this momentum range. The pion flux was varied from 2×10^4 to 5×10^4 per burst. The trigger rate varied from 1 per burst at 5.0 GeV/c to 8 per burst at 3.0 GeV/c. A trigger of the system consisted of $CB_1 B_2 \overline{AHFSA}$ and whenever the trigger logic was satisfied, there was a high voltage pulse of approximately 7.5 KV applied to the chamber. A clearing field of 50 V was applied at all times to reduce the latent image time ($\sim 1 \mu s$) and to reduce recovery time after triggering. In the chambers, a mixture of 90% neon and 10% helium gas was used, 5% of which was bubbled through a quencher of isopropyl alcohol to suppress spurious sparks. The magnetostrictive mechanical pulses produced by sparks traveled with a velocity of approximately 5×10^5 cm/sec through the magnetostrictive read-out ribbon (.016" x .004" Remendur-P). Thus, the time it took to travel along the wire was directly proportional to the length of the portion of the wire it traveled. For each trigger the preamplifier connected to the read-out wire produced an output consisting of two fiducial pulses and the spark pulses between them. The pulse sizes were typically 3 ± 1 V into 50Ω although they varied from preamp to preamp.

FIGURE 6

Runs with both positive and negative pions were made at the four incident momenta of 3.0, 3.5, 4.0 and 5.0 GeV/c. Since the angular region subtended by the F_1 counter was where a large number of pions were scattered (due to the large forward cross-section), most of the runs were made with the F_1 counter in anticoincidence to obtain a sufficient number of events in the large angle region for improved statistics. The total number of events, the number of events with F_1 in coincidence and in anticoincidence, and the number of elastic events at different momenta are given in Table 1 (page 14).

B. Monitoring

The number of pulses from each chamber coordinate was displayed on a scaler readout display. By observing these numbers and by adjusting the SAC 1124 input pulse shaper threshold for the pulses, it was possible to minimize the number of spurious sparks recorded.

The high voltage settings for the phototubes in various counters, the gas pressure of the Cerenkov counter, the status of the liquid hydrogen target and the gas flow in the chambers were regularly monitored. In addition, a detailed monitoring of the experiment was done through a partial analysis of each data tape immediately after its completion. To accomplish this we made diagnostic runs using an Advanced Scientific Instrument 6020 computer. A program CHECK printed out scaler readings of the sparks and the second fiducial of each wire plane. By studying histograms of these quantities, many parts of the system could be checked. Also printed out were the total number of sparks in each wand along with the number of single sparks in each wand (for typical values refer to Table 2, page 15). Also available was the number of events with n single sparks per arm ($n = 1$ to 4). By

TABLE 1

NUMBER OF EVENTS WITH F_1 IN COINCIDENCE AND IN ANTICOINCIDENCE AT DIFFERENT MOMENTUM

Mom.	π^+p				π^-p			
	Total Events	F_1		Elastic Events	Total Events	F_1		Elastic Events
		Coincidence	Anticoincidence			Coincidence	Anticoincidence	
3.0	167,336	60,680	106,656	25,966	100,226	20,001	80,225	12,895
3.5	147,558	40,014	107,544	13,504	98,885	20,003	78,882	7,393
4.0	132,444	36,495	95,949	9,501	138,693	20,001	118,692	6,995
5.0	93,248	25,197	68,051	3,980	126,098	23,023	103,075	3,535
Total	540,586	162,386	378,200	52,951	463,902	83,028	380,874	30,818

TABLE 2

Total Number of Sparks in Each Wand
and Number of Single Sparks in Each Wand

(π^- P 5.0 GeV/c, F_1 Out. Total Number of Events 5000)

<u>WAND Number</u>	<u>Chamber Number</u>	<u>Coordinate</u>	<u>Total Number of Sparks</u>	<u>Number of Single Sparks</u>
1	1	x	5138	4901
2		y	6004	4529
3	2	x	5194	4867
4		y	5568	4723
5	3	x	5624	4670
6		y	5366	4804
7	4	x	5429	4755
8		y	5171	4881
9	5	x	7076	3894
10		y	6116	4141
11	6	x	6610	4014
12		y	6027	4196
13	7	x	7306	3730
14		y	6564	3707
15	8	x	7242	3836
16		y	6082	4294
17	9	x	5666	4298
18		y	5389	4266
19	10	x	5528	4362
20		y	7349	2751
21	11	x	5978	4176
22		y	6223	4068
23	12	x	6208	4039
24		y	6166	4067

studying histograms of the position of sparks along each wand, it was possible to check various parts of the system, detect their malfunctions and correct them. As is explained in detail in Chapter IV, section C, cold spots in two of the chambers and a malfunction of the multiple time-digitizer caused by overheating of the scaler boards were detected by analyzing these histograms.

IV. ANALYSIS OF DATA

The analysis of the data was carried out using the PEPR project's Sigma 5 computer at Indiana University.

The events were first reconstructed in three dimensions. Next, two kinematical constraints, coplanarity and opening angle cuts, were imposed on them to separate the elastic events from the background. These good events were weighted according to the geometrical detection efficiency, and finally the cross sections were determined, taking various correction factors into account.

A. Reconstruction of Events

The reconstruction of the events in three dimensional space was done in a program called WIRES. It first checked for fiducials and selected sparks which fell within the fiducial volume. An event was accepted as a possible candidate for an elastic event if at least three chambers in the beam arm registered one and only one spark and at least two chambers in one of the two final state detecting arms had single sparks. This restriction on the beam arm eliminated multiple beam tracks and ensured a well defined incident beam direction. The percentage of beam tracks satisfying the last criterion

was typically about 90%, and this fraction of the total number of beam particles crossing the target was used in subsequent analysis of the data.

The location of the vertex was the first step in the reconstruction of an event. This was done using the beam arm and one of the final state arms having two or more sparks. If both arms had two or more sparks, the S arm was used because it was at a larger angle to the beam line and permitted the determination of the vertex location more accurately. Next all of the possible combinations of x and y coordinates of the sparks in the other arm were made and their locations in three-dimensional space were determined. Of these possible spark locations, the one which lay closest to the trajectory predicted by two body kinematics was chosen and the line connecting the vertex previously determined and this spark was accepted as the true trajectory in that arm.

This process had the danger of accepting as an elastic event an occasional random background event which was accidentally located closest to the kinematically determined trajectory. To determine this effect and to make appropriate corrections, a special version of WIRES was made in which an artificial kinematic curve lay at an arbitrary angle, and an artificial peaking of events under this kinematic curve was sought. There was no appreciable peaking, indicating that this effect was negligible.

B. Various Cuts and Constraints

The detection system covered only the forward portion of the space about the target, and the two final state arms were not symmetrical with respect to the beam line. The detection efficiency was dependent on the geometry of the detection system, i.e., it was a function of θ_{lab} and ϕ_{lab} .

