DOORWAY STATES AND INELASTIC ELECTRON SCATTERING

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We present below a discussion of some simple properties of transition densities as they appear in e.g. inelastic electron scattering experiments. We introduce and describe the concept of doorway states¹ and the nature of their participation in the excitation process; derive sum rules that impose limitations on the properties of these states and, finally, show how approximations on the number of participating doorways can lead to simple predictions which can be compared with experimental results.

In order to define our problem we introduce a density operator

$$\rho(\vec{x}) = \sum_{i=1} c_i \delta(\vec{x} - \vec{r}_i) \quad (1)$$

and consider the function

$$\begin{aligned} g(\vec{x}; \vec{y}) &= \langle 0 | \rho^\dagger(\vec{x}) \rho(\vec{y}) | 0 \rangle \\ &= \sum \rho_{no}^*(\vec{x}) \rho_{no}(\vec{y}) \end{aligned} \quad (2)$$

The symbol c_i will be left unspecified for the time being, it can be $c_i = 1$ or $(1 \pm \tau_3(i))/2$ or $\tau_\lambda(i)$ (the spherical components of the isospin vector $\vec{t}(i)$). The states n appearing in $\rho_{no}(\vec{x}) = \langle n | \rho(\vec{x}) | 0 \rangle$ are exact eigenstates of the Hamiltonian. They can, as suggested by the one-body nature of ρ , be replaced in the sum of Eq. (2) by a set of simpler (but non-stationary)

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densities (for a given nucleus and multipolarity) are proportional to each other. b) The common shape $\rho(\vec{x})$ is given by Eq. (4a) which, however, is in general not easy to obtain. c) The amplitude s_n^{on} is determined by the long-wave-length matrix element M_{on} through Eq. (6) if the r.h.s. of Eq. (4b) is known which in general is not the case. We note also that a) The usefulness of knowing the right-hand-sides of Eqs. (4a,b) is obviously independent of the validity of Eq. (7), b) If the DDA is not valid Eq. (4a) provides an interesting limit for the properly weighted sum of all the transition densities, c) It is possible that Eq. (7) may be valid but the operator M may not happen to be as familiar a quantity as e.g. the radiative-transition operator and d) The limits of Eqs. (4a,b) which in general are available only through model calculations (e.g. in the harmonic-oscillator shell model) become quite trivial in one case, namely when $\epsilon_1 = \tau_{+1}(1)$ and $M = T$, the operator that transforms neutrons into protons. The state $|s\rangle$ is in this case the analog of the ground state $|0\rangle$, the r.h.s. of Eq. (4b) is equal to T , the total isospin of $|0\rangle$, while that of Eq. (4a) is proportional to the isovector density of the ground state of the system. We will not discuss here the possible usefulness and the well known limitations of the DDA in this case but will rather restrict ourselves to isoscalar excitations i.e. assume from now on $\epsilon_1 = 1$. The normalization of Eq. (1) is then

$$\int d^3x \rho(\vec{x}) = A \quad (8)$$

the mass number of the nucleus. We would like now to proceed with an attempt to find an alternative to the program outlined above that does not have the difficulty presented by our lack of knowledge of Eq. (4). We first notice, however, that if we consider the function

$$\begin{aligned} t_E(\vec{x}; \vec{y}) &= \langle 0 | \rho^+(\vec{x}) \frac{1}{E^+ - H} \rho(\vec{y}) | 0 \rangle \\ &= \sum_n \rho_{\text{no}}^*(\vec{x}) \frac{1}{E^+ - E_n} \rho_{\text{no}}(\vec{y}) \end{aligned} \quad (9)$$

take its imaginary part (the numerator can be shown to be real by e.g. time-reversal invariance) and integrate over all energies E , we simply reproduce Eq. (2). Since again only the one-particle-one-hole components of $|n\rangle$ will contribute in ρ_{no} , we introduce operators Λ , $\bar{\Lambda}$ and a reduced Hamiltonian $\mathcal{H}$

$$\Lambda = \sum_{ph} |ph\rangle \langle ph| ; \quad \bar{\Lambda} = 1 - \Lambda \quad (10)$$

$$\mathcal{H} = \Lambda [H + H \frac{\bar{\Lambda}}{E^+ - \bar{\Lambda} H \bar{\Lambda}} H] \Lambda \quad (11)$$

