Conf-730713--1

TITLE: THE COMPARATIVE EFFICIENCY OF CERTAIN FINITE ELEMENT
AND FINITE DIFFERENCE METHODS FOR A HYPERBOLIC PROBLEM

AUTHOR(S): Blair Swartz and Burton Wendroff

SUBMITTED TO: The Proceedings of the Conference on the Numerical
Solution of Differential Equations, University of
Dundee, Dundee, Scotland, July 3-6, 1973

By acceptance of this article for publication, the publisher recognizes the Government's (license) rights in any copyright and the Government and its authorized representatives have unrestricted right to reproduce in whole or in part said article under any copyright secured by the publisher.

The Los Alamos Scientific Laboratory requests that the publisher identify this article as work performed under the auspices of the U. S. Atomic Energy Commission.

los alamos
scientific laboratory
of the University of California
LOS ALAMOS, NEW MEXICO 87544

NOTICE

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

MASTER

UNITED STATES
ATOMIC ENERGY COMMISSION
CONTRACT W-7405-ENG. 36

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

THE COMPARATIVE EFFICIENCY OF CERTAIN FINITE ELEMENT AND
FINITE DIFFERENCE METHODS FOR A HYPERBOLIC PROBLEM

Blair Swartz and Burton Wendroff
Los Alamos Scientific Laboratory of the
University of California
Los Alamos, New Mexico 87544

ABSTRACT

The notion of the resolving power of approximation methods, i.e., the number of intervals (or function values) per wave length necessary to attain a preassigned error when approximating a given frequency permits evaluation of the comparative efficiency of three types of space differencing used in setting up differential-difference equations for the hyperbolic problem $u_t = u_x$. The three types considered are the high order centered differences of Kreiss and Olliger, the extra high order spline-Galerkin schemes discovered by Thomeé, and the quite high order implicit centered difference schemes communicated to us by Kreiss. Trapezoidal, leap-frog (midpoint rule), and 4th order Padé time discretizations are introduced. The work necessary to solve each full discretization is minimized, for given error requirements, by balancing the number of intervals per wave length against the number of intervals per time period. The resulting data is used to compare the efficiency of these finite element and finite difference methods. We also discuss the efficacy of smooth spline interpolation of the approximations arising from the difference schemes.

THE COMPARATIVE EFFICIENCY OF CERTAIN FINITE ELEMENT AND
FINITE DIFFERENCE METHODS FOR A HYPERBOLIC PROBLEM

Blair Swartz* and Burton Wendroff†
Los Alamos Scientific Laboratory of the
University of California
Los Alamos, New Mexico 87544

1. Introduction.

Numerical analysts are ultimately concerned with the efficiency of the computational schemes they devise.

One way to evaluate the relative efficiency of numerical schemes for evolutionary partial differential equations is to carefully program them for a computer and then compare running times with the observed precision; see, e.g., Culham and Varga [1].

A second approach, as observed by Douglas [2], involves the use of the first few terms of the truncation error. That is, the error and the total computational work are assumed given, respectively, by

$$e = c_p (\Delta x)^p + c_q (\Delta t)^q, \quad W = c_W / (\Delta x \Delta t),$$

with coefficients here assumed independent of Δx and Δt . For given e , W is minimized. We find $c_p (\Delta x)^p = q c_q (\Delta t)^q$ at the minimum. The resulting

$$(1.1) \quad W_{\min} = c_W (c_p/q)^{1/p} (c_q/p)^{1/q} [(q+p)/e]^{1/p+1/q}$$

can be used to compare various schemes. Strang [8] and Walsh [14] explore the use of this notion in the design of difference schemes.

