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MASTER

A UNIQUENESS THEOREM CONCERNING MOMENT DISTRIBUTIONS*

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ABSTRACT

Let the random variable Y have the density function $p(y)$. Let $f(x) = e^{tx} p(x) / E(e^{tx})$ where $E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} p(x) dx$, then $f(x)$ is a density function. The random variables associated with these density functions have the same variances if and only if $p(y)$ is a normal density function.

1. INTRODUCTION AND NOTATION

This paper presents a theorem which is useful in the study of particle size distributions. While certain aspects of this theorem seem to be known to workers in the field of small particle statistics, proofs of these properties do not appear to be readily available in the literature. The following notation shall be used in this paper.

$N(x; \mu, \sigma^2)$ A random variable having a normal density function with mean μ and variance σ^2 .

$\phi(t) = E(e^{tx})$ The moment generating function of a random variable.

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$\phi'(t), \phi''(t)$ The 1st and 2nd derivative of the moment generating function.

We shall assume throughout the paper that the moment generating function exists and there is a unique density function associated with this moment generating function and its moments are given by $\phi^{(i)}(0)$; the i th derivative of the moment generating function evaluated at zero.

2. THEOREM

If $p(y)$ is a density function such that its moment generating function exists, then $f(x) = \frac{e^{tx} p(x)}{E(e^{tx})}$ is a density function. Also, the variance of the random variable associated with f is equal to the variance of the random variable associated with p if and only if $p(y)$ is $N(y, \mu, \sigma^2)$.

Proof: That f is a density function is obvious since

$$\int_{-\infty}^{\infty} \frac{e^{tx} p(x) dx}{E(e^{tx})} = \frac{E(e^{tx})}{E(e^{tx})} = 1$$

If p is $N(x; \mu, \sigma^2)$ then

$$f(x) = e^{tx} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} / e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

$$f(x) \text{ is } N(x; \mu + \sigma^2 t, \sigma^2) \quad [1].$$

Hence, f and p have the same variance if p is $N(x; \mu, \sigma^2)$.

If the variance of the random variable associated with f is equal to the random variable associated with p , we have

$$\int_{-\infty}^{\infty} x^2 f(x) dx - \left[\int_{-\infty}^{\infty} x f(x) dx \right]^2 = \phi''(0) - [\phi'(0)]^2 = \sigma^2.$$

Consequently

$$\frac{\varphi''(t)}{\varphi(t)} - \left[\frac{\varphi'(t)}{\varphi(t)} \right]^2 = \sigma^2$$

or,

$$\varphi''(t) \varphi(t) - [\varphi'(t)]^2 - \sigma^2 [\varphi(t)]^2 = 0.$$

The solution of this differential equation is

$$\varphi(t) = e^{C_1 t + C_2 t^2 + \frac{\sigma^2 t^2}{2}}.$$

Since

$$\varphi(0) = 1, \quad C_1 = 0$$

$$\varphi'(0) = \mu, \quad C_2 = \mu$$

$$\varphi(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

which is the moment generating function for $N(x; \mu, \sigma^2)$. Q.E.D.

3. EXAMPLE

In small particle statistics the assumption is often made that the basic distribution of particle diameters is log normal with mean μ and variance σ^2 . If this is true and if one makes some assumptions concerning the shape of the particles such as cubical, spherical, etc., it follows that the distribution $e^{tx} p(x) / E(e^{tx})$ where $p(x)$ is $N(x; \mu, \sigma^2)$ is also $N(x; \mu + t\sigma^2, \sigma^2)$. This is sometimes referred to as the moment distribution [1]. For $t = 2$, this would be surface area, $t = 3$ that of particle volume. Thus, if one had two sets of observations, one set based on

the frequency of particles vs diameters and another set based upon weight vs diameters, if the variances of the two sets after the logarithmic transformation were the same, then the assumption of a log-normal distribution for particle size would be justified. Table I gives a set of data obtained on tantalum powder. One sample was used to obtain particle size distribution by microscopic count. The second sample was used to obtain the percent cumulative weights using the Micromerograph. The values of the variances of the two sets are $s_1^2 = .2187$ and $s_2^2 = .1932$. Since these variances are not very different, one would assume that a log-normal distribution can be used to describe the basic distribution of particle sizes. Since in actual practice it is often much easier to obtain weights of particle sizes, rather than counts, one can obtain estimates of μ and σ^2 if the assumption is made that the basic distribution is log-normal. In the above example, since the mean of the log-normal based upon weights is 3.3957 and the variance is .1932, an estimate of μ is given by $\mu = 3.3957 - 3(.1932) = 2.8162$.

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REFERENCES

- [1] Aitchison, J. and Brown, J.A.C., The Lognormal Distribution, Cambridge: At the University Press, 1937, 12.
- [2] Herdan, G., Small Particle Statistics, New York: Academic Press Inc., 1960.

TABLE I.

Microscopic Count			Micromerograph Weights	
midpoint diameter d_1	$\ln d_1$	number of particles n_1	$\ln d_1$	% weight
4.5	1.504077	20	2.35138	1.50
7.5	2.014903	47	2.60269	5.23
10.5	2.351375	80	2.80336	6.28
13.5	2.602690	165	2.97042	10.50
16.5	2.803360	187	3.11352	16.71
19.5	2.970415	150	3.45000	18.80
22.5	3.113515	129	3.62434	13.39
25.5	3.238678	73	3.77276	10.08
28.5	3.349904	49	3.90197	7.41
31.5	3.449997	36	4.01638	5.10
34.5	3.540959	32	4.11900	2.89
37.5	3.624341	31	4.21210	2.11
40.5	3.701302	21		
43.5	3.772761	13		
46.5	3.839452	9		
49.5	3.901973	4		
52.5	3.960813	4		
55.5	4.016383	5		
58.5	4.069027	3		

$$\bar{x}_1 = \frac{\sum (\ln d_1) n_1}{N}$$

$$= 2.9199$$

$$s_1^2 = .2187$$

$$\bar{x}_2 = 3.3957$$

$$s_2^2 = .1932$$