Figure 7 (page 19) shows the two body kinematic curve for πp elastic scattering at 3.5 GeV/c. Here θ_1 is the smaller laboratory scattering angle and θ_2 is the larger scattering angle. There are two branches in the plot. The lower one represents the case in which θ_1 is the pion direction, and the upper one represents events in which θ_1 is the proton direction. The detection efficiency which was obtained by a Monte-Carlo type generation of events is shown in Figure 8 (page 20). Altogether 100,000 events were generated for each bin at each momentum. The steep rise at small θ c.m. and the rapid drop at large θ c.m. are both due to the events striking the edges of the counter arrays. The hump at the middle reflects the fact that the pion (or proton) can be scattered either into the F-arm or the S-arm for this particular range of θ c.m., increasing the detection efficiency by a factor of almost two. Also shown in the same figure is the detection efficiency with counter F_1 in anticoincidence. The F_1 anticoincidence events which came from the angular regions where the detection efficiency changed rapidly, were discarded because there was a danger of introducing a large error due to improper weighting. In the figure, these regions have been indicated by hatched areas near θ c.m. = 54° and θ c.m. = 112° . In these regions the majority of the events were taken with counter F_1 in anticoincidence and a much heavier contribution from these events with improper weighting could cause an erroneous cross section determination.

Since the detection efficiency dropped rapidly at the edges of counter arrays and since the position of the edges were not precisely known, a fiducial cut of $\frac{1}{2}$ " all around the counter array was made. A strip of chamber an inch wide along the edges was also removed from the fiducial volume. Edge sparking was also excluded from further analysis by the

FIGURE 7

FIGURE 8

fiducial volume cuts.

The beam spot averaged about $\frac{3}{4}$ " in diameter in the region of the liquid hydrogen target which was 12" in length and 1.75" in diameter. The location of the beam spot was dependent on the adjustment of the focusing magnets at each momentum, and was not always well centered in the target. However, the target was sufficiently large so that no beam particles struck the wall of the target and $\pm .7$ " cuts in X and Y directions (1.4 " in diameter) were found to be adequate to contain most of the beam particles ($>99\%$). During the early part of the experiment the target was found to be misaligned by about 0.5 inches, but detailed analysis showed no beam particles striking the target walls. In Table 3, below, a list is given of tapes taken with the target shifted, with cold spots in the chambers, or with a missing SAC scaler bit. Some of the particles scattered at the upstream end of the target were intercepted by a part of the thick brass target mounting itself. A cut of upstream -3.5 " and downstream 5 " from the center of the target was made in the z (beam) direction accordingly.

TABLE 3
NUMBER OF TAPES TAKEN UNDER VARIOUS CONDITIONS

Momentum	Number of Tapes with Target Shifted	Number of Tapes with Cold Spots	Number of Tapes with the 64 Bit Missing
3.0	7	5	2
3.5	6	6	3
4.0	3	3	3
5.0	2	2	0
Total	18	16	8

Since no magnet was used to momentum-analyze the scattered particles there were two constraints for an elastic event, coplanarity and opening

angle. Coplanarity is defined as the triple scalar product of the vectors,

$$\text{cop} = (\hat{P}_1 \times \hat{P}_2) \cdot \hat{P}_{\text{beam}}$$

where $\hat{P}_1$, $\hat{P}_2$ and $\hat{P}_{\text{beam}}$ stand for unit vectors along the directions of the two final state and beam particles. Events with the absolute value of $\text{cop} \leq 0.012$ were accepted as coplanar. Figure 9, page 23, shows the coplanarity distribution for all reconstructable events at 3.0 GeV/c for the $\pi^+ p \rightarrow \pi^+ p$ reaction.

These coplanar events were sorted into bins of $(\theta_2 - \theta_1)$ intervals, where θ_1 and θ_2 are the smaller and larger laboratory scattering angles as was mentioned before. For each interval, the distribution of events vs. the quantity DL

$$\text{DL} = \frac{\text{measured opening angle } (\theta_1 + \theta_2) - \text{kinematically predicted openings angle } (\theta_1 + \theta_2)}{\text{angle } (\theta_1 + \theta_2)}$$

was made assuming θ_1 to be the pion direction. Then similar plots were made assuming θ_1 to be the proton direction. Figure 10 (page 24) shows such a distribution for the $\pi^+ p \rightarrow \pi^+ p$ reaction at 3.0 GeV/c for a particular interval in $(\theta_2 - \theta_1)$ where θ_1 was taken to be the pion direction. The plot illustrates relative magnitudes of the background and elastic events. In addition to the main peak at 0° , there is a secondary peak corresponding to a reversal of the pion-proton track assignment, and this reflects the fact that in our experimental setup we lacked a means of identifying the pion and proton. The spatial resolution of 1mm for the spark location allowed us to separate the pion from the proton cleanly down to c.m. scattering angles within 7° of the equal-angle ambiguous case.

Using these opening angle distributions, the background was manually estimated for each bin by interpolating the tails under the elastic peak.

Figure 9

Figure 10

The background level varied considerably with both angle and momentum, being the lowest at small angle and low momentum. It varied from typically 6% for cross sections of $100 \mu\text{b}/(\text{GeV}/c)^2$ to 50% for $1 \mu\text{b}/(\text{GeV}/c)^2$ cross sections.

The elastic events thus obtained were used to determine the differential cross section ($\mu\text{b}/\text{str}$) for each bin using the equation:

$$\frac{d\sigma}{d\Omega} = \frac{10^{30} A N}{N \pi R_0 \rho l d\Omega}.$$

AN is the number of elastic events modified to include the system detection efficiency and to include other corrections mentioned below. A is the atomic weight of hydrogen. $N\pi$ is the total number of incident pions (with proper corrections as previously mentioned), R_0 is Avogadro's number, ρ is the density of the liquid hydrogen ($0.07 \text{ gm}/\text{cm}^3$), l is the effective length of the target after cuts (21.6 c.m.) and $d\Omega = 2\pi \sin\theta \text{ c.m.}$, the element of the solid angle. The cross section was also expressed in terms of the square of the momentum transfer $t = -2k^2 (1 - \cos\theta \text{ c.m.})$, where k is the center of mass momentum,

$$\frac{d\sigma}{dt} = \frac{\pi}{k^2} \left(\frac{d\sigma}{d\Omega} \right).$$

C. Corrections to the Data

The efficiencies of the scintillation counters were determined by placing each counter in the beam in coincidence with the beam telescope $B_1 B_2 \overline{AH}$. The percent efficiency at a single point of a counter C is then given by

$$\frac{B_1 B_2 \overline{AH} C}{B_1 B_2 \overline{AH}} 10^2.$$

The counter inefficiencies were thus found to average about $1 \pm 1\%$.

Angular dependent corrections to the data included the chamber inefficiencies and the effects of nuclear interactions of the final state particles. Since there were four spark chambers in each arm and the minimum requirement for reconstruction is two sparks in the beam arm, two sparks in one final state arm and one spark in the other final arm, there was redundancy in the detection system. Utilizing this, the efficiencies of spark chambers could be determined by examining the effects on the data of reconstructing events with information from one chamber omitted. Through this procedure, it was possible to check for angular biases and chamber inefficiencies. If a chamber was inefficient or produced a large number of spurious sparks the total number of events with a good spark in each remaining gap in a given set of data increased when that particular chamber was omitted in reconstructing events. The chamber inefficiencies thus determined ranged from 2 to $5 \pm 2\%$.

Some of the particles after scattering in the target underwent nuclear interactions in the material of the target and chamber themselves. A $3 \pm 2\%$ correction was made which included interactions in the hydrogen of the target, in mylar windows of the target, in the spark chamber mylar sheets, acclar sheets, wire mesh and in chamber wires.