A third method of comparing efficiencies is to devise a characteristic but elementary model problem and explore how well a class of its solutions is approximated by those of the difference schemes; e.g. Thompson [12], Fromm [4], Kreiss and Oliger [7]. The periodic hyperbolic problem

$$(1.2) \quad \partial u / \partial t = \partial u / \partial x, \quad u(0, t) = u(1, t), \quad u(x, 0) = \exp(2\pi i \omega x),$$

(ω integral) has $u = \exp[2\pi i \omega(x+t)]$ as its solution. Most approximating algorithms possess similar solutions. Using the notation

$$v = (v_0, \dots, v_J)^T, \quad v_j \approx u(x_j), \quad x_j \equiv j/J \equiv jh,$$

*Work supported by the U. S. Atomic Energy Commission.

†Work supported partly by the U. S. Atomic Energy Commission and partly by the National Science Foundation.

the differential-difference approximation to (1.2)

$$dv/dt = Sv, \quad v_0(t) = v_j(t), \quad v_j(0) = \exp(2\pi i \omega x_j)$$

typically is satisfied by

$$[v_\omega(t)]_j = \exp[2\pi i \omega(x_j + ct)] ;$$

where the velocity $c = c(\omega, h)$ is explicitly computable from the difference operator S . That is, the component of each frequency in the initial data travels undiminished, but at its own speed.

As an example, let $(Sv)_k = (D_0 v)_k \equiv (v_{k+1} - v_{k-1})/(2h)$. Then

$$dv_\omega/dt = (2\pi i \omega c)v_\omega, \quad \text{while } Sv_\omega = [i \sin(2\pi \omega h)/h] v_\omega.$$

The difference in phase angle between v_ω and u at the mesh points is

$$2\pi(1-c)\omega t = \epsilon \cdot P, \quad P \equiv \omega t = \# \text{ time periods computed.}$$

ϵ , the phase error per time period, is given in this example by

$$\begin{aligned} \epsilon &= 2\pi \left[1 - i \sin(2\pi \omega h) / (2\pi i \omega h) \right] \\ &= 2\pi \left[1 - b(\theta) / (i\theta) \right], \quad \theta \equiv 2\pi \omega h. \end{aligned}$$

$b(\theta)$ is the amplification factor, or symbol, of hD_0 .

Suppose an implicit approximation to $\partial/\partial x$ is used:

$$(\partial/\partial x)v \approx (A^{-1}Bv)/h,$$

where each row of A is the translate of the previous one and similarly for B :

$$A_{jk} = \alpha_{k-j}, \quad B_{jk} = \beta_{k-j}.$$

If we use the semi-discretization

$$(1.3) \quad A dv/dt = Bv/h, \quad v_0(t) = v_j(t), \quad v(0) = v_\omega(0) ;$$

then the phase error per period is

$$\epsilon = 2\pi \left\{ 1 - b(\theta) / [i\theta a(\theta)] \right\},$$

where a and b are the amplification factors of A and B ,

$$(1.4) \quad a(\theta) = \sum \alpha_j e^{ij\theta}, \quad b(\theta) = \sum \beta_j e^{ij\theta},$$

(The role of rational trigonometric approximations of θ is here clear, as ϵ is simply the relative error in such an approximation.)

We can and shall express this phase error per period in terms of a more picturesque variable, namely the number of intervals per wavelength,

$$N \equiv 1/(\omega h) : \\ (1.5) \quad \epsilon = 2\pi - Nb(2\pi/N)/[ia(2\pi/N)] .$$

For a given ϵ , each competing difference scheme will require a certain N to yield that phase error per period. We may regard those needing smaller N as more efficient (for that error tolerance ϵ).

2. Three semi-discretizations.

We shall compare, in this fashion, three differential-difference schemes (1.3).