and rewrite Eq. (9) as

$$\begin{aligned} t_E(\vec{x}; \vec{y}) &= \langle 0 | \rho^+(\vec{x}) \frac{\Lambda}{E^+ - \mathcal{H}} \rho(\vec{y}) | 0 \rangle \\ &= \sum_d \rho_{do}^*(\vec{x}) \frac{1}{E^+ - \mathcal{E}_d} \rho_{do}(\vec{y}) \end{aligned} \quad (12)$$

where $\mathcal{E}_d$ and $|d\rangle$ are the energy dependent eigenvalues and eigenfunctions of $\mathcal{H}$ in the particle-hole space. The states $|d\rangle$ are again the appropriate doorway excitations but their detailed structure is determined by the dynamics of the system. In particular, their coupling to complicated multiparticle excitations, represented by the second term in Eq. (11) is formally included in Eq. (12). If we now multiply again by $M(y)$ and integrate we can argue that one of the states $|d\rangle$ will dominate if it contains most of the long-wavelength strength. The r.h.s. of Eq. (10) reduces then to a single term and from this it is easy to show that the approximation of Eq. (7) can be obtained.

In addition to just reproducing the results obtained previously, we may at this point consider other possibilities e.g. before performing the energy integration we can multiply by a power of E (the first is obviously the simplest) and then integrate. The result is in the familiar form of an energy weighted sum rule i.e.

$$\begin{aligned} f(\vec{x}; \vec{y}) &\equiv -\frac{1}{\pi} \int_0^\infty E dE \operatorname{Im} t_E(\vec{x}; \vec{y}) \\ &= \sum_n E_n \rho_{no}(\vec{x}) \rho_{no}(\vec{y}) \\ &= \langle 0 | \rho(\vec{x}) [H, \rho(\vec{y})] | 0 \rangle \\ &= \frac{1}{2} \langle 0 | [\rho(\vec{x}), [H, \rho(\vec{y})]] | 0 \rangle \end{aligned} \quad (13)$$

The last step in this equation took advantage of the Hermiticity of the isoscalar density $\rho(x)$ and is not possible for e.g. some isovector densities. The relations

$$f(\vec{x}; \vec{y}) = f^*(\vec{x}; \vec{y}) = f(\vec{y}; \vec{x}) \quad (14)$$

follow from Hermiticity and time reversal invariance.

The double commutator in Eq. (13) is now easy to evaluate if the interaction part of H is assumed to commute with ρ (the isoscalar nature of ρ helps us avoid the familiar exchange-current difficulties encountered with isovector operators). The result then is

$$f(\vec{x}; \vec{y}) = \frac{\hbar^2}{2m} \nabla_x \cdot \nabla_y (\delta(\vec{x} - \vec{y}) \rho_{oo}(\vec{x})) \quad (15)$$

which is exceptionally simple since it only involves the (isoscalar) ground-state density $\rho_{oo}(x)$ and from which all other (isoscalar) energy weighted sum rules can be obtained by single or double integration. We may for example consider the familiar operator

$$\begin{aligned} Q(lm) &= \sum_i r_i^l Y_{lm}(\Omega_i) \\ &= \int d^3x x^l Y_{lm}(\Omega_x) \rho(x) \end{aligned} \quad (16)$$

which is a more precise version of what was earlier denoted by M . Performing the appropriate integrations we immediately obtain

$$\begin{aligned} \sum_n E_n |Q_{no}(lm)|^2 &= \frac{\hbar^2}{2m} l(2l+1) \int_0^\infty dx x^{2l} \rho_{oo}(x) \\ &= A \frac{\hbar^2}{2m} l(2l+1) \frac{1}{4\pi} \langle r^{2l-2} \rangle \end{aligned} \quad (17a)$$

and

$$\begin{aligned} \sum_n E_n \rho_{no}^*(\vec{x}) Q_{no}(lm) \\ = - \frac{\hbar^2}{2m} l x^{l-1} \frac{d\rho_{oo}(x)}{dx} Y_{lm}(\Omega_x) \end{aligned} \quad (17b)$$