There was also a loss in the number of incident beam particles, due to the interaction of the beam in the target and in the last beam counter B_2 . The intensity of the beam as a function of the distance x traversed in the target is

$$I(x) = I_0 e^{-x/\lambda}$$

where λ (≈ 310 inches) is the mean free path of a pion in liquid hydrogen for the momentum range of this experiment. Since x is 12 inches at most, $-x/\lambda \ll 1$ and expanding $e^{-x/\lambda}$ and taking the first two terms as an approximation,

$$I(x) \approx I_0(1 - x/\lambda).$$

The average value of x is 6.75 inches which is the distance from the upstream end of the target to the center of the target cuts. Using this value the beam attenuation in the target was found to be 2 to 2.5%. An additional correction of 0.5% was needed to account for interactions of beam particles in the counter B_2 . Thus a total of 2.5 to $3 \pm 1\%$ was subtracted from the measured beam flux $CB_1B_2\overline{AH}$.

The muon contamination in the beam was measured in an earlier experiment¹⁰ by studying the beam attenuation in lead. Various amounts of lead were placed between the beam counters B_1 and B_2 and the recorded beam flux $B_1B_2\overline{CAH}$ was observed as a function of the thickness of lead. From this, the muon content was estimated to be $0 \pm 0.5\%$ for both $\pi^\pm$ beams for our momentum range.

The electron content of the beam was estimated from the beam Cerenkov counter pressure curve. The result is given in Table 4 below.

TABLE 4
ELECTRON CONTENT IN THE BEAM

Momentum	π^+p	π^-p
3.0	$2.5 \pm 1\%$	$5 \pm 2\%$
3.5	$2.0 \pm 1\%$	$5 \pm 2\%$
4.0	$2.0 \pm 1\%$	$5 \pm 2\%$
5.0	$1.0 \pm 0.5\%$	$5 \pm 2\%$

The fact that we used, in our analysis, the same values for lepton

contamination, can be justified because the same external beam and the same production targets were used for this experiment and for that of Reference 10.

The total normalization correction to the data was equal to $16 \pm 8\%$.

The uncertainty in the normalization was computed by

$$\frac{dN}{N} = \left[\frac{\Delta\gamma_1}{1-\gamma_1} + \frac{\Delta\gamma_2}{1-\gamma_2} + \frac{\Delta\gamma_3}{1-\gamma_3} + \frac{\Delta\gamma_4}{1-\gamma_4} + \frac{\Delta\gamma_5}{1-\gamma_5} \right]$$

where

$$\begin{aligned} \gamma_1 \pm \Delta\gamma_1 &= 5 \pm 2\% \text{ lepton contamination,} \\ \gamma_2 \pm \Delta\gamma_2 &= 1 \pm 1\% \text{ counter inefficiency,} \\ \gamma_3 \pm \Delta\gamma_3 &= 2 \pm 2\% \text{ chamber inefficiency,} \\ \gamma_4 \pm \Delta\gamma_4 &= 3 \pm 1\% \text{ incident beam attenuation,} \\ \gamma_5 \pm \Delta\gamma_5 &= 3 \pm 2\% \text{ final state interaction.} \end{aligned}$$

Corrections were also needed to account for accidentals in various counters.

Accidentals in beam counters B_1 , B_2 and Cerenkov counter C were caused principally by protons in the beam and by stray particles and background in the Cerenkov counter. Let τ_B and τ_C be the width of pulses in beam counters and in the Cerenkov counter and let N_C , N_1 , and N_2 be the number of particles per second recorded at the Cerenkov counter and beam counters B_1 and B_2 . If N_π is the number of pions per second defined by $B_1 B_2 \overline{CAH}$, then the accidentals (in percent of pions) due to stray particles other than protons is determined by

$$\frac{N_1 N_2 \tau_B N_C \tau_C}{N_\pi}$$

Calculation shows that this is quite small ($<0.001\%$) and can be neglected.

Accidentals due to protons is given by

$$\frac{N_p N_c \tau_c}{N_\pi}$$

where $N_p = N_2 - N_\pi$ is the number of protons which can cause accidentals.

In Table 5 below, the values of accidentals due to protons at various momenta are given in percentage of pions.

TABLE 5
ACCIDENTALS

Momentum	Accidentals (%)
3.0	0.2%
3.5	0.2%
4.0	0.25%
5.0	0.53%

Accidentals due to anti-counters A_1 , A_2 , A_3 , A_4 , and $\overline{AH}$ result in a fraction of the time that good events are antied out by accident. Let N_1 , N_2 , N_3 , N_4 , and $N_{\overline{AH}}$ be the number of particles per second striking the anti-counters A_1 , A_2 , A_3 , A_4 , and $\overline{AH}$. Then

$$\left[N_1 + N_2 + N_3 + N_4 + N_{\overline{AH}} \right] \cdot \tau_A$$

is the fraction of time good events are antied out, where τ_A is the pulse width of the anti-counters. The average number of events accidentally antied out was found to be about 0.8% of the pions in the beam.

During the beginning part of the experiment "cold spots" had developed in two of the chambers (a strip of chamber approximately 3 inches wide in the first chamber in the F-arm and in the last chamber in the S-arm). The electrical contact of the liquid solder used to glue the chamber wires to the brass bar deteriorated thus rendering these sections of the chambers partially insensitive. The cold spots were repaired by directly soldering the chamber wires to a strip of copper ribbon.

To correct the situation for data taken during this period, these portions of the chambers were removed from the fiducial volume in our analysis. Several tapes, taken after the cold spots were repaired, were analyzed with and without the cold spot correction. From the comparison of the two results, it was found that, in these small angular regions including such a cold spot, a compensation for tapes with cold spots had to be made by scaling down the geometric detection efficiency by a factor of .999 for events with F_1 counter in coincidence and by a factor of .993 for events with F_1 counter in anticoincidence. It was also found that when one scaler board in the Multiple Time Digitizer (Model 1124) was overheated, the distribution of the spark locations from the corresponding wand showed oscillatory type behavior, that is, the 64 bit dropped.

The tapes plagued with this problem were analyzed with the corresponding scaler readings (x coordinates of the sparks of the first chamber in F-arm) completely deleted. The results of these tapes were corrected by scaling down the detection efficiency by a factor of .943 for events with F_1 counter in coincidence, and by a factor of .947 for events with F_1 counter in anti-coincidence. It was determined in a manner similar to the one mentioned above (refer to Table 3, page 21).

V. RESULTS

Tables 6 and 7 (beginning on page 31) present the differential cross sections for π^+p and π^-p elastic scattering at the four momenta studied in this experiment. They are expressed both in terms of $\frac{d\sigma}{dt}$ vs. t and $\frac{d\sigma}{d\Omega}$ vs. $\cos \theta$. Only statistical errors are given. They were