The first spatial discretization is the family of high-order explicit (i.e. $A = I$) centered difference schemes presented in Kreiss and Oliger [7]. They will be represented here simply by the amplification factor of B :

$$b_{C-d}(m, \theta) \equiv i \sin \theta \sum_{v=0}^{m-1} [v! 2 \sin^v(\theta/2)]^2 / (2v+1)! .$$

Each scheme may be reconstituted from its symbol as follows: with $D_0 v$ as before and with $D_+ D_- v \equiv [v(x+h) - 2v(x) + v(x-h)]/h^2$, replace $i \sin \theta$ by D_0 and $\sin^2(\theta/2)$ by $-h^2 D_+ D_- / 4$. From the fact [6, Section 9.121, relation 14] that

$$(2.1) \quad \theta / \sin \theta = \arcsin \tau / (\tau \sqrt{1-\tau^2}) = \sum_{v=0}^{\infty} (v! 2\tau^v)^2 / (2v+1)!$$

if $\tau = \sin(\theta/2) < 1$, we may conclude that the relative error in using b_{C-d} as an approximation to $i\theta$ is $O(\theta^{2m})$. Furthermore, the associated phase error (1.5) is

$$(2.2) \quad \epsilon_{C-d}(m, N) \equiv N \sin(2\pi/N) \sum_{v=m}^{\infty} [v! 2 \sin^v(\pi/N)]^2 / (2v+1)! .$$

The second approximations to $\partial/\partial x$ we consider are the difference schemes Thomée has associated with the matrices involved in Galerkin's method (in space) using smooth splines (discussed in detail elsewhere in this volume by Thomée [10] and Wendroff [15]). The matrices arise by asking that u be approximated by W , of the

form

$$W(x, t) = \sum_{j=1}^J v_j(t) \varphi_j(x),$$

determined by requiring the error in satisfying (1.2) be $\perp$ to the span of $\{\varphi_k\}_{k=1}^J$:

$$\langle W_t - W_x, \varphi_k \rangle = 0, \quad 1 \leq k \leq J.$$

The φ_j are selected as $\varphi_j(x) \equiv \varphi[(x - x_j)/h]$, where φ is Schoenberg's basic, smooth B-spline of order m (degree $m-1$) [9]. The small support of the φ_j , centered on x_j , means that

$$A_{ij} \equiv \langle \varphi_i, \varphi_j \rangle / h \equiv \alpha_{j-i}, \quad B_{ij} \equiv \langle \varphi_i, d\varphi_j/dx \rangle \equiv \beta_{j-i}$$

have bandwidth $2m-1$. As $\varphi_i(x_j) \neq \delta_{ij}$, the v_j are not $W(x_j, t)$. Thomeé observed that if, in the resulting semi-discrete scheme, the v_j 's were taken to be function values v_j (as in (1.3)), then one could prove phase errors at the x_j whose order of accuracy is twice that of the order of global accuracy of best approximation using the same smooth splines. From the known Fourier transform of φ one may find $a(\theta)$ and $b(\theta)$ (1.4) via the Poisson summation formula. The resulting phase error is

$$(2.3) \quad \epsilon_{sp}(m, N) = -2\pi N \sum_{-\infty}^{\infty} v(v + 1/N)^{-2m} / \sum_{-\infty}^{\infty} (v + 1/N)^{-2m} > 0,$$

which is easily shown [10] to be $O(N^{-2m})$, $N \rightarrow \infty$. This order of accuracy is roughly equal to the bandwidth of the matrices (as in central differencing).

The existence of a third class of schemes, implicit central differencing, was recently pointed out to us by Kreiss. Like (2.1) it arises from a trigonometric approximation of $\theta/\sin\theta$; this time from a Padé rather than a Taylor approximate [13, p. 345, p. 380]

$$\begin{aligned} \arcsin \tau / (\tau \sqrt{1-\tau^2}) \approx g_m(\tau) &= \frac{1}{1} - \frac{1 \cdot 2\tau^2}{3} - \frac{1 \cdot 2\tau^2}{5} - \\ &\dots - \frac{(2j-1)2j\tau^2}{(4j-1)} - \frac{(2j-1)2j\tau^2}{(4j+1)} - \dots - \frac{(2m-1)2m\tau^2}{(4m-1)}. \end{aligned}$$