We have for simplicity considered the ground state of an even nucleus and assumed ρ_{oo} to be spherically symmetric. In addition to the rather familiar limit for the energy-weighted

strength given by Eq. (17a), we have thus obtained a sum rule for the transition densities. In contrast to the r.h.s. of Eq. (4a), this result, Eq. (17b), is quite simple and familiar. If at this point we introduce again the single doorway dominance approximation we find that the common shape of the $\rho_{0n}^{(x)}$ i.e., $\rho_{0n}^{(x)}$ is now proportional to the r.h.s. of Eq. (17b). It is interesting to notice that this result gives transition densities identical to those previously obtained through specific nuclear models e.g. that of Tassie³ (based on hydrodynamical considerations) or of Ferrell and Visscher⁴ (coherent particle-hole excitations and harmonic oscillator wave functions). We have, therefore, demonstrated that the results of these models can be obtained simply by the DDA imposed on the energy weighted sum rule. We should emphasize, however, that the r.h.s. of Eq. (17b) being a properly weighted sum of all transition densities would retain its interest even if it failed to reproduce accurately the transition densities of individual excited states.

The above discussion can be extended easily to the case where instead of the operator Q one introduces

$$Q'(lm) = \sum_i r_i^{l+2} Y_{lm}(\Omega_i) \quad (18a)$$

$$Q''(lm) = \sum_i r_i^{l+4} Y_{lm}(\Omega_i) \quad (18b)$$

etc. One obtains for example

$$\sum_n E_n |Q'_{no}(lm)|^2 = \frac{\hbar^2}{2m} [(l+2)^2 + l(l+1)] \frac{A}{4\pi} \langle r^{2l+2} \rangle \quad (19a)$$

and

$$\begin{aligned} \sum_n E_n \rho_{no}^*(x) Q'_{no}(lm) &= -\frac{\hbar^2}{2m} (2(2l+3) x^l \rho_{00}(x) \\ &+ (l+2) x^{l+1} \frac{d}{dx} \rho_{00}(x)) Y_{lm}(\Omega_x) \end{aligned} \quad (19b)$$

$$= -\frac{\hbar^2}{m} \nabla_x \cdot (\vec{x} \rho_{00}(\vec{x})) \quad \text{for } l=0. \quad (19c)$$

Since the ratio of the matrix elements of Q' and Q is simply the transition radius,⁵ we may consider the "mixed" sum rule

$$\sum_n E_n Q_{no}^*(lm) Q'_{no}(lm) = \frac{\hbar^2}{2m} l(2l+3) \frac{A}{4\pi} \langle r^{2l} \rangle \quad (20)$$

and define an average transition radius $\langle R_{tr}^2(l) \rangle$ as the ratio of the left-hand-sides of Eq. (20) and (17a). We then obtain

$$\langle R_{tr}^2(l) \rangle = \frac{(2l+3)\langle r^{2l} \rangle}{(2l+1)\langle r^{2l-2} \rangle}, \quad (l \neq 0) \quad (21)$$

which if the DDA is assumed will also give the transition radius of each individual excitation. For monopole transitions this result is not valid (Eqs. 17(a,b) are identically zero in this case) and the natural definition of $\langle R_{tr}^2 \rangle$ is the ratio of sum rules involving Q'' and Q' . The result is

$$\begin{aligned} \langle R_{tr}^2(l=0) \rangle &= 2 \frac{\langle r^4 \rangle}{\langle r^2 \rangle} \\ &= 1.43 \langle R_{tr}^2(l=2) \rangle \end{aligned} \quad (22)$$

Instead of continuing the discussion of transition densities we shall at this point turn our attention to their form factors. With

$$\rho(\vec{q}) = \int d^3x e^{i\vec{q} \cdot \vec{x}} \rho(\vec{x}) \quad (23)$$

we find

$$\begin{aligned} f(\vec{q}'; \vec{q}) &= \langle 0 | \rho^\dagger(\vec{q}') [H, \rho(\vec{q})] | 0 \rangle \\ &= \sum_n E_n \rho_{no}^*(\vec{q}') \rho_{no}(\vec{q}) \\ &= \frac{\hbar^2}{2m} \vec{q} \cdot \vec{q}' \rho_{oo}(\vec{q} - \vec{q}') \end{aligned} \quad (24)$$

a result also discussed recently by J. V. Noble.⁶ This is a limiting case (with recoil neglected) of the more general relation