TABLE 6
 $\pi^+ p$ Differential Cross Section
 $\pi^+ p$ 3.0 GeV/c

$-t$ (GeV/c) ²	$\frac{d\sigma}{dt}$ (μb)/(GeV/c) ²	$\Delta \frac{d\sigma}{dt}$ (μb)/(GeV/c) ²	$\cos\theta_{\text{c.m.}}$	$\frac{d\sigma}{d_A}$ $\mu\text{b}/\text{str}$	$\Delta \left(\frac{d\sigma}{d_A}\right)$ $\mu\text{b}/\text{str}$
.240	9956.0	168.0	.901	3841.9	64.8
.340	4984.0	111.0	.860	1923.3	42.8
.440	2798.0	81.0	.819	1079.7	31.3
.540	1260.0	54.0	.777	486.2	20.8
.640	760.0	41.0	.736	293.3	15.8
.740	445.0	43.0	.695	171.7	16.6
.840	315.0	36.0	.654	121.6	13.9
.940	322.0	34.0	.612	124.3	13.1
1.040	376.0	34.0	.571	145.1	13.1
1.140	346.5	11.0	.530	133.7	4.2
1.240	310.7	11.0	.489	119.9	4.2
1.340	331.4	13.0	.447	127.9	5.0
1.440	315.1	10.0	.406	121.6	3.9
1.540	295.8	8.8	.365	114.1	3.4
1.640	248.0	7.6	.324	95.7	2.9
1.740	218.3	8.5	.282	84.2	3.3
1.840	185.1	5.9	.241	71.4	2.3
2.140	98.9	2.7	.117	38.2	1.0
2.440	30.3	3.8	-.006	11.7	1.5
2.540	20.8	3.7	-.048	8.0	1.4
2.640	8.0	3.7	-.089	3.1	1.4
2.740	14.9	4.2	-.130	5.7	1.6
2.840	20.1	5.5	-.171	7.8	2.1
2.940	22.3	4.6	-.213	8.6	1.8
3.040	26.1	4.5	-.254	10.1	1.7
3.140	28.9	5.3	-.295	11.2	2.0
3.240	45.3	6.4	-.336	17.5	2.5
3.340	44.7	6.6	-.378	17.2	2.5
3.440	43.3	8.4	-.419	16.7	3.2
3.540	51.7	8.8	-.460	20.0	3.4
3.640	47.2	8.8	-.501	18.2	3.4
3.740	52.2	9.2	-.543	20.1	3.6

TABLE 6
 π^+p Differential Cross Section
 (continued)

π^+p 3.5 GeV/c

t (GeV/c) ²	$\frac{d\sigma}{dt}$ (nb)/(GeV/c) ²	$\Lambda \frac{d\sigma}{dt}$ (nb)/(GeV/c) ²	$\cos\theta_{c.m.}$	$\frac{d\sigma}{d\Omega}$ nb/str	$\Lambda \left(\frac{d\sigma}{d\Omega}\right)$ nb/str
.337	4727.1	109.0	.883	2171.4	50.1
.462	2201.3	73.0	.840	1011.2	33.5
.587	887.1	45.0	.797	407.5	20.7
.712	471.5	33.0	.753	216.6	15.2
.837	311.7	26.0	.710	143.2	11.9
.962	275.0	32.0	.667	126.3	14.7
1.087	312.8	26.0	.623	143.7	11.9
1.212	245.7	22.0	.580	112.9	10.1
1.337	173.0	28.0	.537	79.5	12.9
1.462	180.0	7.6	.493	82.7	3.5
1.587	145.7	5.8	.450	66.9	2.7
1.712	118.0	7.6	.407	54.2	3.5
1.837	93.1	4.2	.364	42.8	1.9
1.962	57.9	2.9	.320	26.6	1.3
2.087	38.5	2.5	.227	17.7	1.1
2.212	26.8	2.6	.234	12.3	1.2
2.587	9.8	.7	.104	4.5	.3
2.962	4.7	1.5	-.026	2.2	.7
3.087	6.0	1.9	-.070	2.7	.9
3.212	7.6	2.3	-.113	3.5	1.1
3.337	8.5	2.2	-.156	3.9	1.0
3.462	10.5	2.9	-.200	4.8	1.3
3.587	13.8	3.4	-.243	6.3	1.6
3.712	14.6	4.2	-.286	6.7	1.9
3.837	12.8	3.1	-.329	5.9	1.4
3.962	13.6	2.6	-.373	6.2	1.2
4.087	12.5	3.3	-.416	5.7	1.5
4.212	15.4	2.5	-.459	7.1	1.1
4.337	15.6	3.1	-.503	7.2	1.4

TABLE 6
 π^+p Differential Cross Section
 (continued)

π^+p 4.0 GeV/c

$-t$ (GeV/c) ²	$\frac{d\sigma}{dt}$ (μb)/(GeV/c) ²	$\Lambda \frac{d\sigma}{dt}$ (μb)/(GeV/c) ²	$\cos\theta_{\text{c.m.}}$	$\frac{d\sigma}{d\Lambda}$ $\mu\text{b/str}$	$\Lambda \left(\frac{d\sigma}{d\Lambda}\right)$ $\mu\text{b/str}$
.410	3000.0	75.7	.878	1599.3	40.4
.560	1050.0	43.6	.833	559.7	23.4
.710	438.0	26.6	.788	233.5	14.2
.860	282.0	21.2	.743	150.3	11.3
1.010	203.0	18.4	.698	108.2	9.8
1.160	172.0	16.6	.654	91.7	8.8
1.310	149.0	18.0	.609	79.4	9.6
1.460	128.0	17.0	.564	68.2	9.1
1.610	93.5	14.0	.519	49.8	7.5
1.760	68.7	3.2	.475	36.6	1.7
1.910	50.5	3.5	.430	26.9	1.9
2.060	34.6	3.0	.385	18.4	1.6
2.210	14.9	1.9	.340	7.9	1.0
2.360	7.8	.9	.295	4.1	.5
2.510	2.6	.6	.251	1.4	.3
3.035	1.8	.3	.094	1.0	.2
3.560	2.6	.9	-.063	1.4	.5
3.710	3.3	.8	-.108	1.7	.4
3.860	2.8	1.1	-.152	1.5	.6
4.010	3.7	1.1	-.197	2.0	.6
4.160	3.1	1.4	-.242	1.7	.7
4.310	3.4	1.2	-.287	1.8	.6
4.460	4.1	1.5	-.332	2.2	.8
4.610	3.1	1.3	-.376	1.7	.7
4.760	2.8	1.4	-.421	1.5	.7
4.910	3.5	1.3	-.466	1.9	.7

TABLE 6
 $\pi^+ p$ Differential Cross Section
 (continued)

$\pi^+ p$ 5.0 GeV/c

$-t$ (GeV/c) ²	$\frac{d\sigma}{dt}$ (μb)/(GeV/c) ²	$\frac{1}{s} \frac{d\sigma}{dt}$ (μb)/(GeV/c) ²	$\cos\theta_{c.m.}$	$\frac{d\sigma}{d_n}$ $\mu\text{b/str}$	$N \left(\frac{d\sigma}{d_n} \right)$ $\mu\text{b/str}$
.650	650.0	26.4	.848	442.7	18.0
.950	165.0	12.9	.778	112.4	8.8
1.250	96.7	9.5	.708	65.9	6.5
1.550	37.0	6.0	.638	25.2	4.1
1.850	27.7	5.8	.568	18.9	4.0
2.150	7.2	3.0	.498	4.9	2.1
2.450	1.4	2.2	.427	.95	1.5
2.750	.65	.26	.357	.44	.18
3.050	.22	.20	.287	.15	.14
3.950	.51	.07	.077	.35	.05
4.850	.39	.16	-.133	.26	.11
5.150	.51	.20	-.203	.35	.13
5.450	.67	.27	-.274	.46	.18
5.750	.42	.28	-.344	.29	.19