We then have $b_m(\theta)/a_m(\theta) \approx i \sin \theta g_m[\sin(\theta/2)]$, with relative accuracy $O(\theta^{4m})$ in approximating $i\theta$. The case $m=1$ coincides with Thomeé's piecewise linear, tridiagonal, $O(h^4)$ scheme

$$(A v)_j = (v_{j+1} + 4v_j + v_{j-1})/6, \quad (B v)_j/h = (v_{j+1} - v_{j-1})/2h$$

while the second pentadiagonal scheme, $O(h^8)$ accurate, is

$$Av = (1, 16, 36, 16, 1)v/70, \quad Bv/h = (-5, -32, 0, 32, 5)v/(84h).$$

To compare these three families of schemes we tabulate, in Table I(a)-(c), the number of mesh intervals per wavelength, N , required to attain phase errors per period of 10^{-2} and of 10^{-4} . An integer p in the first column means the order of accuracy of the phase error is N^{-p} for large N .

Table I: Intervals per Wavelength for

(a) Spline-Galerkin Differencing (2.3)

Type Spline (Accuracy)	$\epsilon = 10^{-2}$	$\epsilon = 10^{-4}$
Linear (4)	8.7	27
Quad. (6)	4.8	9.7
Cubic (8)	3.6	6.0
Quartic (10)	3.1	4.5
Quintic (12)	2.9	3.8
Sextic (14)	2.7	3.4

(b) Implicit Centered Differencing

Order of Accuracy	$\epsilon = 10^{-2}$	$\epsilon = 10^{-4}$
4	8.7	27
8	3.9	6.7
12	3.1	4.3
16	2.7	3.5
20	2.6	3.1
24	2.5	2.3

(c) Explicit Central Differencing (2.2)

Order of Difference Formula	$\epsilon = 10^{-2}$	$\epsilon = 10^{-4}$
2	64.4	644
4	13.3	42.5
6	7.9	17.3
8	6.0	11.0
12	4.6	7.0
16	4.0	5.5
20	3.6	4.8

(d) Smooth Spline Interpolation (5.1)

Order of Interpolant	$\epsilon = 10^{-2}$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-4}$
4	5.0	8.4	14.6
6	3.4	4.6	6.4
8	3.0	3.6	4.5
10	2.7	3.1	3.7
12	2.6	2.9	3.3
14	2.5	2.7	3.0

The spline differences appear to be more efficient than explicit central differences. For a given order of accuracy, they also have slightly stronger resolving power than the implicit central differences. However, comparing results for the same bandwidth (same position in the Tables I(a), (b)) the improved resolving power of implicit central differencing is apparent.

3. Full space-time discretization.

We now discretize (1.3) in its time variable.

The trapezoidal rule is appropriate for implicit approximations to $\partial/\partial x$:

$$A(w^{n+1} - w^n)/\Delta t = B(w^{n+1} + w^n)/2h, \quad w^0 = v_\omega(0), \quad w_j^n \approx u(jh, n\Delta t) \quad .$$

For this scheme we find

$$w^n = (R)^n w^0, \quad n = t/\Delta t$$

where, with $M \equiv 1/(\omega\Delta t)$ = number of (time) intervals per period,

$$R = \left[a(\theta) + \pi b(\theta)/(M\theta) \right] / \left[a(\theta) - \pi b(\theta)/(M\theta) \right] \\ \equiv \exp(i\sigma_{\text{Trap}}) \quad (\theta \equiv 2\pi\omega h = 2\pi/N) \quad .$$

Since a is real and b is imaginary, σ is real; the scheme is stable. Again different frequencies jump along, undiminished, at their own velocities. We further find, with ϵ given by (1.5), that

$$\tan(\sigma_{\text{Trap}}/2) = \xi/2, \quad \xi \equiv (2\pi - \epsilon)/M = \xi(N, M) \quad .$$