$$\begin{aligned} \sum_{n \neq 0} (E_n + \frac{\hbar^2}{2mA} \vec{q} \cdot \vec{q}') \rho_{no}^*(\vec{q}') \rho_{no}(\vec{q}) \\ = \frac{\hbar^2}{2m} \vec{q} \cdot \vec{q}' (\rho_{oo}(\vec{q} - \vec{q}') - \frac{1}{A} \rho_{oo}(\vec{q}') \rho_{oo}(\vec{q})) \end{aligned} \quad (24')$$

which is obtained when coordinates relative to the center of mass rather than absolute coordinates are used in the evaluation of the double commutator. We notice that for large values of q (and $q' = q$) the correction in the l.h.s. may be important but that in the r.h.s. is negligible. For $\vec{q}' = \vec{q} = 0$ on the other hand it is the second term on the l.h.s. that is of interest reducing to zero the term in the bracket i.e., reproducing the familiar selection rule for isoscalar E1 transitions. For simplicity in what follows, we will ignore Eq. (24') and present results consistent with Eq. (24).

We define form factors by

$$F_{no}^J(q) = [4\pi(2J+1)]^{1/2} \int d^3x j_J(qx) Y_{Jm}(\hat{q}_x) \rho_{no}(\vec{x})$$

$$\xrightarrow{q \rightarrow 0} \frac{q^J}{(2J+1)!!} M_{no}^J \quad (25)$$

with

$$F_{el}(q) \equiv F_{oo}^0(q); \quad F_{el}(0) = A \quad (26)$$

$$M_{no}^J \equiv \sqrt{4\pi} \langle n || Q(J) || 0 \rangle \quad (27)$$

and similar expressions defining M^i in terms of $Q^i(lm)$ etc.

The most general sum rule for the form factors can be derived easily from the results obtained above and is

$$\sum_n^{(J)} E_n F_{no}^J(q') F_{no}^J(q)$$

$$= \frac{\hbar^2}{2m} (2J+1) 4\pi \int_0^\infty dx \rho_{oo}(x) \{ x^2 \frac{dj_J(q'x)}{dx} \frac{dj_J(qx)}{dx}$$

$$+ J(J+1) j_J(q'x) j_J(qx) \} \quad (28)$$

where the summation on the left is restricted to states of fixed angular momentum J . Although the r.h.s. of Eq. (28) involves the ground-state expectation value of a one-body operator and is in principle not too difficult to evaluate for a given $\rho_{oo}(\vec{x})$, it does not have the physical simplicity of some of the results we exhibited earlier. Simpler expressions may be obtained from it, however, in special cases e.g. when $q' \rightarrow 0$.

We then find

$$\begin{aligned} \sum_n^{(J)} E_n M_{no}^J F_{no}^J(q) \\ = \frac{\hbar^2}{2m} (2J+1) J q \int d^3x \rho_{oo}(x) j_{J-1}(qx) x^{J-1} \\ = (-1)^{J+1} \frac{\hbar^2}{2m} J (2J+1) q^J \left(\frac{1}{q} \frac{d}{dq} \right)^{J-1} F_{el}(q) \end{aligned} \quad (29)$$

i.e., the limit of this sum is simply related to the elastic-scattering form factor. In the single doorway approximation all the form factors become proportional to each other. In this case we can write

$$F_{no}^J(q) = \frac{q^J}{(2J+1)!!} M_{no}^J G^J(q), \quad (\text{DDA}) \quad (30)$$

with $G^J(0) = 1$

$$\text{and } A \langle r^{2J-2} \rangle G^J(q) = (-1)^{J-1} (2J-1)!! \left(\frac{1}{q} \frac{d}{dq} \right)^{J-1} F_{el}(q). \quad (31)$$

For the quadrupole case the form factor reduces to

$$F_{no}^{J=2}(q) = - M_{no}^{J=2} [5 A \langle r^2 \rangle]^{-1} q \frac{d}{dq} F_{el}(q) \quad (32)$$

which has been shown⁷ to agree quite well with experimental data in a number of cases.