TABLE 7
 $\pi^- p$ Differential Cross Section
 $\pi^- p$ 3.0 GeV/c

t (GeV/c) ²	$\frac{d\sigma}{dt}$ (μb)/(GeV/c) ²	$\Lambda \frac{d\sigma}{dt}$ (μb)/(GeV/c) ²	$\cos \theta_{c.m.}$	$\frac{d\sigma}{d\Omega}$ μb/str	$\Lambda \left(\frac{d\sigma}{d\Omega} \right)$ μb/str
.240	9220.0	270.0	.901	3557.9	104.2
.340	4555.0	180.0	.860	1757.7	69.5
.440	2158.0	120.4	.819	832.8	46.5
.540	995.0	83.0	.777	384.0	32.0
.640	563.0	60.0	.736	217.3	23.2
.740	312.0	44.6	.695	120.4	17.2
.840	250.0	40.2	.654	96.5	15.5
.940	261.0	39.4	.612	100.7	15.2
1.040	372.0	46.1	.571	143.6	17.8
1.140	319.0	11.6	.530	123.1	4.5
1.240	296.0	11.6	.489	114.2	4.5
1.340	281.0	10.8	.447	108.4	4.2
1.440	259.0	10.0	.406	100.0	3.9
1.540	224.0	8.3	.365	86.4	3.2
1.640	191.0	7.0	.324	73.7	2.7
1.740	168.0	6.0	.282	64.9	2.3
1.840	123.0	5.2	.241	47.5	2.0
2.140	77.5	1.9	.117	29.9	.7
2.440	45.2	5.1	-.006	17.4	2.0
2.540	27.5	2.9	-.048	10.6	1.1
2.640	16.4	3.0	-.089	6.3	1.2
2.740	12.4	3.2	-.130	4.8	1.2
2.840	11.5	2.9	-.171	4.4	1.1
2.940	16.9	3.7	-.213	6.5	1.4
3.040	18.1	4.1	-.254	7.0	1.6
3.140	18.4	3.1	-.295	7.1	1.2
3.240	33.1	6.0	-.336	12.8	2.3
3.340	26.2	5.7	-.378	10.1	2.2
3.440	29.8	4.6	-.419	11.5	1.8
3.540	21.4	3.6	-.460	8.3	1.4
3.640	24.3	3.8	-.501	9.4	1.5
3.740	14.7	4.1	-.543	5.7	1.6

TABLE 7
 $\pi^- p$ Differential Cross Section
 (continued)

$\pi^- p$ 3.5 GeV/c

t (GeV/c) ²	$\frac{d\sigma}{dt}$ (μb)/(GeV/c) ²	$\Lambda \frac{d\sigma}{dt}$ (μb)/(GeV/c) ²	$\cos\theta_{\text{c.m.}}$	$\frac{d\sigma}{dN}$ $\mu\text{b/str}$	$\Lambda \left(\frac{d\sigma}{dN}\right)$ $\mu\text{b/str}$
.337	4460.0	150.0	.883	2048.7	68.9
.462	1690.0	91.2	.840	776.3	41.9
.587	649.0	55.5	.797	298.1	25.5
.712	299.0	36.9	.753	137.3	17.0
.837	250.0	33.1	.710	114.8	15.2
.962	165.0	28.8	.667	75.8	13.2
1.087	240.0	33.2	.623	110.2	15.3
1.212	196.0	35.0	.580	90.0	16.1
1.337	234.0	35.0	.537	107.5	16.1
1.462	192.0	7.2	.493	88.2	3.3
1.587	143.0	6.1	.450	65.7	2.8
1.712	115.0	5.4	.407	52.8	2.5
1.837	83.4	4.1	.364	38.3	1.9
1.962	56.6	3.0	.320	26.0	1.4
2.087	33.3	2.3	.277	15.3	1.1
2.212	22.5	2.0	.234	10.3	.9
2.587	6.3	.6	.104	2.9	.3
2.962	1.6	.9	-.026	.73	.43
3.087	2.3	1.2	-.070	1.1	.6
3.212	3.3	1.2	-.113	1.5	.6
3.337	5.1	1.5	-.156	2.3	.7
3.462	8.9	2.2	-.200	4.1	1.0
3.587	9.9	2.2	-.243	4.5	1.0
3.712	13.7	2.5	-.286	6.3	1.1
3.837	11.6	2.4	-.329	5.3	1.1
3.962	9.1	2.0	-.373	4.2	.9
4.087	10.2	2.1	-.416	4.7	1.0
4.212	4.2	1.6	-.459	1.9	.7
4.337	2.5	1.5	-.503	1.1	.7

TABLE 7
 π^-p Differential Cross Section
 (continued)

π^-p 4.0 GeV/c

t (GeV/c) ²	$\frac{d\sigma}{dt}$ (μb)/(GeV/c) ²	$\Lambda \frac{d\sigma}{dt}$ (μb)/(GeV/c) ²	$\cos\theta_{c.m.}$	$\frac{d\sigma}{d_n}$ $\mu\text{b/str}$	$\Lambda \left(\frac{d\sigma}{d_n}\right)$ $\mu\text{b/str}$
.410	2600.0	86.0	.878	1386.0	45.8
.560	828.0	49.3	.833	441.4	26.3
.710	316.0	28.0	.788	168.5	14.9
.860	212.0	28.0	.743	113.0	14.9
1.010	170.0	20.7	.698	90.6	11.0
1.160	149.0	19.2	.654	79.4	10.2
1.310	130.0	22.0	.609	69.3	11.7
1.460	129.0	22.0	.564	68.8	11.7
1.610	84.5	17.0	.519	45.0	9.1
1.760	65.9	2.9	.475	35.1	1.5
1.910	45.7	2.1	.430	24.4	1.1
2.060	31.2	1.9	.385	16.6	1.0
2.210	17.2	1.5	.340	9.2	.8
2.360	8.4	.7	.295	4.5	.4
2.510	3.8	.5	.251	2.0	.3
3.035	1.0	.2	.094	.55	.08
3.560	1.6	.4	-.063	.84	.22
3.710	1.2	.4	-.108	.63	.23
3.860	1.6	.5	-.152	.84	.28
4.010	2.2	.7	-.197	1.2	.4
4.160	2.0	.5	-.242	1.1	.3
4.310	3.0	.7	-.287	1.6	.4
4.460	4.1	.7	-.332	2.2	.4
4.610	2.8	.7	-.376	1.5	.4
4.760	2.5	.6	-.421	1.3	.3
4.910	1.5	.5	-.466	.8	.3

TABLE 7
 $\pi^- p$ Differential Cross Section
 (continued)

$\pi^- p$ 5.0 GeV/c

$-t$ (GeV/c) ²	$\frac{d\sigma}{dt}$ (μb)/(GeV/c) ²	$\Lambda \frac{d\sigma}{dt}$ (μb)/(GeV/c) ²	$\cos\theta_{\text{c.m.}}$	$\frac{d\sigma}{d\Lambda}$ $\mu\text{b/str}$	$\Lambda \left(\frac{d\sigma}{d\Lambda}\right)$ $\mu\text{b/str}$
.650	496.0	22.5	.848	337.8	15.3
.950	142.0	11.5	.778	96.7	7.8
1.250	80.6	8.5	.708	54.9	5.8
1.550	49.1	6.4	.638	33.4	4.3
1.850	19.6	4.3	.568	13.3	2.9
2.150	6.8	2.6	.498	4.6	1.7
2.450	1.3	1.3	.427	.90	.90
2.750	.21	.13	.357	.15	.09
3.050	.28	.22	.287	.19	.15
3.950	.43	.12	.077	.29	.08
4.850	.58	.18	-.133	.39	.12
5.150	.36	.17	-.203	.25	.12
5.450	.63	.25	-.274	.43	.17
5.750	.24	.24	-.344	.16	.16

calculated using

$$\Delta\left(\frac{d\sigma}{d\Omega}\right) = \frac{\sqrt{N + N_B}}{N - N_B} \cdot \frac{d\sigma}{d\Omega}$$

where N is the number of total events and N_B is the number of the background events to be subtracted from N .