The phase difference, per period per wavelength, between u and w is

$$(3.1) \quad E_{\text{Trap}} \equiv 2\pi - M\sigma_{\text{Trap}} \quad .$$

The fourth-order in time Padé-diagonal scheme,

$$(A^2 - \lambda AB/2 + \lambda^2 B^2/12)w^{n+1} = (A^2 + \lambda AB/2 + \lambda^2 B^2/12)w^n, \quad \lambda \equiv \Delta t/h \quad ,$$

is likewise stable, with pure phase error

$$(3.2) \quad E_{P_4} \equiv 2\pi - M\sigma_4, \quad \tan(\sigma_4/2) = \xi / \left[2(1 - \xi^2/12) \right] \quad .$$

The leap-frog scheme (midpoint rule), $w^{n+1} - w^{n-1} = 2\lambda B w^n$, is more appropriate for explicit centered differencing. Assuming stability, (see Fornberg [3]), we have

$$(3.3) \quad E_{L-f} \equiv 2\pi - M\sigma_{L-f}, \quad \sin \sigma_{L-f} = \xi \quad .$$

In each case we have expressed the phase error, per wavelength per period, as $E = E(N, M)$. To compare various schemes we let d be the computational work per mesh point in space-time; and, for a given E , we attempt to find the pair (N, M) which minimizes

$$W = dMN = \text{the work per wavelength per period.}$$

We assume d is independent of N and M , as other computations associated with FFT ($d = \text{const.} \log \omega N$, a slowly varying function of N) lead to similar comparisons.

Table II displays some of the optimal pairs (N,M) for splines with the midpoint and Padé discretizations.

The leap-frog error (3.3) is not a monotone decreasing function of M and N. The difficulty stems from the cancellation of the space and time parts of the phase error so that indeed, there may be no error at all (e.g. $\partial/\partial x \approx D_0$, $\Delta t/h = 1$). This non-monotonicity makes the constrained minimization difficult and its results unrealistic. To solve this problem we expand (3.3) in $1/M$, drop high-order terms, and make the rest positive:

$$(3.4) \quad \hat{E}_{L-f} \approx \epsilon + (2\pi)^3/(6M^2) .$$

This estimate is identical with the more conventional estimates [10] of $|u-w|$; we use it in constructing Table III. Similar estimates of (3.1), (3.2) [10] change Table II very little.

Table II: (N,M) for Spline-differencing in Space and

Spline Type (Accuracy)	(a) Trapezoidal time differencing (3.1)		(b) 4 th -order Padé time differencing (3.2)	
	$E = 10^{-2}$	$E = 10^{-4}$	$E = 10^{-2}$	$E = 10^{-4}$
Linear (4)	11.4, 55	36, 557	10.4, 7.1	32, 23
Quad. (6)	5.9, 52	12, 524	5.5, 6.7	11.3, 22
Cubic (8)	4.3, 50	7.2, 505	4.1, 6.5	6.8, 21
Quartic (10)	3.6, 49	5.3, 493	3.5, 6.4	5.1, 20.6
Quintic (12)	3.3, 48.1	4.4, 485	3.2, 6.3	4.2, 20.4
Sextic (14)	3.0, 47.6	3.9, 479	2.9, 6.2	3.7, 20.2

Table III: (N,M) for Leap-frog in Time (3.4)

(a) Explicit Centered Space Differencing

Accuracy	$\hat{E} = 10^{-2}$	$\hat{E} = 10^{-4}$
2	91, 91	909, 909
4	17.6, 79	56, 788
6	10.0, 75	21.8, 743
8	7.5, 72	13.5, 720
12	5.5, 70	8.3, 697
16	4.6, 68.8	6.4, 685
20	4.2, 68.1	5.5, 678

(b) Spline-differencing in Space

Spline Type (Accuracy)	$\hat{E} = 10^{-2}$	$\hat{E} = 10^{-4}$
Linear (4)	11.4, 79	36, 787
Quad. (6)	5.9, 73	12, 741
Cubic (8)	4.3, 71	7.2, 714
Quartic (10)	3.6, 69	5.3, 697
Quintic (12)	3.3, 68	4.4, 686
Sextic (14)	3.0, 67.5	3.9, 678

It is possible to approximate E even more simply [10] as

$$E \approx c_p N^{-p} + c_q M^{-q}.$$

The approach leading to (1.1) then yields estimates of the minimizing pairs (N, M) in the tables; estimates which turn out to be accurate except that the estimates of N in the lower parts of the spline tables are up to 25% low.