A sum rule involving M' and F can also be written down. The right hand side can again be expressed in terms of derivatives of the elastic form factor, but since the general result is somewhat more complicated than Eq. (29) we present here only the special case of monopole excitations:

$$\sum_n^{(J=0)} E_n M_{no}' F_{no}^0(q) = \frac{\hbar^2}{2m} q \frac{d}{dq} F_{el}(q). \quad (33)$$

It is interesting to notice that the functional dependence of the r.h.s. on q is identical to the one found for the quadrupole case in Eq. (32). A number of experimentally available monopole form factors have been found⁷ to compare well with the

shape indicated by Eq. (33). It is clear that the DDA of Eq. (30) (and a similar expression for monopoles) should be a reasonable approximation for relatively small values of q . For very small momentum transfers it reduces to an identity and we can easily assume that there must exist a region of q within which Eq. (30) is still valid. For large momentum transfers, however, there is no reason to expect that the DDA based on the long-wavelength collective modes will be a meaningful approximation. It is not easy to determine a priori the critical value of q , beyond which the DDA will fail, as a function of e.g. the mass number of the nucleus or of the multipolarity of the transition. We can of course obtain this information by detailed comparisons with experimental results. We shall, instead, present here a theoretical argument which illustrates that Eq. (30) leads to inconsistencies in the high momentum-transfer region: If we insert the approximate expression of Eq. (30) into Eq. (28) and let $q' = q$ we find

$$\sum_n^J E_n |F_{no}^J(q)|^2 = \sum_n E_n |M_{no}^J|^2 \left\{ \frac{q^3}{(2J+1)!!} G^J(q) \right\}^2$$

which after some oscillations tends to zero for large values of q . From Eq. (28), however, we obtain

$$\sum_n^J E_n |F_{no}^J(q)|^2 \xrightarrow{q \rightarrow \infty} \frac{\hbar^2}{4m} (2J+1) A \langle r^{-2} \rangle \quad (34)$$

i.e. the energy-weighted sum of the squared form factors becomes a non-vanishing constant when $qR \gg J$. This sum on the other hand is quite consistent with Eqs. (30) and (31) for small values of q .

Probably the most interesting problem at this point is to determine a generalized version of Eq. (30) which can be used for momentum transfers larger than those required for the validity of the DDA. The natural answer is to use a combination of Eq. (30) and the r.h.s. of the sum rule involving M' and $F(q)$. Actually it is preferable to use instead of M' the quantity $\bar{M}$ defined by

$$\bar{M}_{no}^J = M_{no}^{'J} - \langle R^2 \rangle_{tr}(J) M_{no}^J \quad (35)$$

This quantity by definition will give

$$\sum_n^{(J)} E_n \bar{M}_{no}^J M_{no}^J = \langle 0 | \bar{M}^J H M^J | 0 \rangle = 0 \quad (36)$$

i.e. the nodes created by M and $\bar{M}$ are diagonal. For simplicity, instead of writing down the most general result we will present here the special cases obtained for a uniform distribution of $\rho_{00}(x)$. The results are

$$\sum_n^{(J)} E_n M_{no}^J F_{no}^J(q) = 3A \frac{\hbar^2}{2m} J(2J+1) R^{J-2} j_J(qR) \quad (37)$$

$$\sum_n^{(J)} E_n \bar{M}_{no}^J F_{no}^J(q) = 3A \frac{\hbar^2}{m} (2J+1) R^J j_{J+2}(qR) \quad (38)$$

where R is the nuclear radius. The expression in Eq. (37) is of course quite familiar except that, as mentioned earlier, it appears now as the limit of a sum rule rather than as a model for individual form factors.

For values of q not too far above the region of validity of the DDA it should be fairly trivial to obtain good fits of the data with a proper combination of Eqs. (37) and (38) (or their more general versions for a non-uniform distribution). Whereas, however, the DDA form factor was uniquely determined from the radiative-transition matrix element, the generalized version will require fitting the transition radius as well. If this procedure is carried out for a number of cases, namely several states of the same J in a given nucleus, the result will hopefully provide useful information about the second doorway state i.e. the mode generated by the quantity $\bar{M}$.

We may note in conclusion that the plane-wave Born approximation implied by the use of form factors in the last part of this discussion is not a real limitation since for each result there exists an equivalent expression for the transition densities. A real limitation, however, is due to the fact that we considered here only the isoscalar part of the Coulomb form factor. Similar considerations involving isovector excitations as well as those induced by the current terms are clearly necessary.

Most of the results presented here appear in the Ph.D. thesis of T. J. Deal. The author wishes to thank Mr. Deal for many stimulating discussions.

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