Differential cross section for $\pi^+p \rightarrow \pi^+p$ from this experiment are plotted in Figure 11 (page 40) along with results from other experiments for comparison. Hand drawn curves connect data points to guide the eye. The experiments of Coffin et al.⁵ (The University of Michigan) and Chandler et al.⁶ (Indiana University) are in good agreement with our data. As has been observed before⁵⁻⁷, the dip near $t \approx -0.9 \text{ (GeV/c)}^2$ is less pronounced for π^+p , becoming a shoulder-like structure at 3.0 and 3.5 GeV/c, and almost non-existent at 5.0 GeV/c. However, the dip near $t \approx -3.0 \text{ (GeV/c)}^2$ is much deeper and present at all momenta.¹²

Similar plots of differential cross sections for $\pi^-p \rightarrow \pi^-p$ are shown in Figure 12 (page 41). Results from Coffin et al.⁵ (The University of Michigan), Crittenden et al.⁶ (Indiana University), Fellingner et al. (Iowa-Minnesota Collaboration), and Owen et al. (Cornell-BNL Collaboration) are also presented for comparison, and again we note that agreement with these data is good. The overall characteristics are almost identical to those for the $\pi^+p \rightarrow \pi^+p$ reaction. The deep minimum near $t \approx -3.0 \text{ (GeV/c)}^2$ is present at all energies, and the dip at $t \approx -0.9 \text{ (GeV/c)}^2$ is more pronounced than the dip for π^+p at 2.5 and 3.0 GeV/c. It becomes a shoulder-like structure at higher momentum and almost disappears at 5.0 GeV/c.

Figure 11

Figure 12

VI. THEORY

In recent years a large number of models¹³ have been proposed to explain pion-nucleon scattering data. However, each of these models usually explains only a part of the total angular distribution, reflecting the difficulty associated with the complicated structure in $\pi^{\pm}p$ elastic scattering. In this study we are mainly concerned with two models. One is an optical model¹⁴ for all scattering angles, which can be applied to not only πp scattering, but also to other two-body reactions with arbitrary spin. The other one is a Regge exchange model¹⁵ which describes the data through the entire region $0 < |t| < 4$ (GeV/c)² for incident momenta from 2.5 to 13.5 GeV/c.

A. Optical Model

Chu and Hendry have developed the following model based on the optical and absorptive models¹⁶. It is intended to describe the cross-sections at all scattering angles. For the sake of completeness, we will present in this section the essential part of their theory.

Following Jacob and Wick¹⁷ the s-channel helicity flip and non-flip amplitudes may be written in the form

$$f_{+-}(E, \theta) = \frac{1}{k} \sum_j (j + \frac{1}{2}) T_{+-}^j(E) d_{-\frac{1}{2}\frac{1}{2}}^j(\theta)$$

$$f_{++}(E, \theta) = \frac{1}{k} \sum_j (j + \frac{1}{2}) T_{++}^j(E) d_{\frac{1}{2}\frac{1}{2}}^j(\theta)$$

where k is the center of mass momentum, $d_{mn}^j(\theta)$ is the reduced rotation matrix, and T_{mn}^j is a matrix related to the scattering matrix S through the equation $S = 1 + T$.

According to the absorptive model, in the summation over j , there is

little or no contribution other than diffraction from the low values of j since they are absorbed, while above a certain value of j there is likewise no contribution since these are unscattered, and a particular band of j -values gives almost all of the scattering for any given energy. To illustrate the model, the authors discuss pion-nucleon charge exchange scattering. One of the well-established features in the differential cross-section¹⁸ is a dip at $t = 0$, which implies that the helicity flip amplitude of $f_{+-}(\theta)$ is the dominant one for this process. The other well-known features of this process are a valley near the forward direction at $t \approx -0.6 \text{ (GeV/c)}^2$, another valley near the backward direction at $u \approx -0.2 \text{ (GeV/c)}^2$ accompanied by a sharp rise on approaching 180° . These features can be explained in a straightforward manner by considering

$$d_{\lambda\mu}^j(\theta) \sim J_{\mu-\lambda}\left((2j+1) \sin \frac{\theta}{2}\right)$$

which is the Bessel function approximation for the d-function for large j and small θ , and making use of the relation¹⁷

$$d_{\lambda\mu}^j(\theta) = (-1)^{j+\lambda} d_{\lambda-\mu}^j(\pi-\theta) = d_{-\mu-\lambda}^j(\theta)$$

where it is assumed that only one value of j is involved for simplicity.

For the forward direction,

$$f_{+-}(\theta) \sim J_1\left((2j+1) \sin \frac{\theta}{2}\right) \approx J_1\left(b(-t)^{\frac{1}{2}}\right)$$

where b is the impact parameter and t is the square of the momentum transfer.

$J_1(x)$ has its first two zeros at $x = 0$ and $x = 3.83$ and one can see that the forward scattering features are associated with these zeros since $x = 3.83$ corresponds to $t \approx -0.6 \text{ (GeV/c)}^2$ for $b \approx 5 \text{ (GeV/c)}^{-1} = 1 \text{ Fermi}$.

For the backward direction,

$$f_{+-}(\theta) \sim J_0\left((2j+1) \cos \frac{\theta}{2}\right) \approx J_0\left(b(-u)^{\frac{1}{2}}\right).$$

Using the same value for b , one can also correlate the backward dip at $u \approx -0.2 \text{ (GeV/c)}^2$ to the first zero of $J_0(x)$ at $x = 2.41$ where $J_0\left(b(-u)^{\frac{1}{2}}\right)$ is the Bessel function approximation for the d -function near $\theta = 180^\circ$. Thus $b \approx 1F$ appears to be a quite meaningful value and it will be regarded as a slowly varying parameter, independent of energy in the fit to πp elastic scattering data. In general there are several j -values which are important, and these are centered around $j + \frac{1}{2} \approx kb$ for any given energy. By using several adjacent values of j , one can get the correct ratio of forward-backward magnitudes.

In terms of Legendre functions,

$$d_{-\frac{1}{2}}^j(\theta) = (j+\frac{1}{2})^{-1} \sin \frac{\theta}{2} (P'_{j+\frac{1}{2}} + P'_{j-\frac{1}{2}}).$$

The d functions are always positive in the forward direction while they alternate in sign with successive j in the backward direction, thus causing an enhancement in the forward direction but bringing about a cancellation in the backward direction. The dip positions in the forward and backward directions remain roughly the same because the zeros of adjacent $d_{-\frac{1}{2}}^j$ functions in this region are approximately at the same places. They will be also constant in t , u respectively, independent of energy with b taken constant. Dips at intermediate angles are expected to move slowly with energy.

Application of this model to πp elastic scattering is a little more complicated since both f_{++} and f_{+-} are now important.