Computations were done, but are omitted, for implicit centered differencing in space. The remarks at the end of Section 2 apply.

4. Conclusions.

First we make an observation concerning Tables II and III. The only place in the Tables where the typical stability limit, $M > N$, is equalled or violated is in the two locations where the spatial order of accuracy is as low as the temporal order, i.e., the top of Table III (a) and the top right of Table II (b). This is predictable from the relation above (1.1).

We are certainly attracted by the 4th order time differencing, at least for $E=10^{-4}$. We could surely invert the pentadiagonal matrix involved in the upper-right corner of Table II (b) in fewer than the $(909)^2/(32 \times 23) \approx 1200$ operations per mesh point which lets it compete with the diamond difference scheme (Table III (a), upper-right). Indeed, this 4th order piecewise linear Padé scheme competes well with most of the $E=10^{-4}$ column of Table III (a), especially if we contemplate differencing a problem $\partial u/\partial t = \partial(a u)/\partial x + b$ in the form $dv/dt = A^{-1}B(av)/h + b$, so that the main work in the Padé scheme is a back substitution.

More general conclusions are more difficult, and it might be wiser to let the reader form his own. Nevertheless, suppose it takes most of the computational time to simply evaluate the coefficient of the differential equation at the points of the space-time net. Then we may take $d=1$ for all schemes, and compare only products MN . Suppose $E=10^{-2}$. Any leap-frog scheme will have $N > 2$ and $M > [100(2\pi)^3/6]^{1/2} \approx 64$ for an $MN > 128$. This product exceeds all of the products MN in the $E=10^{-2}$ column of Table II (b); the latter schemes would be judged more efficient.

Indeed, coefficient evaluations and storage are both considerations favoring comparison of schemes with the same N . If we use the FFT to evaluate all difference operators and their inverses, then (in many problems) the work per mesh point is the same for all schemes with the same N . For each E (and same N) it is the ratio of the M values which then measures the relative efficiencies of the types of differencing. For example, the cubic spline-trapezoidal scheme requires $N = 4.3$ for $E=10^{-2}$. This N is matched by a 20th order centered difference, leap-frog scheme. The ratio of the M values, $68/50 \approx 1.4$, slightly favors the cubic-spline, trapezoidal scheme. The ratio $68/6.5 \approx 10$ similarly compares the cubic spline, Padé scheme with the same 20th order centered-difference, leap-frog scheme. For $E = 10^{-4}$, $685/23 \approx 32$ compares the cubic spline, 4th order Padé scheme with

the 16th order centered difference, leap-frog scheme. On this basis we would conclude that the spline, 4th order Padé schemes are best.

We attempt to extrapolate our conclusions to nonlinear problems in [10].

5. The resolving power of smooth spline interpolation.

The reader may wonder how to convert a high-order accurate result at the equally-spaced mesh points into a high order global approximation. One way is to find a trigonometric approximation, using as many frequencies as one has confidence in. Another way is to use a smooth spline interpolant.

Golomb [5] has considered in detail the errors involved in smooth spline interpolation, at uniformly placed knots, of $\exp[2\pi i \omega x]$. From (5.9) and Lemma 6.3 of that paper, if e is the max norm over $[0, 1]$ of the interpolation error using splines of order $2m$ (degree $2m-1$), then

$$(5.1) \quad e \leq 2\gamma/(1+\gamma) \leq 4(2/N)^{2m},$$

where the number of intervals per wavelength $N > 2$, and where

$$\gamma \equiv \sum_{v=-\infty, v \neq 0}^{\infty} 1/(1+vN)^{2m}.$$

In table I (d) we exhibit the number of intervals per wavelength which, according to the first inequality of (5.1), yield a maximum error of 10^{-2} , 10^{-3} , and 10^{-4} , for orders of accuracy compatible with the orders of accuracy of our other spline differencing tables.

Let s_u and s_w be, respectively, the splines of order $2m$ interpolating the solution $u = \exp[2\pi i \omega(x+t)]$ and its approximation w . Then

$$\|u - s_w\|_{\infty} \leq \|u - s_u\|_{\infty} + \|s_u - s_w\|_{\infty} = e + \|s_u - s_w\|_{\infty}.$$

But $s_u - s_w$ is the spline interpolant, at the knots, of

$$u - w = [1 - \exp(-iPE)] \cdot \exp[2\pi i \omega(x+t)] \equiv Cu$$

where P is the number of periods computed and E , the phase error of Section 3, is a constant over the spatial mesh. Since $|C| \leq PE$,

$$\|s_u - s_w\|_{\infty} = \|Cs_u\|_{\infty} \leq PE(1+e);$$

so that

$$\|u - s_w\|_{\infty} \leq e + PE + ePE.$$

Thus, if N and M are picked so that e and PE are less than 10^{-2} , then $\|u - s_w\| \leq 2.01 \times 10^{-2}$.

For example, if we compute with a cubic spline trapezoidal scheme for one period desiring an error of $E = 10^{-2}$, we would use $N = 4.3$ (Table II). According to Table I (d), any septic spline (order 8) interpolant of u using more than 3.6 intervals per wavelength satisfies $e \leq 10^{-3}$. Hence, in fact, the septic interpolant of w gives a global error less than 1.1×10^{-2} . Indeed, the quintic interpolant of w yields errors better than 2×10^{-2} .

REFERENCES

- [1] Culham, W. E., and Varga, R. S., "Numerical methods for time-dependent, nonlinear boundary value problems," Soc. of Petroleum Engineers Journal, v. 11, 1971, pp. 374-388.
- [2] Douglas, J., A survey of numerical methods for partial differential equations, in Advances in Computers, v. 2, Academic Press, New York, 1961, pp. 1-54.
- [3] Fornberg, B., "On high order approximations of hyperbolic partial differential equations," Dept. of Computer Sciences Rpt. MR 39, Uppsala University, 1972.
- [4] Fromm, J. E., Numerical solution of the Navier-Stokes equations at high Reynolds numbers and the problem of discretization of convective derivatives, in Numerical Methods in Fluid Dynamics (J. J. Smolderen, ed.), AGARD Lecture Series, LS-48, Tech. Editing and Reprod., Ltd., Harford House, London, 1972.
- [5] Golomb, M., "Approximation by periodic spline interpolants on uniform meshes," J. Approx. Thy., v. 1, 1968, pp. 26-65.
- [6] Gradshteyn, I. S., and Ryzhik, I. M., Table of Integrals, Series, and Products; Academic Press, New York, 1965.
- [7] Kreiss, H.-O., and Oliger, J., "Comparison of accurate methods for the integration of hyperbolic equations," Tellus, v. 24, 1972, pp. 199-215.
- [8] Strang, G., "On the construction and comparison of difference schemes," SIAM J. Numer. Anal., v. 5, 1968, pp. 506-517.
- [9] Schoenberg, I. J., "Contributions to the problem of approximation of equidistant data by analytic functions," parts A and B, Quart. Appl. Math., v. 4, 1946, pp. 45-99, pp. 112-141.
- [10] Swartz, B., and Wendroff, B., "The relative efficiency of finite difference and finite element methods, I hyperbolic problems and splines"; submitted for publication.
- [11] Thomeé, V., these notes.
- [12] Thompson, P. D., Numerical Weather Analysis and Prediction, Macmillan, New York, 1961.
- [13] Wall, H. S., Analytic Theory of Continued Fractions, van Nostrand, New York, 1948.

- [14] Walsh, R. J., "Optimization and comparison of partial difference methods, to appear SIAM J. Numer. Anal., 1973.
- [15] Wendroff, B., "Spline-Galerkin methods for initial-value problems with variable coefficients," these notes.