Chu and Hendry made a fit to $\pi^+ p$ differential cross sections at 3.0

GeV/c and it is presented in Figure 13 (page 46). Data used in this plot are from this experiment, from Coffin et al., and from Chandler et al., and they cover the entire angular region from forward to backward directions. Seven partial waves, $l = 0, 1, 2, 3, 4, 5, 6$, were included in the summation. As is expected, $f_{++}(\theta)$ is dominant in the forward direction, while it is $f_{+-}(\theta)$ which is the main contributor in the backward direction (recall that $f_{++}(\theta) \sim d_{22}^j(\theta)$ has maximum value for $\theta = 0^\circ$, and approaches zero for $\theta = 180^\circ$, and $f_{+-}(\theta) \sim d_{11}^j(\theta)$ is zero for $\theta = 0^\circ$ and has maximum for $\theta = 180^\circ$). The deep valley near $t = -3.0 \text{ (GeV/c)}^2$ is due to the remarkable fact that this position is close to dips in both helicity amplitudes.

B. Regge Exchange Model

One of the well known characteristics of the Regge-pole model is that the forward scattering peak shrinks when only a single pole is dominant. However in the case of π -N elastic scattering, there is hardly any shrinking in the forward peak with energy, and this was once taken as evidence against Regge theory. But it is now known that when there are several poles contributing, the shrinking effect may be reversed.¹⁹ In the model which was developed by Barger and Phillips,¹⁵ and is discussed below, three poles are used to explain the π -p data in which there is little shrinkage in the forward (diffraction) peak.

The three Regge poles are the Pommeranchuk pole P which describes the asymptotic limit of $\sigma_{\text{tot}}(\pi^\pm p)$, a second Pommeranchuk pole P' and the ρ pole which are needed to explain the difference of $\pi^\pm p$ amplitudes from the asymptotic limit and from each other.

Figure 13

The ρ has a negative signature and contributes with different signs to π^+p and π^-p scattering, which is a direct consequence of the signature concept and crossing symmetry, thus the ρ trajectory is needed to explain the finite difference $\sigma_{\text{tot}}(\pi^-p) - \sigma_{\text{tot}}(\pi^+p)$. The second vacuum trajectory ϕ is needed in order to explain the experimental fact that both $\sigma_{\text{tot}}(\pi^+p)$ and $\sigma_{\text{tot}}(\pi^-p)$ approach their common Pommeranchuk limit from above.

Following Singh²⁰, the differential cross section can be expressed in terms of a helicity-flip amplitude B and a nonflip amplitude A' ,

$$\frac{d\sigma}{dt}(s,t) = \frac{1}{\pi s} \left(\frac{M}{4k}\right)^2 \left\{ \left(1 - \frac{t}{4M^2}\right) |A'|^2 + \frac{t}{4M^2} \left(s - \frac{s+p^2}{1-t/(4M^2)}\right) |B|^2 \right\}$$

where M is the nucleon mass, p is the pion laboratory momentum, and k is the c.m. momentum. With P , P' and ρ contributing, the $\pi^\pm p$ elastic amplitudes are

$$A'(\pi^+p \rightarrow \pi^+p) = A'_P + A'_{P'} - A'_\rho$$

$$A'(\pi^-p \rightarrow \pi^-p) = A'_P + A'_{P'} + A'_\rho$$

with similar expressions for helicity-flip amplitudes. As was noted before, π^+p elastic scattering data show dips near $t = -0.9$ and $t = -3.0$ (GeV/c)², and this has been ascribed by the authors to a possible cyclic behavior of the P' Regge residue.^{21,22} The contributions of P , P' and ρ poles to the invariant amplitudes A' and B can be expressed in the form

$$\begin{aligned} A'_j &= -\gamma_j \exp(-i\pi\alpha_j/2) v^{\alpha_j} & \text{for } j = P, P' \\ &= i\gamma_j \exp(-i\pi\alpha_j/2) v^{\alpha_j} & \text{for } j = \phi \\ B_j &= -\beta_j \exp(-i\pi\alpha_j/2) v^{\alpha_j-1} & \text{for } j = P, P' \\ &= i\beta_j \exp(-i\pi\alpha_j/2) v^{\alpha_j-1} & \text{for } j = \phi \end{aligned}$$

where $\alpha_j(t)$ is the trajectory, $\beta_j(t)$ and $\gamma_j(t)$ are essentially the residues divided by $\sin(\pi\alpha_j/2)$ for P and P' , or by $\cos(\pi\alpha_j/2)$ for ρ , and $v = (s - u)/(4M)$. They assume, for simplicity, trajectories linear in t

$$\alpha(t) = \alpha(0) + \alpha' t$$

with

$$\begin{aligned} \alpha(0) &= 1 && \text{for } P \\ &\approx 0.5 && \text{for } P' \text{ and } \rho \text{ and} \\ \alpha' = \frac{d\alpha}{dt} &\approx 0 && \text{for } P \\ &\approx 1(\text{GeV})^{-2} && \text{for } P' \text{ and } \rho. \end{aligned}$$

The cyclic residue for P' can be written

$$\gamma_{P'} = \lambda(t) \sin(\pi\alpha_{P'}/2)$$

where $\lambda(t)$ is a smooth function with no zeros. Thus $\gamma_{P'}$ has recurring zeros at right signature points $\alpha_{P'} = 0, 2, 4, \dots$

Some of the amplitudes can be neglected. This can be seen by discussing π^+p polarization data. First they show approximate mirror symmetry $P(\pi^+p) \approx -P(\pi^-p)$, and second, there are zeros near $t = -0.5 (\text{GeV}/c)^2$ and possibly further zeros²³ beyond $t = -2(\text{GeV}/c)^2$ (refer to Figure 14, page 49). The polarization is $P = 2 \operatorname{Im} A'^* B / (d\sigma/d\Omega)$,

$$\text{or } P \frac{d\sigma}{d\Omega} = 2 \operatorname{Im} A'^* B = 2 \operatorname{Im} (A'_P + A'_{P'} - A'_{\rho})^* (B_P + B_{P'} - B_{\rho}).$$

A'_P can be neglected since it is known to be small from analysis of charge exchange differential cross section. In the above expression, terms involving B can contribute to $P(\pi^+p)$ with opposite signs, thus giving the characteristic mirror symmetry. P - P' interference is small since it contributes with the same sign. The polarization is then given by

Figure 14

$$P(\pi^{\pm}p) \frac{d\sigma}{d\alpha}(\pi^{\pm}p) = \pm 2 \sum_{i=P, P'} \beta_p \gamma_i \cos \frac{\pi}{2} (\alpha_i - \alpha_0) \alpha_i + \alpha_0 - 1$$

at $t = -0.5 \text{ (GeV/c)}^2$, P-P term vanish since both β_0 and the cosine factor are equal to zero. In order for the $P'-\rho$ term to vanish at $t = -0.5 \text{ (GeV/c)}^2$ which is required by the data, $\gamma_{P'}$ must have a zero since the cosine factor does not vanish, and this is ensured by the cyclic nature of $\gamma_{P'}$. There are no zeros near $t = -1.5 \text{ (GeV/c)}^2$, that is β_0 has no zero at the right-signature point $\alpha_0 = -1$. The data suggest the existence of zeros near $t = 2.5 \text{ (GeV/c)}^2$ which implies β_0 vanishing at $\alpha_0 = -2$. Therefore it seems to be reasonable to assume that β_0 has recurring zeros at wrong-signature points

$$\beta_0 = \xi(t) \sin \left(\frac{\pi \alpha_0}{2} \right)$$

where $\xi(t)$ is a smooth function of t with no zeros.

Barger and Phillips have made a fit to πp scattering data. They first found the ρ parameters by fitting to charge exchange data and the difference between $\sigma_{\text{tot}}(\pi^- p)$ and $\sigma_{\text{tot}}(\pi^+ p)$. P and P' parameters were found by fitting to elastic scattering and polarization data. As the result of this fit, they obtained the following expressions for the various parameters,

$$\alpha_P = 1.0 + 0.21 t$$

$$\alpha_{P'} = 0.55 + 0.92 t$$

$$\alpha_\rho = 0.55 + 1.0 t$$

$$\gamma_P = 18.2 e^{4.9t} + 34.9 e^{2.4t} + 0.09 e^{-0.30t}$$

$$\gamma_{P'} = (51.5 e^{2.1t} + 13.1 e^{-0.41t}) \sin \left(\frac{\pi}{2} \alpha_{P'} \right)$$

$$\gamma_\rho = (7.7 (1 + \frac{t}{0.23}) e^{2.9t} \sin \left(\frac{\pi}{2} \alpha_\rho \right) / \alpha_\rho$$

$$\beta_\rho = (105.6 e^{3.4t} + 37.9 e^{0.12t}) \sin \left(\frac{\pi}{2} \alpha_\rho \right)$$

where the γ 's are in $(\text{GeV})^{-1}$ and β_0 is in $(\text{GeV})^{-2}$.

Using these expressions for the trajectories and residue functions we have made a quantitative comparison to the data,^{5,7,24} and the result is shown in Figures 15 and 16 (pages 52 and 53). Although this model was originally intended to explain the πp data up to $t = -4.0 (\text{GeV}/c)^2$, we extended its application beyond this value of t for momenta larger than 3.0 GeV/c .

The fit to the data through the region $0 < -t < 2.5 (\text{GeV}/c)^2$ is remarkably good for both $\pi^\pm p$ where the only structure is the plateau or shoulder near $t = -.9 (\text{GeV}/c)^2$. Near $t = -3.0 (\text{GeV}/c)^2$ where there is a minimum, the theoretical curves dip down much deeper than the data at momenta $\leq 3.5 \text{ GeV}/c$ for both $\pi^\pm p$. At larger momenta, the second theoretical minimum near $t = -5.0 (\text{GeV}/c)^2$ goes deeper than the data for both $\pi^\pm p$. However, this is in the region beyond $t = -4.0 (\text{GeV}/c)^2$ where a good fit should not be expected. The agreement of the theoretical curve to the data becomes quite good up to $t = -6.0 (\text{GeV}/c)^2$ at 6.0 GeV/c for $\pi^- p$.

C. The Effective Trajectory

If only a single Regge pole is being exchanged in the t -channel the differential cross section takes the form^{22,25}

$$\frac{d\sigma}{dt} = f(t) s^{2\alpha(t) - 2}$$

where s is in units of $s_0 = 1 \text{ GeV}$, an arbitrary scale factor. In πp elastic scattering, it was found that several poles are contributing, but if one trajectory is dominant in a particular region of t , this should become clear in a plot of $\alpha_{\text{eff}}(t)$ vs. t , where $\alpha_{\text{eff}}(t)$ is an experimentally

$$\pi^+ p \rightarrow \pi^+ p$$

• THIS EXPERIMENT
+ COFFIN $\&$ $\&$ (MICHIGAN)

Figure 15

Figure 16

determined function of t defined by the above equation. One notices that,

$$\frac{d\sigma}{dt}(s_1, t) / \frac{d\sigma}{dt}(s_2, t) = \left(\frac{s_1}{s_2} \right)^{2\alpha(t)-2}$$

Thus, measuring the differential cross sections at fixed t and various energies, one can determine $\alpha(t)$ unambiguously. Figures 17 and 18 (pages 55 and 56) show the behavior of $\alpha_{\text{eff}}^{\pm}(t)$ for the elastic $\pi\pi$ scattering cross section^{5,6,7,24,26} in the region of negative t .

No data for $\theta_{\text{c.m.}} > 120^\circ$ have been used. For comparison, we have drawn the trajectory $\alpha(t) = 0.5 + t$.

One remarkable feature of the effective trajectories is that, as t becomes more negative, both $\alpha_{\text{eff}}^{\pm}(t)$ continue to fall off quite sharply with t , although one might expect an effective trajectory to flatten out as cuts become more important. The exchange of two Regge poles having trajectories $\alpha_i(t)$, $i = 1, 2$, implies a cut with the effective trajectory given by

$$\alpha_c(t) = \alpha_1(0) + \alpha_2(0) - 1 + \frac{\alpha_1' \alpha_2'}{\alpha_1' + \alpha_2'} t$$

where

$\alpha_1' \alpha_2' / (\alpha_1' + \alpha_2')$ is always less than α_1' and α_2' , provided that none of them is greater than unity.

For $|t| < 1.0 \text{ (GeV/c)}^2$, $\alpha_{\text{eff}}^{\pm}(t)$ is quite flat indicating that Pommeranchuk trajectory is dominant in this region. For $1.0 \leq |t| \leq 2.5 \text{ (GeV/c)}^2$, $\alpha_{\text{eff}}^{\pm}(t)$ come close to the extrapolated straight line P' , ρ trajectories indicating the importance of one or both of these trajectories in this t -range. Beyond $t = 2.5 \text{ (GeV/c)}^2$, the $\alpha_{\text{eff}}^{\pm}(t)$ trajectories begin

Figure 17

Figure 18

to diverge: $\alpha_{\text{eff}}^+(t)$ drops below the linear trajectory shown while $\alpha_{\text{eff}}^-(t)$ appears to level off. As was suggested by Chu and Hendry, this unusual structure can be explained in terms of the interference between two Regge contributions, one of which dominates over the other for an intermediate range of energies²⁷. Schematically writing

$$\frac{d\sigma}{dt}(\pi^\pm p) \sim \frac{1}{s^2} |s^{\alpha_1(t)} \pm e_\pm(t) s^{\alpha_2(t)}|^2$$

with $e_\pm$ small, $e_\pm$ being the relative strength of the two contributions, one can obtain splitting of the $\alpha_{\text{eff}}^\pm(t)$ if $\alpha_2(t)$ lies above $\alpha_1(t)$. Thus in our case, this can be realized by having a dominant $P'(p)$ pole interfere with a higher lying $\nu(P')$ pole or cut. Such a model would imply that, at energies above this intermediate region of $\text{Flab.} = 2.5$ to 8 GeV/c , $\alpha_{\text{eff}}^+(t)$ and $\alpha_{\text{eff}}^-(t)$ would again become equal in this t -range, coinciding with the higher trajectory or cut. We also evaluated the effective trajectories, $\alpha_{\text{eff}}^\pm(t)$ using the expression

$$\frac{d\sigma}{dt} = \frac{1}{P_1^2} f(t) v^{2\alpha(t)}$$

where $v = \frac{s-u}{2}$ and P_1 is the pion laboratory momentum²². They are plotted in Figures 19 and 20 (pages 58 and 59) and the trajectories show the same characteristics as those in Figures 17 and 18 (pages 55 and 56).

Figure 19

Figure 20

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